A DECOMPOSITION THEOREM FOR FRAMES AND THE WEAK FEICHTINGER CONJECTURE

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We will show that every bounded Bessel sequence can be decomposed into two subsets each of which is an arbitrarily small perturbation of a sequence with a finite orthogonal decomposition. As an application of this result we prove that the weak Feichtinger Conjecture is equivalent to the Feichtinger Conjecture.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space and let I be a countable index set. A family $\{f_i\}_{i\in I}$ is a *frame* for $\mathcal{H}$, if there exist $0 < A \le B < \infty$ such that for all $g \in \mathcal{H}$,

$$A\,\|g\|^2 \le \sum_{i\in I} |\langle g, f_i\rangle|^2 \le B\,\|g\|^2 . \tag{1}$$

The constants A and B are called *lower* and *upper frame bounds* for the frame. We call a frame $\{f_i\}_{i\in I}$ *bounded*, if there exists $\delta > 0$ such that $\|f_i\| \ge \delta$ for all $i \in I$ (the norms of the frame elements are always uniformly bounded from above [**3**, Proposition 4.6]), and *unit norm*, if $\|f_i\| = 1$ for all $i \in I$. If $\{f_i\}_{i\in I}$ is a frame only for its closed linear span, we call it a *frame sequence*. Those sequences which satisfy the upper inequality in (1) are called *Bessel sequences*. A family $\{f_i\}_{i\in I}$ is a *Riesz basis* for $\mathcal{H}$ if the sequence is complete and there exist $0 < A \le B < \infty$ such that for all sequences of scalars $c = \{c_i\}_{i\in I}$,

$$A \sum_{i\in I} |c_i|^2 \le \left\| \sum_{i\in I} c_i f_i \right\|^2 \le B \sum_{i\in I} |c_i|^2.$$

If $\{f_i\}_{i\in I}$ is a Riesz basis only for its closed linear span, we call it a *Riesz basic sequence*.

Now we can state the main conjecture we will be addressing in this paper.

Conjecture 1.1 (Feichtinger Conjecture)**.** Every bounded frame can be written as a finite union of Riesz basic sequences.

Much work has been done on the Feichtinger Conjecture in just the last few years [**1**, **2**, **4**, **7**, **10**, **13**]. This is because the conjecture is not just interesting and important for frame theory but also is connected to the infamous 1959 Kadison-Singer Conjecture [**11**], which is known to be equivalent to the paving conjecture. In [**4**], it was shown that the Kadison-Singer Conjecture implies the Feichtinger Conjecture. It is unknown if these two problems are equivalent - but the results of [**4**] indicate they are certainly very close. In particular, it is proven in [**4**] that the Feichtinger Conjecture is equivalent to the conjectured (weak) generalization of the Bourgain-Tzafriri restricted-invertibility theorem.

It is easily seen, by just normalizing the frame vectors, that we may assume that the frame is a unit norm frame in Conjecture 1.1. It also follows easily that we only need to assume that the sequence is a bounded Bessel sequence in Conjecture 1.1. That is, by adding an orthonormal basis to the Bessel sequence we obtain a bounded frame which can be written as a finite union of Riesz basic sequences if and only if the original Bessel sequence can be written this way. Thus the Feichtinger Conjecture reduces to the conjecture that every unit norm Bessel sequence can be written as a finite union of Riesz basic sequences.

In this paper we will deal with the following version of Conjecture 1.1, which we call the *weak Feichtinger Conjecture*.

Conjecture 1.2 (Weak Feichtinger Conjecture)**.** Every unit norm Bessel sequence can be written as a finite union of frame sequences.

In fact we will prove the following result.

Theorem 1.3. *The Feichtinger Conjecture is equivalent to the weak Feichtinger Conjecture.*

Part of the motivation of this work is a surprising result of Casazza, Christensen, and Kalton [**5**] concerning frames of translates, which shows that the set of translates of a function in $L^2(\mathbb{R})$ with respect to a subset of $\mathbb{N}$ is a frame sequence if and only if it is a Riesz basic sequence.

As a main ingredient for the proof of Theorem 1.3, we will prove a decomposition theorem for frames, which is very interesting in its own right. By providing an explicit construction, we will show that even each unit norm Bessel sequence can be decomposed into two subsequences in such a way that both are small perturbations of "ideal" sequences. Our idea of an ideal sequence is a sequence for which there exists a partition of its elements into finite sets such that the spans of the elements of those sets are mutually orthogonal; thus, properties of the sequence are completely determined by properties of its local components. This definition is inspired by a more general notion called frames of subspaces [**6**].

