

Fourier Transforms of Finite Chirps

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Abstract—Chirps arise in many signal processing applications. While chirps have been extensively studied both as functions over the real line and the integers, less attention has been paid to the study of chirps over finite groups.

We study the existence and properties of chirps over finite cyclic groups of integers. In particular, we introduce a new definition of a finite chirp which is slightly more general than those that have previously been used. We explicitly compute the discrete Fourier transforms of these chirps, yielding results that are number-theoretic in nature. As a consequence of these results, we determine the degree to which the elements of certain finite tight frames are well-distributed.

I. INTRODUCTION

A *linear chirp* is a function whose frequency changes linearly with time. For example, while a wave function of the form $\exp(2\pi ixt)$ has constant frequency x , the chirp $\exp(2\pi i(xt + yt^2/2))$ has frequency $x + yt$ at time $t \in \mathbb{R}$. Chirps often arise in nature as a consequence of the Doppler effect, the phenomenon by which the perceived frequency of a wave is altered whenever the wave is emanating from or reflecting off a moving body. As such, chirps have historically been of great interest in applications such as radar and sonar.

However, the study of chirps has mostly been confined to the real line and the integers, in the context of integral transforms and the chirp \mathcal{Z} -transform, respectively. Less attention has been paid to the study of chirps over finite cyclic groups, that is, to chirps over $\mathbb{Z}_a \equiv \mathbb{Z}/a\mathbb{Z} = \{0, \dots, a-1\}$, where a is a positive integer. This is in contrast to wave functions which, in the context of Fourier transforms, have been studied for many decades on arbitrary locally compact abelian groups.

At the same time, the concept of a finite chirp is by no means new. An excellent discussion of a modern application of finite chirps is given in [8], in which a *discrete chirp-Fourier transform* is introduced. *Chirplets* have been used in image processing for over a decade [5]. For that matter, discrete chirps, under a different name, were investigated by Gauss in his study of quadratic reciprocity. Indeed, the computation of *Gauss sums* [2] is equivalent to finding the Discrete Fourier Transform (DFT) of a finite chirp, and was a subject of great interest in the mid-nineteenth century. This connection between modern signal processing and classical number theory is emphasized in [1], in which the trace of the DFT matrix,

$$\text{Tr}(F_a) = \frac{1}{\sqrt{a}} \sum_{t=0}^{a-1} e^{2\pi i t^2/a},$$

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is noted to be the canonical example of a Gauss sum. An independent derivation of this trace is given in [6], giving an interesting example of when the concepts of applied signal processing may be used to prove results in pure mathematics.

The following section provides a brief introduction to the mathematics of finite signal processing, with an emphasis upon the finite form of the Poisson Summation Formula. In Section III, we introduce our definition of finite chirps, and discuss their elementary properties. In Section IV, we begin the task of computing the Fourier transforms of our chirps. In particular, we compute the magnitudes of the Fourier coefficients, and discuss their significance with respect to finite tight frames. Finally, in the last section, we complete our derivation of the Fourier transforms of these chirps.

II. PRELIMINARIES

Let \mathbb{Z}^+ be the set of all positive integers. For any $a \in \mathbb{Z}^+$, let $\mathbb{Z}_a \equiv \mathbb{Z}/a\mathbb{Z}$ be the finite group of integers in which addition is performed modulo a . Consider the space of all complex-valued functions over \mathbb{Z}_a ,

$$\ell(\mathbb{Z}_a) = \{f : \mathbb{Z}_a \rightarrow \mathbb{C}\},$$

in which function addition, scalar multiplication, and the inner product are defined in the usual fashion. We shall equivalently regard elements of $\ell(\mathbb{Z}_a)$ as a -periodic, complex-valued sequences over \mathbb{Z} , so as to consider $f(t)$ for any $t \in \mathbb{Z}$. The *Dirac basis* for $\ell(\mathbb{Z}_a)$ consists of the *delta functions* $\{\delta_a^x\}_{x \in \mathbb{Z}_a}$,

$$\delta_a^x(t) = \begin{cases} 1 & t = x \pmod{a}, \\ 0 & t \neq x \pmod{a}, \end{cases}$$

and is an orthonormal basis for $\ell(\mathbb{Z}_a)$. Meanwhile, the *Fourier basis* for $\ell(\mathbb{Z}_a)$ consists of the *wave functions* $\{m_a^x\}_{x \in \mathbb{Z}_a}$,

$$m_a^x(t) = e^{2\pi ixt/a} = w_a^{xt},$$

where $w_a \equiv e^{2\pi i/a}$ is the “first” a th root of unity. When suitably scaled by a factor of \sqrt{a} , the Fourier basis is also an orthonormal basis for $\ell(\mathbb{Z}_a)$.

For $x \in \mathbb{Z}$, the corresponding *translation* and *modulation* operators are $T_a^x, M_a^x : \ell(\mathbb{Z}_a) \rightarrow \ell(\mathbb{Z}_a)$, respectively, where,

$$\begin{aligned} (T_a^x f)(t) &= f(t-x), \\ (M_a^x f)(t) &= m_a^x(t)f(t). \end{aligned}$$

Next, the *Fourier transform* on \mathbb{Z}_a is $F_a : \ell(\mathbb{Z}_a) \rightarrow \ell(\mathbb{Z}_a)$,

$$(F_a f)(x) = \frac{1}{\sqrt{a}} \sum_{t \in \mathbb{Z}_a} f(t) w_a^{-xt}.$$

Finally, the *product* and *convolution* of any two elements $f, g \in \ell(\mathbb{Z}_a)$ are $fg, f * g \in \ell(\mathbb{Z}_a)$ respectively, where,

$$\begin{aligned} (fg)(t) &= f(t)g(t), \\ (f * g)(t) &= \sum_{s \in \mathbb{Z}_a} f(t-s)g(s). \end{aligned}$$

Several well-known properties of these operations are summarized in the following result. Due to their ubiquity, these properties are stated without proof, and will be used throughout our work without explicit reference.

Proposition 1: For any $f, g \in \ell(\mathbb{Z}_a)$ and any $x, y \in \mathbb{Z}$,

- 1) $M_a^x T_a^y = m_a^y(x) T_a^y M_a^x$,
- 2) $F_a T_a^x = M_a^{-x} F_a$,
- 3) $F_a M_a^x = T_a^x F_a$,
- 4) $F_a(fg) = (1/\sqrt{a})(F_a f) * (F_a g)$,
- 5) $F_a(f * g) = \sqrt{a}(F_a f)(F_a g)$.

We shall also make use of a finite version of the Poisson Summation Formula. Though not nearly as well-known as its infinite-dimensional cousins, this version is easily proved using finite geometric series. Here and throughout, we write $x \mid y$ to denote integers x and y such that x divides y .

Proposition 2: Let $a, b \in \mathbb{Z}^+$. Then, for all $f \in \ell(\mathbb{Z}_{ab})$,

$$\sqrt{a} \sum_{\{t \in \mathbb{Z}_{ab}, a \mid t\}} f(t) = \sqrt{b} \sum_{\{x \in \mathbb{Z}_{ab}, b \mid x\}} (F_{ab} f)(x).$$

III. FINITE CHIRPS

Perhaps the most difficult aspect of finite chirps is finding their ‘‘proper’’ definition. In [8], the lowest-order nontrivial finite chirp is defined to be $f \in \ell(\mathbb{Z}_a)$,

$$f(t) = w_a^{t^2} = e^{2\pi i t^2/a}. \quad (1)$$

We note that this chirp is well-defined over \mathbb{Z}_a , that is,

$$\begin{aligned} f(t+a) &= e^{2\pi i(t+a)^2/a} \\ &= e^{2\pi i t^2/a} e^{2\pi i(2t+a)} \\ &= e^{2\pi i t^2/a} \mathbf{1} \\ &= f(t). \end{aligned}$$

However, in the case of continuous chirps, we note that the squared term in the exponent is usually accompanied by an additional factor of $1/2$. This is done to compensate for the fact that the frequency of a wave is obtained by looking at the derivative of the wave function. From this perspective, the finite chirp should instead be defined as

$$g(t) = w_a^{t^2/2} = e^{\pi i t^2/a}.$$

Indeed, much of the original work on Gauss sums [2] takes $g(t)$ as the ‘‘canonical’’ chirp, rather than $f(t)$. However, under this definition,

$$\begin{aligned} g(t+a) &= e^{\pi i(t+a)^2/a} \\ &= e^{\pi i t^2/a} e^{\pi i(2t+a)} \\ &= e^{\pi i t^2/a} (-1)^a \\ &= (-1)^a g(t), \end{aligned}$$

and thus g is not well-defined over \mathbb{Z}_a when a is odd. When dealing with Gauss sums, this problem is usually avoided by simply summing over $\{0, \dots, a-1\}$, rather than an arbitrary collection of coset representatives. This approach is inadequate from the point of view of harmonic analysis and

signal processing, as one often needs to make translation-based changes of variables in various sums.