This paper is organized as follows. In Section 2 we will give the definition of ϵ-perturbation, formalize the notion of an ideal sequence, and state some basic results. Section 3 contains the Decomposition Theorem and a discussion concerning an improvement of its proof and concerning the necessity of decomposing into *two* subsequences, whereas the proof of Theorem 1.3 will be given in Section 4.

2. Definitions and basic results

For the remainder let $\mathcal{H}$ be a separable Hilbert space and let I be an index set.

There exist many different definitions for a sequence being a perturbation of a given sequence. In this paper we will use the following.

Definition 2.1. Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be sequences in $\mathcal{H}$ with $g_j \in \overline{\text{span}}_{i\in I}\{f_i\}$ for all $j \in I$, and let $\epsilon > 0$. If

$$\sum_{i\in I} \|f_i - g_i\|^2 \leq \epsilon,$$

then $\{g_i\}_{i\in I}$ is called an *ϵ-perturbation of* $\{f_i\}_{i\in I}$.

One advantage of this definition is that it implies the hypotheses of a perturbation theorem of Christensen [**8**, Corollary 15.1.5] (see also Theorem 4.1), which we will employ for the proof of Theorem 1.3.

For finite frames, it is precisely the interaction of the frame vectors which makes them interesting and applicable to a broad spectrum of applied problems. For infinite frames, the ideal situation is for the global properties of the frame to be determined locally. We formalize this idea in the next definition in the more general setting of sequences.

Definition 2.2. Let $\{f_i\}_{i\in I}$ be a sequence in $\mathcal{H}$. We say $\{f_i\}_{i\in I}$ possesses a *finite orthogonal decomposition,* if I can be partitioned into finite sets $\{I_j\}_{j=1}^{\infty}$ so that

$$\text{span}_{i\in I}\{f_i\} = \left(\sum_{j=1}^{\infty} \oplus \, \text{span}_{i\in I_j}\{f_i\}\right)_{\ell_2}.$$

Equivalently, there must exist an orthogonal family of finite dimensional subspaces $\{\mathcal{H}_j\}_{j=1}^{\infty}$ of $\mathcal{H}$ so that for all $i \in I$ there exists $j \in \mathbb{N}$ with $f_i \in \mathcal{H}_j$.

Notice that the orthogonal family of finite dimensional subspaces forms an orthonormal basis of subspaces and in this sense is a special case of a frame of subspaces [**6**].

The next proposition expresses the fact that global properties of a sequence which possesses a finite orthogonal decomposition are determined locally. Recall that a sequence $(f_i)_{i\in I}$ is called *ω-independent* if, whenever $c = (c_i)_{i\in I}$ is a sequence of scalars and $\sum_{i\in I} c_i f_i = 0$, it follows that $c = 0$.

Proposition 2.3. *Let $\{f_i\}_{i\in I}$ possess a finite orthogonal decomposition given by the finite sets $\{I_j\}_{j=1}^{\infty}$. Then the following hold.*

(i) *$\{f_i\}_{i\in I}$ is ω-independent if and only if each $\{f_i\}_{i\in I_j}$ is linearly independent.*

(ii) *Let A and B be the optimal lower and upper frame bounds of $\{f_i\}_{i\in I}$, respectively. For each $j\in\mathbb{N}$, let A_j and B_j be the optimal lower and upper frame bound of $\{f_i\}_{i\in I_j}$, respectively. Then*

$$A = \inf_{1\le j<\infty} A_j, \qquad B = \sup_{1\le j<\infty} B_j.$$

Proof. To prove (i), just note that

$$\sum_{i\in I} c_i f_i = 0 \text{ if and only if, for all } j\in\mathbb{N}, \text{ we have } \sum_{i\in I_j} c_i f_i = 0.$$

It remains to show (ii). For this, let $g\in\mathcal{H}$. Then there exist $g_j\in\mathcal{H}_j$, $j\in\mathbb{N}$ such that we can write $g=\sum_{j=1}^{\infty}\oplus g_j$. We compute

$$\sum_{i\in I} |\langle g, f_i\rangle|^2 = \sum_{j=1}^{\infty}\sum_{i\in I_j} |\langle g_j, f_i\rangle|^2$$

$$\ge \sum_{j=1}^{\infty} A_j \|g_j\|^2 \ge \left(\inf_{1\le j<\infty} A_j\right)\left(\sum_{j=1}^{\infty}\|g_j\|^2\right) = \left(\inf_{1\le j<\infty} A_j\right)\|g\|^2.$$

This proves that $\inf_{1\le j<\infty} A_j$ is a lower frame bound for $\{f_i\}_{i\in I}$ in $\mathcal{H}$. The fact that it is optimal follows by choosing for any $j\in\mathbb{N}$, some $g\in\mathcal{H}_j$ with

$$\sum_{i\in I_j} |\langle g, f_i\rangle|^2 = A_j\|g\|^2.$$

The claim concerning the upper frame bound can be proven using a similar argument. □

The following lemma is well-known, but since the proof is short, we include it for completeness.

Lemma 2.4. *Let $\{f_i\}_{i\in I}$ be a Bessel sequence in $\mathcal{H}$, and let P be a finite rank projection on $\mathcal{H}$. Then*

$$\sum_{i\in I}\|Pf_i\|^2 < \infty.$$

Proof. Let $\mathcal{K}$ be the projection space of P with dimension d, and let S denote the frame operator of $\{Pf_i\}_{i\in I}$, i.e., $S(g)=\sum_{i\in I}\langle g, f_i\rangle f_i$ for all

$g \in \mathcal{H}$. Further, let $\{e_j\}_{j=1}^d$ be an orthonormal eigenvector basis for $\mathcal{K}$ with respect to S and respective eigenvalues $\{\lambda_j\}_{j=1}^d$. Then we obtain

$$\sum_{i\in I} \|Pf_i\|^2 = \sum_{j=1}^{d} \sum_{i\in I} |\langle f_i, e_j\rangle|^2 = \sum_{j=1}^{d} \lambda_j < \infty.$$

□

3. The Decomposition Theorem

The following theorem states that we can decompose each unit norm Bessel sequence into two subsequences such that both are ϵ-perturbations of sequences which possess a finite orthogonal decomposition. In fact, we will even derive an explicit algorithm for generating this partition. The proof is inspired by *blocking arguments* from Banach space theory [**12**]. The Decomposition Theorem is also the main ingredient for the proof of Theorem 1.3.

Theorem 3.1 (Decomposition Theorem). *Let $\{f_i\}_{i\in I}$ be a unit norm Bessel sequence in $\mathcal{H}$, and let $\epsilon > 0$. Then there exists a partition $I = I_1 \cup I_2$ such that, for $j = 1, 2$, the sequence $\{f_i\}_{i\in I_j}$ is an ϵ-perturbation of some sequence $\{g_i\}_{i\in I_j}$ in $\mathcal{H}$, which possesses a finite orthogonal decomposition.*

Proof. Let $\{f_i\}_{i\in I}$ be a unit norm Bessel sequence in $\mathcal{H}$, and let $\epsilon > 0$. Without loss of generality we may assume that $I = \mathbb{N}$, since if I is finite we are done.

In the first step we will define a strictly increasing sequence $\{n_i\}_{i=1}^\infty$ in $\mathbb{N}$ by an induction argument. In the second step, we show that by defining $I_j := \bigcup_{i=0}^\infty \{n_{2i+(j-1)} + 1, \ldots, n_{2i+j}\}$, for each $j = 1, 2$, the sequence $\{f_i\}_{i\in I_j}$ is an ϵ-perturbation of some sequence $\{g_i\}_{i\in I_j}$ in $\mathcal{H}$, which possesses a finite orthogonal decomposition.

For the initial induction step, we set $n_1 := 1$. Further, we define S_1 by $S_1 := \{n_1\}$, and let P_1 denote the orthogonal projection onto $\operatorname{span}_{i\in S_1}\{f_i\}$. To construct n_2, observe that, by Lemma 2.4, we have

$$\sum_{i=1}^{\infty} \|P_1 f_i\|^2 < \infty.$$

Therefore we can choose $n_2 > n_1$ so that

$$\sum_{i=n_2+1}^{\infty} \|P_1 f_i\|^2 < \frac{\epsilon}{2}. \tag{2}$$

Using this new element of our sequence, we define T_1 by $T_1 := \{n_1 + 1, \ldots, n_2\}$ and let Q_1 denote the orthogonal projection onto $\operatorname{span}_{i\in T_1}\{f_i\}$.