We remedy the problem with $g(t)$ in another way: by replacing the quadratic t^2 with $t(t-a)$. That is, we consider $c_a \in \ell(\mathbb{Z}_a)$,

$$c_a(t) = w_a^{t(t-a)/2} = e^{\pi i t(t-a)/a}.$$

This definition was originally proposed by N. Kaiblinger [4], as was presented publicly at the conference ‘‘Harmonic Analysis and Applications’’ in honor of John J. Benedetto’s 60th birthday. As Kaiblinger noted, this chirp is well-defined since

$$\begin{aligned} c_a(t+a) &= e^{\pi i(t+a)t/a} \\ &= e^{\pi i t^2/a} (-1)^t \\ &= e^{\pi i t^2/a} (-1)^{-t} \\ &= e^{\pi i t(t-a)/a} \\ &= c_a(t). \end{aligned}$$

A similar technique may be used to produce discrete versions of $e^{2\pi i t^n/n}$ for any $n \in \mathbb{Z}^+$. For example, when $n = 3$ the function $e^{2\pi i t(t-a)(t-2a)/3a}$ is well-defined over \mathbb{Z}_a .

Returning to the quadratic, we have that for any $x \in \mathbb{Z}$, we may raise c_a to the x th power in a pointwise fashion in order to obtain a well-defined x th-order chirp on \mathbb{Z}_a ,

$$c_a^x(t) = w_a^{xt(t-a)/2} = (-1)^{xt} w_a^{xt^2/2},$$

and with it, a *chirp-modulation operator* $C_a^x : \ell(\mathbb{Z}_a) \rightarrow \ell(\mathbb{Z}_a)$,

$$(C_a^x f)(t) = c_a^x(t) f(t).$$

We note that for even integers x , c_a^x reduces to a power of the finite chirp given in equation (1), and thus our definition is in fact an extension of that given in [8]. We now state and prove some elementary properties of these chirps.

Proposition 3: For $a \in \mathbb{Z}^+$, and any $t, x, y \in \mathbb{Z}$,

- 1) $\overline{c_a^x} = c_a^{-x}$,
- 2) $c_a^{x+a} = c_a^x$ when a is odd,
- 3) $c_a^{x+a} = m_a^{a/2} c_a^x$ when a is even,
- 4) $c_a^x(yt) = c_a^{xy^2}(t)$,
- 5) $c_a^x(-t) = c_a^x(t)$,
- 6) $c_a^y(t-x) = c_a^y(t) m_a^{-xy}(t) c_a^y(x)$,
- 7) $T_a^x C_a^y = c_a^y(x) M_a^{-xy} C_a^y T_a^x$,
- 8) $M_a^x C_a^y = C_a^y M_a^x$.

Proof: (1) This follows immediately from the fact that $\overline{w_a} = w_a^{-1}$. (2,3) Since $t \in \mathbb{Z}$ is even if and only if t^2 is even,

$$w_a^{at^2/2} = e^{\pi i at^2/a} = (-1)^{t^2} = (-1)^t.$$

Therefore, for any $a \in \mathbb{Z}$ we have,

$$\begin{aligned} c_a^{x+a}(t) &= (-1)^{(x+a)t} w_a^{(x+a)t^2/2} \\ &= (-1)^{xt} w_a^{xt^2/2} (-1)^{at} w_a^{at^2/2} \\ &= c_a^x(t) (-1)^{at} (-1)^t \\ &= (-1)^{(a+1)t} c_a^x(t). \end{aligned}$$

Thus, when a is odd, $c_a^{x+a}(t) = c_a^x(t)$. If, on the other hand, a is even, then $a/2 \in \mathbb{Z}$ and

$$c_a^{x+a}(t) = (-1)^t c_a^x(t) = e^{2\pi i(a/2)t/a} c_a^x(t) = m_a^{a/2} c_a^x.$$

(4) As before, note that $(-1)^y = (-1)^{y^2}$. Thus,

$$c_a^x(yt) = (-1)^{xyt} w_a^x(yt)^{2/2} = (-1)^{xy^2t} w_a^{xy^2t^2/2} = c_a^{xy^2}(t).$$

(5) Apply the previous result to $y = -1$. (6) Simply note,

$$\begin{aligned} c_a^y(t-x) &= (-1)^{y(t+x)} w_a^{y(t-x)^2/2} \\ &= (-1)^{yt} (-1)^{yx} w_a^{yt^2/2} w_a^{-yxt} w_a^{yx^2/2} \\ &= c_a^y(t) m_a^{-xy}(t) c_a^y(x). \end{aligned}$$