We proceed by induction. Notice that in each induction step we will define two new elements of our sequence. Let $k \in \mathbb{N}$ and suppose that we have already constructed $n_1, \ldots, n_{2k}$ and defined $\{S_m\}_{m=1}^k$, $\{T_m\}_{m=1}^k$, $\{P_m\}_{m=1}^k$, and $\{Q_m\}_{m=1}^k$. In the following induction step we will construct n_{2k+1} and n_{2k+2}, and define S_{k+1}, T_{k+1}, P_{k+1}, and Q_{k+1}. First, we employ Lemma 2.4, which implies that

$$\sum_{i=1}^{\infty} \|Q_k f_i\|^2 < \infty.$$

Therefore we can choose $n_{2k+1} > n_{2k}$ so that

$$\sum_{i=n_{2k+1}+1}^{\infty} \|Q_k f_i\|^2 < \frac{\epsilon}{2^{2k}}. \tag{3}$$

Now let S_{k+1} be defined by $S_{k+1} := \{n_{2k}+1, \ldots, n_{2k+1}\}$, and let P_{k+1} denote the orthogonal projection of $\mathcal{H}$ onto $\operatorname{span}_{i \in \bigcup_{m=1}^{k+1} S_m}\{f_i\}$. Secondly, again by Lemma 2.4, we have

$$\sum_{i=1}^{\infty} \|P_{k+1} f_i\|^2 < \infty.$$

Thus there exists $n_{2k+2} > n_{2k+1}$ such that

$$\sum_{i=n_{2k+2}+1}^{\infty} \|P_{k+1} f_i\|^2 < \frac{\epsilon}{2^{2k+1}}. \tag{4}$$

Hence we define the set T_{k+1} by $T_{k+1} := \{n_{2k+1}+1, \ldots, n_{2k+2}\}$, and let Q_{k+1} denote the orthogonal projection of $\mathcal{H}$ onto $\operatorname{span}_{i \in \bigcup_{m=1}^{k+1} T_m}\{f_i\}$. Iterating this procedure yields a sequence $\{n_i\}_{i=1}^{\infty}$ and, in particular, we obtain a partition $\{S_m\}_{m=1}^{\infty} \cup \{T_m\}_{m=1}^{\infty}$ of $\mathbb{N}$.

For the second step let $\{S_m\}_{m=1}^{\infty}$ and $\{T_m\}_{m=1}^{\infty}$ be defined as in the induction argument. Then we define I_1 and I_2 by

$$I_1 := \bigcup_{m=1}^{\infty} S_m \quad \text{and} \quad I_2 := \bigcup_{m=1}^{\infty} T_m.$$

It remains to prove that, for each $j = 1, 2$, we can construct a sequence $\{g_i\}_{i \in I_j}$ in $\mathcal{H}$ such that $\{g_i\}_{i \in I_j}$ has a finite orthogonal decomposition and $\{f_i\}_{i \in I_j}$ is an ϵ-perturbation of it.

In the following we will prove the claim only for $j = 1$. The case $j = 2$ can be dealt with in a similar manner. Using the sequence $\{P_m\}_{m=1}^{\infty}$ from the induction argument, we define $\{g_i\}_{i \in I_1}$ by

$$g_i := \begin{cases} f_i & : \ i \in S_1, \\ f_i - P_{m-1} f_i & : \ i \in S_m,\ m > 1. \end{cases}$$

Since the sequence $\{\mathcal{H}_m\}_{m=1}^\infty$ defined by

$$\mathcal{H}_m := \begin{cases} P_1\mathcal{H} & : \quad m = 1, \\ (P_m - P_{m-1})\mathcal{H} & : \quad m > 1 \end{cases}$$

satisfies

$$\operatorname{span}_{1\le m<\infty}\mathcal{H}_m = \left(\sum_{m=1}^\infty \oplus \mathcal{H}_m\right)_{\ell_2}$$

and we have

$$g_i \in \mathcal{H}_m \quad \text{for all } m \in \mathbb{N},\ i \in S_m,$$

it follows that $\{g_i\}_{i\in I_1}$ possesses a finite orthogonal decomposition. Further, for all $m \in \mathbb{N}$ and $i \in S_m$, we have

$$P_{m-1}f_i \in \operatorname{span}_{k\in\bigcup_{l=1}^{m-1} S_l}\{f_k\},$$

which implies that $\operatorname{span}_{i\in I_1}\{g_i\} = \operatorname{span}_{i\in I_1}\{f_i\}$. Finally, applying (2) and (4) yields

$$\sum_{i\in I_1}\|g_i - f_i\|^2 = \sum_{m=1}^\infty \sum_{i\in S_m}\|g_i - f_i\|^2 = \sum_{m>1}\sum_{i\in S_m}\|P_{m-1}f_i\|^2 \le \sum_{m=2}^\infty \frac{\epsilon}{2^m} < \epsilon.$$

Thus $\{f_i\}_{i\in I_1}$ is an ϵ-perturbation of $\{g_i\}_{i\in I_1}$. □

Remark 3.2. The decomposition argument can be done simultaneously on two frames at once — for example on a frame $\{f_i\}_{i\in I}$ and its dual frame, which is $\{S^{-1}f_i\}_{i\in I}$, S being the frame operator of $\{f_i\}_{i\in I}$. We will not address this here, since we do not have any serious application at this time.

Next we observe that it is necessary to divide our index set into two subsets in Theorem 3.1. That is, the Bessel sequence itself need not be an ϵ-perturbation of any sequence with a finite orthogonal decomposition.

Example 3.3. The unit norm Bessel sequence $\{f_i\}_{i=1}^\infty$ defined by $f_i = \frac{e_i+e_{i+1}}{\sqrt{2}}$ is not an ϵ-perturbation of any sequence with a finite orthogonal decomposition for small $\epsilon > 0$.

Proof. If we partition $\mathbb{N}$ into finite sets $\{I_j\}_{j=1}^\infty$, then there exists a natural number $i_0 \in \mathbb{N}$ so that $i_0 \in I_j$ and $i_0 + 1 \in I_k$ where $j \ne k$. Assume, by way of contradiction, that $\{g_i\}_{i=1}^\infty$ is an ϵ-perturbation of $\{f_i\}_{i=1}^\infty$ and $\{g_i\}_{i=1}^\infty$ has a finite orthogonal decomposition given by $\{I_j\}_{j=1}^\infty$. Then $\|f_{i_0}\| = 1$ and $\|f_{i_0} - g_{i_0}\| < \sqrt{\epsilon}$, which implies

$$\|g_{i_0}\| \ge \|f_{i_0}\| - \|f_{i_0} - g_{i_0}\| \ge 1 - \sqrt{\epsilon}.$$

Similarly, $\|g_{i_0+1}\| \ge 1-\sqrt{\epsilon}$. Since $\operatorname{span}_{i\in I_j}\{g_i\}$ is orthogonal to $\operatorname{span}_{i\in I_k}\{g_i\}$, we have

$$\|g_{i_0} - g_{i_0+1}\|^2 = \|g_{i_0}\|^2 + \|g_{i_0+1}\|^2 \ge 2(1-\sqrt{\epsilon})^2.$$

Using this estimate and the fact that $\|f_{i_0} - f_{i_0+1}\|^2 = 1$ and $\|f_{i_0} - g_{i_0}\| + \|f_{i_0+1} - g_{i_0+1}\| < 2\sqrt{\epsilon}$, it follows that

$$\begin{aligned}\sqrt{2}(1-\sqrt{\epsilon}) &\le \|g_{i_0} - g_{i_0+1}\| \\ &\le \|f_{i_0} - f_{i_0+1}\| + \|f_{i_0} - g_{i_0}\| + \|f_{i_0+1} - g_{i_0+1}\| \\ &\le 1 + 2\sqrt{\epsilon}.\end{aligned}$$

This is a contradiction for small $\epsilon > 0$. □

Remark 3.4. Our proof of the Decomposition Theorem relies on the ordering of the elements of the sequence. This can sometimes cause problems as we will see below. However, it is possible to do an *optimal* construction which removes this assumption. We first choose $i_0 \in I$ and let $S_1 = \{i_0\}$. Now, following the proof,

$$\sum_{i \in I \setminus S_1} \|P_1 f_i\|^2 < \infty.$$

So choose $T_1 \subset I \setminus S_1$ with $|T_1|$ minimal and

$$\sum_{i \in I \setminus (T_1 \cup S_1)} \|P_1 f_i\|^2 < \frac{\epsilon}{2}.$$

So we have put the f_i, $i \in I \setminus S_1$ with $\|P_1 f_i\|$ maximal into T_1. In the induction step (equation (3)), choose

$$S_{k+1} \subset I \setminus \left(\bigcup_{m=1}^{k} S_m \cup \bigcup_{m=1}^{k} T_m \right)$$

with $|S_{k+1}|$ minimal and

$$\sum_{i \in I \setminus \left(\bigcup_{m=1}^{k+1} S_m \cup \bigcup_{m=1}^{k} T_m\right)} \|Q_k f_i\|^2 < \frac{\epsilon}{2^{2k}}.$$

Similarly, we now construct the next T_{k+1} and then iterate the procedure.