(7) This follows immediately from the previous result. (8) This follows immediately from the fact that both M_a^x and C_a^y are multiplicative operators. ■

We next discuss a simple way in which our finite chirps may be used to provide overcomplete decompositions of elements of $\ell(\mathbb{Z}_a)$. In particular, we consider the collection of a^2 functions obtained by taking every modulation of $\{c_a^y\}_{y=0}^{a-1}$. That is, we consider the collection:

$$\{m_a^x c_a^y\}_{y=0, x \in \mathbb{Z}_a}^{a-1}.$$

Note that by the previous result, this collection is “complete,” namely that it contains every possible modulation of every possible chirp. Furthermore, this collection is effectively translation invariant, as the translation of any element is a unit-scalar multiple of another element.

The next result shows that this collection is a finite *tight frame* for $\ell(\mathbb{Z}_a)$, and thus possesses an exceptionally efficient means by which any $f \in \ell(\mathbb{Z}_a)$ may be decomposed into a linear combination of the collection’s elements. The argument used is essentially the same idea used to show that any union of orthonormal bases is a tight frame.

Proposition 4: For any $a \in \mathbb{Z}^+$,

$$f = \frac{1}{a^2} \sum_{y=0}^{a-1} \sum_{x \in \mathbb{Z}_a} \langle f, m_a^x c_a^y \rangle m_a^x c_a^y$$

for all $f \in \ell(\mathbb{Z}_a)$.

Proof: We begin by recalling that the normalized Fourier basis $\{(1/\sqrt{a})m_a^x\}_{x \in \mathbb{Z}_a}$ is an orthonormal basis for $\ell(\mathbb{Z}_a)$. For any $y \in \mathbb{Z}$, the fact that $|c_a^y(t)| = 1$ for all $t \in \mathbb{Z}_a$ automatically implies that $\{(1/\sqrt{a})m_a^x c_a^y\}_{x \in \mathbb{Z}_a}$ is also an orthonormal basis for $\ell(\mathbb{Z}_a)$. Thus, for any $y \in \mathbb{Z}$, Parseval’s identity gives

$$af = \sum_{x \in \mathbb{Z}_a} \langle f, m_a^x c_a^y \rangle m_a^x c_a^y$$

for all $f \in \ell(\mathbb{Z}_a)$. Summing these orthonormal basis decompositions over all $y = 0, \dots, a-1$ gives

$$a^2 f = \sum_{y=0}^{a-1} \sum_{x \in \mathbb{Z}_a} \langle f, m_a^x c_a^y \rangle m_a^x c_a^y$$

for all $f \in \ell(\mathbb{Z}_a)$. ■

We conclude this section by noting that not all tight frames are equally useful. As discussed in detail in [7], tight frames tend to perform better in applications when the frame elements

are designed to be as *uncorrelated* as possible. In our context, the degree to which our frame elements are uncorrelated is measured by the quantity:

$$\max_{(x_1, y_1) \neq (x_2, y_2)} |\langle m_a^{x_1} c_a^{y_1}, m_a^{x_2} c_a^{y_2} \rangle|.$$

As demonstrated below, this quantity may be explicitly computed by first finding the magnitudes of the Discrete Fourier Transforms of an arbitrary finite chirp.

IV. INNER PRODUCTS OF MODULATED CHIRPS

The remainder of this paper is dedicated to the computation of the Fourier transforms of our finite chirps. The following result, though simple, nevertheless demonstrates the advantage of having an extra “1/2” term present in the definition of our chirps, as it allows us to easily complete a square that arises in the exponents.

Lemma 5: For any $a \in \mathbb{Z}^+$ and any $b, x, y \in \mathbb{Z}$,

$$(F_a c_a^b)(x+by) = \overline{m_a^x(y) c_a^b(y)} (F_a c_a^b)(x).$$

Proof: We first note that

$$\begin{aligned} (F_a c_a^b)(x+by) &= \frac{1}{\sqrt{a}} \sum_{t \in \mathbb{Z}_a} c_a^b(t) w_a^{-(x+by)t} \\ &= \frac{1}{\sqrt{a}} \sum_{t \in \mathbb{Z}_a} (-1)^{bt} w_a^{bt^2/2} w_a^{-xt} w_a^{-byt}. \end{aligned}$$