This stronger form of the decomposition construction is useful because it eliminates the ordering of the elements. For example, if we work with the $\{f_i\}_{i=1}^{\infty}$ in Example 3.3, then for any permutation of $\{f_i\}_{i=1}^{\infty}$, as long as $f_{i_0} = f_1$ the decomposition we obtain from this stronger form of the proof of the Decomposition Theorem is

$$I_1 = \left\{ \frac{e_{2i-1} + e_{2i}}{\sqrt{2}} \right\}_{i=1}^{\infty} \quad \text{and} \quad I_2 = \left\{ \frac{e_{2i} + e_{2i+1}}{\sqrt{2}} \right\}_{i=1}^{\infty}$$

both of which are orthonormal bases for their spans. But, if we reorder the sequence $\{f_i\}_{i=1}^{\infty}$ by taking $\{f_i\}_{i=2^k}^{2^{k+1}-1}$ into

$$\{f_{2^k+1}, f_{2^k+2}, \dots, f_{2^{k+1}-1}, f_{2^k}\} \quad \text{for each } 0 \le k < \infty,$$

the proof of the Decomposition Theorem produces the partition

$$I_1 = \{2^{2k}, 2^{2k+1}, \ldots, 2^{2k+1} - 1 : 0 \le k < \infty\},$$

and

$$I_2 = \{2^{2k+1}, 2^{2k+1} + 1, \ldots, 2^{2k+2} - 1 : 0 \le k < \infty\}.$$

Now, $\{f_i\}_{i\in I_j}$ is not a frame sequence for $j = 1, 2$. To see this for I_1, we note that the sets

$$J_k = \{2^{2k}, 2^{2k+1}, \ldots, 2^{2k+1} - 1\} \quad (0 \le k < \infty)$$

give a finite orthogonal decomposition of $\{f_i\}_{i\in I_1}$ into linearly independent sets. So if $\{f_i\}_{i\in I_1}$ is a frame sequence, then, by Proposition 2.3 (i) and [**3**, Proposition 4.3], it is a Riesz basic sequence. Let

$$a_{2^{2k}+i} = \frac{(-1)^i}{\sqrt{2^{2k}}}, \quad i = 0, 1, \ldots, 2^{2k} - 1.$$

Then,

$$\sum_{i=0}^{2^{2k}-1} |a_i|^2 = 1,$$

while

$$\left\| \sum_{i=0}^{2^{2k}-1} a_{2^{2k}+i} f_{2^{2k}+i} \right\|^2 = \left\| \frac{1}{\sqrt{2^{2k}}} (e_1 - e_{2^{2k+1}-1}) \right\|^2 = \frac{1}{2^{2k-1}}.$$

This implies that $\{f_i\}_{i\in I_1}$ is not a Riesz basic sequence. Thus $\{f_i\}_{i\in I_1}$ is not a frame sequence.

4. Proof of Theorem 1.3

In order to prove Theorem 1.3, we will require the following three results, two of which are known results, which we state to help make this paper self-contained.

The first result which we will employ is a perturbation theorem of Christensen [**8**, Corollary 15.1.5]. Notice that the hypotheses of it are implied by the definition of ϵ-perturbation which we chose.

Theorem 4.1. *Let $\{f_i\}_{i=1}^\infty$ be a frame for $\mathcal{H}$ with bounds A and B, and let $\{g_i\}_{i=1}^\infty$ be a sequence in $\mathcal{H}$. If there exists a constant $R < A$ such that for every $f \in \mathcal{H}$,*

$$\sum_{i=1}^\infty |\langle f, f_i - g_i\rangle|^2 \le R\|f\|^2,$$

then $\{g_i\}_{i=1}^{\infty}$ *is a frame for* $\mathcal{H}$ *with bounds*

$$A\left(1-\sqrt{\frac{R}{A}}\right)^2 \quad \text{and} \quad B\left(1+\sqrt{\frac{R}{B}}\right)^2.$$

If $\{f_i\}_{i=1}^{\infty}$ *is a Riesz basis, then* $\{g_i\}_{i=1}^{\infty}$ *is a Riesz basis.*

Further, we need the following result, which is [**4**, Theorem 4.2]. As usual, $\lceil B\rceil$ denotes the smallest integer $\geq B$.

Theorem 4.2. *Every unit norm Bessel sequence with Bessel bound* B *can be decomposed into* $\lceil B\rceil$ *linearly independent sets.*

Finally, we will require the following new result. If the weak Feichtinger Conjecture is true, it turns out that we can even decompose unit norm Bessel sequences into a finite union of frame sequences with a uniform lower frame bound which depends only on the Bessel bound.

Lemma 4.3. *Conjecture* 1.2 *implies that for all* $B<\infty$*, there exists* $A = A(B) > 0$ *such that whenever* $\{f_i\}_{i\in I}$ *is a unit norm Bessel sequence in* $\mathcal{H}$ *with Bessel bound* B*, then* $\{f_i\}_{i\in I}$ *is a finite union of frame sequences with lower frame bound* A*.*

Proof. Suppose Conjecture 1.2 is true. Towards a contradiction assume that there exists $B<\infty$ and, for each $n\in\mathbb{N}$, a unit norm Bessel sequence $\{f_i^n\}_{i\in I}$ such that whenever $\{f_i^n\}_{i\in I}$ is partitioned into a finite number of sets, the lower frame bound of at least one of these sets is $\leq \frac{1}{n}$. For each $n\in\mathbb{N}$, define $\mathcal{K}_n := \operatorname{span}_{i\in I}\{f_i^n\}$. Then, by Proposition 2.3 (ii), $\{f_i^n\}_{i\in I,n\in\mathbb{N}}$ is a unit norm Bessel sequence in $\mathcal{K} := (\sum_{n=1}^{\infty}\oplus\mathcal{K}_n)_{\ell_2}$ with Bessel bound B. Now we divide $\{f_i^n\}_{i\in I,n\in\mathbb{N}}$ into a finite number of sets F_j, $j=1,\ldots,M$. For each $j\in\{1,\ldots,M\}$ and $n\in\mathbb{N}$, we set $I_j^n := \{i\in I : f_i^n\in F_j\}$. Since $\{f_i^n\}_{i\in\bigcup_{j=1}^M I_j^n} = \{f_i^n\}_{i\in I}$, by assumption, there exists at least one set $\{f_i^n\}_{i\in I_{j_n}^n}$ with lower frame bound $\leq\frac{1}{n}$. Now notice that there exists some $l\in\{1,\ldots,M\}$ and $N\subset\mathbb{N}$ such that $\{f_i^n\}_{i\in\bigcup_{n\in N}I_{j_n}^n}\subset F_l$. Hence, by Proposition 2.3 (ii), the lower frame bound of F_l is $\leq\inf_{n\in N}\frac{1}{n}=0$. This is a contradiction. □

We are now ready to prove our main theorem.

Proof of Theorem 1.3. Obviously, Conjecture 1.1 implies Conjecture 1.2.

To prove the converse implication suppose that Conjecture 1.2 holds. Let $\{f_i\}_{i\in I}$ be a unit norm Bessel sequence in $\mathcal{H}$ with Bessel bound B, and let $A = A(9B)$ be given by Lemma 4.3. Fix some $\epsilon < \min\{\frac{A}{8},\frac{1}{4}\}$. By Theorem 3.1, there exists a partition $I = I_1\cup I_2$ such that, for all $j=1,2$, the sequence $\{f_i\}_{i\in I_j}$ is an ϵ-perturbation of some sequence $\{g_i\}_{i\in I_j}$ in $\mathcal{H}$, which possesses a finite orthogonal decomposition.

Let $j \in \{1,2\}$ be arbitrarily fixed. Since $\{g_i\}_{i\in I_j}$ is an ϵ-perturbation of a Bessel sequence with Bessel bound B, the computation

$$\begin{aligned}\sum_{i\in I_j} |\langle f, g_i\rangle|^2 &\le \sum_{i\in I_j} (|\langle f, g_i - f_i\rangle| + |\langle f, f_i\rangle|)^2 \\ &\le 2\sum_{i\in I_j} \left(\|f\|^2 \|g_i - f_i\|^2 + |\langle f, f_i\rangle|^2 \right) \\ &\le 2(\epsilon + B)\, \|f\|^2 \quad (f \in \mathcal{H})\end{aligned}$$

shows that it is itself a Bessel sequence with Bessel bound $2(B+\epsilon)$. Further, we have

$$\|g_i\| \ge \|f_i\| - \|f_i - g_i\| \ge 1 - \sqrt{\epsilon} \ge \frac{1}{2} \quad (i \in I_j).$$

By Theorem 4.2, there exists a finite partition $\{J_k^j\}_{k=1}^{M^j}$ of I_j such that $\{g_i\}_{i\in J_k^j}$ is linearly independent for each $k \in \{1,\dots,M^j\}$.