Next, we complete the square in the exponents, that is, we observe that $bt^2 - 2byt = b[(t-y)^2 - y^2]$, and therefore

$$\begin{aligned} (F_a c_a^b)(x+by) &= \frac{1}{\sqrt{a}} \sum_{t \in \mathbb{Z}_a} (-1)^{bt} w_a^{b[(t-y)^2 - y^2]/2} w_a^{-xt} \\ &= (-1)^{by} w_a^{-by^2/2} \frac{1}{\sqrt{a}} \sum_{t \in \mathbb{Z}_a} (-1)^{b(t-y)} w_a^{b(t-y)^2/2} w_a^{-xt} \\ &= \overline{c_a^b(y)} \frac{1}{\sqrt{a}} \sum_{t \in \mathbb{Z}_a} (-1)^{bt} w_a^{bt^2/2} w_a^{-x(t+y)} \\ &= \overline{c_a^b(y)} \frac{1}{\sqrt{a}} \sum_{t \in \mathbb{Z}_a} c_a^b(t) w_a^{-x(t+y)}. \end{aligned}$$

To conclude, note that since c_a^b is well-defined over \mathbb{Z}_a , we may legitimately make a change of variables by translation:

$$\begin{aligned} (F_a c_a^b)(x+by) &= w_a^{-yx} \overline{c_a^b(y)} \frac{1}{\sqrt{a}} \sum_{t \in \mathbb{Z}_a} c_a^b(t) w_a^{-xt} \\ &= \overline{m_a^x(y) c_a^b(y)} (F_a c_a^b)(x). \end{aligned}$$

The next result concerns the magnitudes of the Fourier transforms of our chirps. Here and throughout, we write (a, b) to denote the greatest common divisor of $a, b \in \mathbb{Z}$. This result also requires us to make the following definition: two integers a and b have a *common power of two* if there exists odd integers α and β such that $a = 2^m \alpha$ and $b = 2^m \beta$ for some positive integer m . That is, a and b have a common power of two if the exponents of two in their respective prime factorizations are equal.

Proposition 6: Let $a \in \mathbb{Z}^+$, $b \in \mathbb{Z}$ have greatest common divisor (a, b) . If a and b do not have a common power of two,

$$|(F_a c_a^b)(x)|^2 = \begin{cases} (a, b) & x \in (a, b)\mathbb{Z} \\ 0 & x \notin (a, b)\mathbb{Z} \end{cases}.$$

Alternatively, if a and b do have a common power of two,

$$|(F_a c_a^b)(x)|^2 = \begin{cases} (a, b) & x \in (a, b)(\mathbb{Z} + \frac{1}{2}) \\ 0 & x \notin (a, b)(\mathbb{Z} + \frac{1}{2}) \end{cases}.$$

Proof: To begin, fix $x \in \mathbb{Z}$, and sum the statement of Lemma 5 over all $y \in \mathbb{Z}_a$, rescaling by a factor of \sqrt{a} :

$$\begin{aligned} \frac{1}{\sqrt{a}} \sum_{y \in \mathbb{Z}_a} (F_a c_a^b)(x + by) &= \frac{1}{\sqrt{a}} \sum_{y \in \mathbb{Z}_a} \overline{m_a^x(y) c_a^b(y)} (F_a c_a^b)(x) \\ &= (F_a c_a^b)(x) \frac{1}{\sqrt{a}} \sum_{y \in \mathbb{Z}_a} \overline{c_a^b(y) w_a^{xy}} \\ &= (F_a c_a^b)(x) \overline{(F_a c_a^b)(-x)}. \end{aligned}$$

Next, note that Proposition 3 gives that c_a^b is an even function on \mathbb{Z}_a , namely that $c_a^b(-t) = c_a^b(t)$ for all $t \in \mathbb{Z}_a$. As with any locally compact abelian group, the Fourier transform of an even function is even, and so $(F_a c_a^b)(-x) = (F_a c_a^b)(x)$. Simplifying, we obtain

$$|(F_a c_a^b)(x)|^2 = \frac{1}{\sqrt{a}} \sum_{y \in \mathbb{Z}_a} (F_a c_a^b)(x + by).$$