Fix some $k \in \{1,\dots,M^j\}$. Since the sequence $\{g_i\}_{i\in J_k^j}$ is a Bessel sequence with Bessel bound $2(B+\epsilon)$ and $\|g_i\| \ge \frac{1}{2}$ for all $i \in J_k^j$, the computation

$$\sum_{i\in J_k^j} \left| \left\langle f, \frac{g_i}{\|g_i\|} \right\rangle \right|^2 = \sum_{i\in J_k^j} \frac{1}{\|g_i\|^2} |\langle f, g_i\rangle|^2 \le 4 \sum_{i\in J_k^j} |\langle f, g_i\rangle|^2 \le 8(B+\epsilon)\, \|f\|^2,$$

where $f \in \mathcal{H}$, shows that $\{\frac{g_i}{\|g_i\|}\}_{i\in J_k^j}$ is a unit norm Bessel sequence with Bessel bound $\le 8(B+\epsilon)$. Since $\epsilon < \frac{A}{8} \le \frac{B}{8}$, it follows that the Bessel bound is $\le 9B$. By Lemma 4.3, there exists a finite partition $\{J_{kl}^j\}_{l=1}^{N_k^j}$ of J_k^j such that $\{\frac{g_i}{\|g_i\|}\}_{i\in J_{kl}^j}$ is a frame sequence with lower frame bound A for all $l \in \{1,\dots,N_k^j\}$. The computation

$$A\,\|f\|^2 \le \sum_{i\in J_{kl}^j} \left| \left\langle f, \frac{g_i}{\|g_i\|} \right\rangle \right|^2 \le 4 \sum_{i\in J_{kl}^j} |\langle f, g_i\rangle|^2 \quad (f \in \mathcal{H})$$

implies that $\{g_i\}_{i\in J_{kl}^j}$ is a linearly independent frame sequence with lower frame bound $\frac{A}{4}$.

We now claim that $\{g_i\}_{i\in J_{kl}^j}$ is a Riesz basic sequence. For this, recall that $\{g_i\}_{i\in I_j}$ possesses a finite orthogonal decomposition into linearly independent sets. It follows by Proposition 2.3 (i) that $\{g_i\}_{i\in J_{kl}^j}$ is an ω-independent frame sequence. Hence, by [**3**, Proposition 4.3], it is a Riesz basic sequence.

To finish the proof, observe that, since the sequence $\{f_i\}_{i\in J^j_{kl}}$ is an ϵ-perturbation of $\{g_i\}_{i\in J^j_{kl}}$, we have

$$\sum_{i\in J^j_{kl}} |\langle f, f_i - g_i\rangle|^2 \le \|f\|^2 \sum_{i\in J^j_{kl}} \|f_i - g_i\|^2 \le \|f\|^2 \epsilon.$$

Moreover, $\{g_i\}_{i\in J^j_{kl}}$ possesses the lower frame bound $\frac{A}{4}$, which is strictly greater than ϵ by the definition of ϵ, and $\{g_i\}_{i\in J^j_{kl}}$ forms a Riesz basic sequence. Hence we can apply Theorem 4.1, which yields that $\{f_i\}_{i\in J^j_{kl}}$ is a Riesz basic sequence. Thus the sequence $\{J^j_{kl}\}_{j=1,2,\,k=1,\ldots,M^j,\,l=1,\ldots,N^j_k}$ is a finite partition of I such that, for each $j = 1, 2$, $k = 1, \ldots, M^j$, $l = 1, \ldots, N^j_k$, the sequence $\{f_i\}_{i\in J^j_{kl}}$ is a Riesz basic sequence, which implies Conjecture 1.2. □

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References

[1] R. Balan, P.G. Casazza, C. Heil, and Z. Landau, *Density, overcompleteness, and localization of frames*, preprint.

[2] M. Bownik and D. Speegle, *Feichtinger conjecture for wavelet frames, Gabor frames and frames of translates*, preprint.

[3] P.G. Casazza, *The art of frame theory*, Taiwanese J. of Math. **4** (2000), 129–201.

[4] P.G. Casazza, O. Christensen, A. Lindner, and R. Vershynin, *Frames and the Feichtinger conjecture*, preprint.

[5] P.G. Casazza, O. Christensen, and N. Kalton, *Frames of translates*, Collect. Math. **52** (2001), 35–54.

[6] P.G. Casazza and G. Kutyniok, *Frames of subspaces*, Contemp. Math. **345** (2004), 87–114.

[7] P. Casazza and R. Vershynin, *Kadison-Singer meets Bourgain-Tzafriri*, preprint.

[8] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[9] O. Christensen and A. Lindner, *Decomposition of Riesz frames and wavelets into a finite union of linearly independent sets*, Lin. Alg. Appl. **355** (2002), 147–159.

[10] K. Gröchenig, *Localized frames are finite unions of Riesz sequences*, Adv. Comput. Math. **18** (2003), 149–157.

[11] R. Kadison and I. Singer, *Extensions of pure states*, American Jour. Math. **81** (1959), 547–564.

[12] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I: Sequence Spaces*, Springer-Verlag, Berlin, 1977.

[13] N. Weaver, *The Kadison-Singer problem in discrepancy theory*, Discrete Math. **278** (2004), 227–239.

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