To continue, we use the easily-proven fact that

$$\sum_{t \in \mathbb{Z}_a} f(bt) = (a, b) \sum_{\{x \in \mathbb{Z}_a, (a, b)|x\}} f(x)$$

for any $f \in \ell(\mathbb{Z}_a)$ and for any $b \in \mathbb{Z}$. Applying this fact when $f = F_a M_a^{-x} c_a^b$ gives

$$\begin{aligned} |(F_a c_a^b)(x)|^2 &= \frac{1}{\sqrt{a}} \sum_{y \in \mathbb{Z}_a} (T_a^{-x} F_a c_a^b)(by) \\ &= \frac{1}{\sqrt{a}} \sum_{y \in \mathbb{Z}_a} (F_a M_a^{-x} c_a^b)(by) \\ &= \frac{(a, b)}{\sqrt{a}} \sum_{\{z \in \mathbb{Z}_a, (a, b)|z\}} (F_a M_a^{-x} c_a^b)(z). \end{aligned}$$

Next, we apply the Poisson Summation Formula. Specifically, in terms of Proposition 2, taking “ a ,” “ b ,” and “ f ” to be $a/(a, b)$, (a, b) and $F_a M_a^{-x} c_a^b$, respectively, yields

$$\begin{aligned} |(F_a c_a^b)(x)|^2 &= \frac{(a, b)}{\sqrt{a}} \frac{\sqrt{a/(a, b)}}{\sqrt{(a, b)}} \sum_{\{t \in \mathbb{Z}_a, [a/(a, b)]|t\}} (M_a^{-x} c_a^b)(t) \\ &= \sum_{u=0}^{(a, b)-1} (M_a^{-x} c_a^b)(au/(a, b)) \\ &= \sum_{u=0}^{(a, b)-1} w_a^{-xau/(a, b)} (-1)^{bau/(a, b)} w_a^{ba^2 u^2 / 2(a, b)^2}. \end{aligned}$$

To simplify this expression, we note that

$$w_a^{-xau/(a, b)} = e^{-2\pi i x a u / a(a, b)} = e^{-2\pi i x u / (a, b)}.$$

Furthermore, since $(-1)^{u^2} = (-1)^u$ for all $u \in \mathbb{Z}$, then

$$w_a^{ba^2 u^2 / 2(a, b)^2} = e^{\pi i b a^2 u^2 / a(a, b)^2} = (-1)^{uab/(a, b)^2}.$$

Thus,

$$|(F_a c_a^b)(x)|^2 = \sum_{u=0}^{(a, b)-1} e^{-2\pi i x u / (a, b)} (-1)^{uab[(a, b)+1]/(a, b)^2}.$$

To conclude, note that if a and b do not have a common power of two, then either $a/(a, b)$, $b/(a, b)$ or $(a, b) + 1$ is even. Our expression for $|(F_a c_a^b)(x)|^2$ is a geometric series:

$$\begin{aligned} |(F_a c_a^b)(x)|^2 &= \sum_{u=0}^{(a, b)-1} e^{-2\pi i x u / (a, b)} \\ &= \begin{cases} (a, b) & x/(a, b) \in \mathbb{Z} \\ 0 & x/(a, b) \notin \mathbb{Z} \end{cases} \\ &= \begin{cases} (a, b) & x \in (a, b)\mathbb{Z} \\ 0 & x \notin (a, b)\mathbb{Z} \end{cases}. \end{aligned}$$

If, on the other hand, a and b have a common power of two, then $a/(a, b)$, $b/(a, b)$ and $(a, b) + 1$ are all odd. In this case, we obtain a slightly different geometric series:

$$\begin{aligned} |(F_a c_a^b)(x)|^2 &= \sum_{u=0}^{(a, b)-1} (-1)^u e^{-2\pi i x u / (a, b)} \\ &= \sum_{u=0}^{(a, b)-1} e^{-2\pi i [x/(a, b) - 1/2] u} \\ &= \begin{cases} (a, b) & x/(a, b) - 1/2 \in \mathbb{Z} \\ 0 & x/(a, b) - 1/2 \notin \mathbb{Z} \end{cases} \\ &= \begin{cases} (a, b) & x \in (a, b)(\mathbb{Z} + \frac{1}{2}) \\ 0 & x \notin (a, b)(\mathbb{Z} + \frac{1}{2}) \end{cases}. \end{aligned}$$

We also have the following near-immediate corollary:

Corollary 7: For any $a \in \mathbb{Z}^+$ and any $x_1, x_2, y_1, y_2 \in \mathbb{Z}$, we have $|\langle m_a^{x_1} c_a^{y_1}, m_a^{x_2} c_a^{y_2} \rangle|^2 = a(a, y_1 - y_2)$ when either:

- a and $y_1 - y_2$ do not have a common power of two and $x_1 - x_2$ is an integer multiple of $(a, y_1 - y_2)$,
- a and $y_1 - y_2$ have a common power of two and $x_1 - x_2$ is a half-integer multiple of $(a, y_1 - y_2)$,

and we have $\langle m_a^{x_1} c_a^{y_1}, m_a^{x_2} c_a^{y_2} \rangle = 0$ otherwise. In particular, when $a = p$ is prime,

$$|\langle m_p^{x_1} c_p^{y_1}, m_p^{x_2} c_p^{y_2} \rangle|^2 = \begin{cases} p^2 & \text{if } y_1 = y_2, x_1 = x_2 \\ p & \text{if } y_1 \neq y_2 \\ 0 & \text{if } y_1 = y_2, x_1 \neq x_2 \end{cases},$$

where all equivalences between either x_1 and x_2 , or y_1 and y_2 , are taken mod p .

Proof: For any $x_1, x_2, y_1, y_2 \in \mathbb{Z}$,

$$\begin{aligned} |\langle m_a^{x_1} c_a^{y_1}, m_a^{x_2} c_a^{y_2} \rangle|^2 &= |\langle c_a^{y_1 - y_2}, m_a^{x_2 - x_1} \rangle|^2 \\ &= a |(F_a c_a^{y_1 - y_2})(x_2 - x_1)|^2. \end{aligned}$$

Returning to the issue brought up at the end of the previous section, the level of “uncorrelation” in our tight frames is:

$$\max_{(x_1, y_1) \neq (x_2, y_2)} |\langle m_a^{x_1} c_a^{y_1}, m_a^{x_2} c_a^{y_2} \rangle| = \sqrt{ab},$$

where b is the largest proper divisor of a . As such, the “best” modulated chirp frames are obtained when a is prime, a fact which was originally noted in [8].

V. FOURIER TRANSFORMS OF FINITE CHIRPS

In our final results, we compute the actual values of the Fourier transforms of our finite chirps. Remarkably, we see that the Fourier transform of one chirp is a decimated version of another chirp.

Theorem 8: If $a \in \mathbb{Z}^+$, $b \in \mathbb{Z}$ do not have a common power of two, then,

$$\frac{(F_a c_a^b)(x)}{(F_a c_a^b)(0)} = \begin{cases} c_a^{-bd^2}(x/(a, b)) & x \in (a, b)\mathbb{Z} \\ 0 & x \notin (a, b)\mathbb{Z} \end{cases},$$

where $d \in \mathbb{Z}$, $bd = (a, b) \bmod a$.

Proof: Take any $x \in \mathbb{Z}$. If (a, b) does not divide x , then Proposition 6 implies $(F_a c_a^b)(x) = 0$. If (a, b) does divide x , take $t \in \mathbb{Z}$ such that $x = (a, b)t$, and take $c, d \in \mathbb{Z}$ such that $ac + bd = (a, b)$. Since $F_a c_a^b$ is well-defined over \mathbb{Z}_a ,

$$\begin{aligned} (F_a c_a^b)(x) &= (F_a c_a^b)((a, b)t) \\ &= (F_a c_a^b)(act + bdt) \\ &= (F_a c_a^b)(bdt). \end{aligned}$$

Applying Lemma 5 with “ x ” and “ y ” taken to be 0 and dt , respectively, we have

$$(F_a c_a^b)(x) = \overline{m_a^0(dt)} c_a^b(dt) (F_a c_a^b)(0).$$

Next, note that Proposition 6 implies $(F_a c_a^b)(0) \neq 0$. Also, $m_a^0(dt) = 1$. Thus, by Proposition 3,

$$\frac{(F_a c_a^b)(x)}{(F_a c_a^b)(0)} = \overline{c_a^b(dt)} = c_a^{-bd^2}(t) = c_a^{-bd^2}(x/(a, b)).$$

■

Theorem 9: Let $a \in \mathbb{Z}^+$, $b \in \mathbb{Z}$ have a common power of two, that is, let $a = 2^m \alpha$ and $b = 2^m \beta$ with α and β odd. Then,

$$\begin{aligned} &\frac{1}{2^{m/2}} \frac{(F_a c_a^b)(x)}{(F_a c_a^\beta)(0)} \\ &= \begin{cases} c_a^{-\beta \delta^2}((x - a/2)/(a, b)) & x \in (a, b)(\mathbb{Z} + \frac{1}{2}) \\ 0 & x \notin (a, b)(\mathbb{Z} + \frac{1}{2}) \end{cases}, \end{aligned}$$

where $\delta \in \mathbb{Z}$, $\beta \delta = (\alpha, \beta) \bmod \alpha$.

Proof: Take any $x \in \mathbb{Z}$. If x is not a half-integer multiple of (a, b) , then Proposition 6 implies $(F_a c_a^b)(x) = 0$. If x is a half-integer multiple of (a, b) , let $t = (x - a/2)/(a, b)$, so that $x = (a, b)t + a/2$. Note that since a and b have a common factor of two, then $a/2(a, b) \in \mathbb{Z} + \frac{1}{2}$. Since $x/(a, b) \in \mathbb{Z} + \frac{1}{2}$ as well, then $t \in \mathbb{Z}$.

Take $c, d \in \mathbb{Z}$ such that $ac + bd = (a, b)$. Since $a = 2^m \alpha$ and $b = 2^m \beta$ with α and β odd, we have

$$2^m(\alpha c + \beta d) = (a, b) = (2^m \alpha, 2^m \beta) = 2^m(\alpha, \beta),$$

and thus $\alpha c + \beta d = (\alpha, \beta)$. Letting $\delta = d$, we therefore have $\beta \delta = (\alpha, \beta) \bmod \alpha$.

Continuing, note that since $F_a c_a^b$ is well-defined over \mathbb{Z}_a ,

$$\begin{aligned} (F_a c_a^b)(x) &= (F_a c_a^b)((a, b)t + a/2) \\ &= (F_a c_a^b)(act + b\delta t + a/2) \\ &= (F_a c_a^b)(b\delta t + a/2). \end{aligned}$$

Applying Lemma 5 with “ x ” and “ y ” taken to be $a/2$ and δt , respectively, we have

$$(F_a c_a^b)(x) = \overline{m_a^{a/2}(\delta t)} c_a^b(\delta t) (F_a c_a^b)(a/2).$$

To continue, note that for any $u \in \mathbb{Z}$,

$$m_a^{a/2}(u) = w_a^{(a/2)u} = e^{2\pi i(a/2)u/a} = (-1)^u = (-1)^{\beta u},$$

since β is odd. Similarly, since b is even,

$$\begin{aligned} c_a^b(u) &= (-1)^{bu} w_a^{bu^2/2} \\ &= e^{\pi i b u^2 / a} \\ &= e^{\pi i 2^m \beta u^2 / (2^m \alpha)} \\ &= e^{\pi i \beta u^2 / \alpha}. \end{aligned}$$

Taken together, we have that

$$m_a^{a/2}(u) c_a^b(u) = (-1)^{\beta u} e^{\pi i \beta u^2 / \alpha} = c_\alpha^\beta(u),$$

and consequently,

$$(F_a c_a^b)(x) = \overline{c_\alpha^\beta(\delta t)} (F_a c_a^b)(a/2).$$

Similarly,

$$\begin{aligned} (F_a c_a^b)(a/2) &= \frac{1}{\sqrt{a}} \sum_{u \in \mathbb{Z}_a} c_a^b(u) m_a^{-a/2}(u) \\ &= \frac{1}{\sqrt{a}} \sum_{u \in \mathbb{Z}_a} c_\alpha^\beta(u) \\ &= \frac{2^m}{\sqrt{2^m \alpha}} \sum_{u \in \mathbb{Z}_\alpha} c_\alpha^\beta(u) \\ &= 2^{m/2} F_\alpha c_\alpha^\beta(0), \end{aligned}$$

and therefore,

$$(F_a c_a^b)(x) = \overline{c_\alpha^\beta(\delta t)} 2^{m/2} F_\alpha c_\alpha^\beta(0).$$

Finally, as Proposition 6 implies that $F_\alpha c_\alpha^\beta(0) \neq 0$, then

$$\frac{1}{2^{m/2}} \frac{(F_a c_a^b)(x)}{F_\alpha c_\alpha^\beta(0)} = \overline{c_\alpha^\beta(\delta t)} = c_\alpha^{-\beta \delta^2}((x - a/2)/(a, b)),$$

by Proposition 3. ■

As a final remark, we note that in each of these results, we have only determined the Fourier transforms of our chirps up to a scalar multiple. To resolve the matter further, one must explicitly determine the values $c_a^b(0)$ for all a and b that do not share a common power of two. In fact, c_a^b may be shown to always be $\sqrt{(a, b)}$ times an eighth root of unity. This problem was classically solved using sophisticated techniques of number theory involving special functions and complex analysis. However, the techniques presented in this paper may also be further developed to provide independent, elementary proofs of these same results [3].

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