Excess of Parseval Frames

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ABSTRACT

The excess of a sequence in a Hilbert space H is the greatest number of elements that can be removed yet leave a set with the same closed span. This paper proves that if $\mathcal{F}$ is a frame for H and there exist infinitely many elements $g_n \in \mathcal{F}$ such that $\mathcal{F} \setminus \{g_n\}$ is complete for each individual n and if there is a uniform lower frame bound L for each frame $\mathcal{F} \setminus \{g_n\}$, then for each $\varepsilon > 0$ there exists an infinite subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ of $\{g_n\}_{n\in\mathbb{N}}$ such that $\mathcal{F} \setminus \{g_{n_k}\}_{k\in\mathbb{N}}$ is still a frame for H. Moreover, if the frame is Parseval (i.e., has frame bounds $A = B = 1$), then we show that for each $\varepsilon > 0$ this can be done in a way that changes the lower frame bound to no less than $L - \varepsilon$.

Keywords: Bessel sequences, excess, frames, Gabor systems, Riesz bases, wavelets, Weyl–Heisenberg systems

1. INTRODUCTION

A sequence $\mathcal{F} = \{f_i\}_{i\in I}$ of elements of a Hilbert space H is a *frame* for H if there exist constants A, $B > 0$ such that

$$\forall\, h \in H, \quad A\,\|h\|^2 \;\le\; \sum_{i\in I} |\langle h, f_i\rangle|^2 \;\le\; B\,\|h\|^2. \tag{1}$$

The numbers A, B are called *lower* and *upper frame bounds*, respectively. Frames were first introduced by Duffin and Schaeffer[10] in the context of nonharmonic Fourier series, and today frames play important roles in many applications in mathematics, science, and engineering. We refer to the texts of Daubechies,[9] Christensen,[8] or the research-tutorial of Heil–Walnut[13] for basic properties of frames.

Each frame $\mathcal{F}$ provides basis-like representations of the elements of H. Specifically, there exist vectors $\tilde{f}_i \in H$ such that

$$\forall\, h \in H, \quad h \;=\; \sum_{i\in I} \langle h, f_i\rangle\, \tilde{f}_i \;=\; \sum_{i\in I} \langle h, \tilde{f}_i\rangle\, f_i, \tag{2}$$

with unconditional convergence of these series. In general, however, a frame need not be a basis, and the representations in (2) need not be unique. Frames which are not bases are overcomplete, i.e., there exist proper subsets of the frame which are complete.[10] The *excess* of the frame is the greatest integer n such that n elements can be deleted from the frame and still leave a complete set, or ∞ if there is no upper bound to the number of elements that can be removed. In the former case, it can be shown that the frame is simply a Riesz basis to which finitely many elements have been adjoined.[14] Such frames are called "near Riesz bases" and behave in many respects like Riesz bases. A frame with infinite excess need not contain a Riesz basis as a subset.[6]

Our earlier paper[2] studied the excess of frames and of more general systems. The motivation was the particular case of *Gabor* or *Weyl–Heisenberg* frames. Given $g \in L^2(\mathbb{R})$ and α, $\beta > 0$, the collection $\{e^{2\pi i m\beta x}\, g(x-n\alpha)\}_{m,n\in\mathbb{Z}}$ is called a Gabor frame if it is a frame for the Hilbert space $L^2(\mathbb{R})$. The Balian–Low Theorem states that if a Gabor frame is a Riesz basis for $L^2(\mathbb{R})$, then the window function g must be poorly localized in either time or frequency.[4, 9] Thus, the most useful Gabor frames are overcomplete. It can be shown that if $\alpha\beta > 1$ then any

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Gabor system is incomplete, if $\alpha\beta = 1$ then a Gabor frame is a Riesz basis, and if $\alpha\beta < 1$ then a Gabor frame is overcomplete.[1, 9, 18, 19]

It was proved by Duffin and Schaeffer[10] that if $\mathcal{F}$ is a frame for H and $f \in \mathcal{F}$ is such that $\mathcal{F} \setminus \{f\}$ is complete in H, then $\mathcal{F} \setminus \{f\}$ is a frame for H. By iterating, it follows that in any overcomplete frame at least finitely many elements can be removed yet still leave a frame. It was shown in Balan et al.[2] that in any overcomplete Gabor frame it is possible to find an *infinite* subset that can be deleted yet leave a frame (not merely a complete set), and furthermore the frame bounds of the resulting system can be controlled. Moreover, this result was a corollary of more general results on the excesses and deficits of Bessel sequences and arbitrary frames, also with implications for wavelet frames.

In this paper we give a new proof of a key result from Balan et al.[2] Namely, we prove that if there exist infinitely many elements $g_n \in \mathcal{F}$ such that $\mathcal{F} \setminus \{g_n\}$ is complete for each individual n and if there is a uniform lower frame bound L for each individual frame $\mathcal{F} \setminus \{g_n\}$, then there exists an infinite sequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ of $\{g_n\}_{n\in\mathbb{N}}$ such that $\mathcal{F} \setminus \{g_{n_k}\}_{k\in\mathbb{N}}$ is a frame for H. Moreover, we show that if the frame is Parseval ($A = B = 1$), then this can be done in a way that changes the lower frame bound to no less than $L - \varepsilon$.

Recently, we have shown that if stronger hypotheses on the frame are imposed, namely that $\mathcal{F}$ be a so-called *localized frame*, then stronger results on excess can be obtained.[3] However, the results of this paper apply without the need to assume the localization hypothesis.

2. PRELIMINARIES

$\mathbb{N}$ will denote the set of natural numbers, while I will denote a generic countable index set. $|E|$ denotes the cardinality of a set E. H will always denote a separable, infinite-dimensional Hilbert space.

The finite linear span of a sequence of elements $\mathcal{F} = \{f_i\}_{i\in I}$ of H will be denoted by $\mathrm{span}(\mathcal{F})$. The closure in H of this set will be denoted by $\overline{\mathrm{span}}(\mathcal{F})$. We say that $\mathcal{F}$ is complete if $\overline{\mathrm{span}}(\mathcal{F}) = H$, or, equivalently, if the only vector f satisfying $\langle f, f_i\rangle = 0$ for all i is $f = 0$.

A sequence $\mathcal{F} = \{f_i\}_{i\in I}$ in H is a *Bessel sequence* if there exists a constant $B > 0$ such that

$$\forall\, h \in H, \quad \sum_{i\in I} |\langle h, f_i\rangle|^2 \le B\|h\|^2. \tag{3}$$

In this case the associated *analysis operator* is $T\colon H \to \ell^2(I)$ defined by $T(h) = \big\{\langle h, f_i\rangle\big\}_{i\in I}$, and the *synthesis operator* $T^*\colon \ell^2(I) \to H$ is defined by $T^*(c) = \sum_{i\in I} c_i f_i$ (this series converges unconditionally in the norm of H for any $c = (c_i)_{i\in I} \in \ell^2(I)$). These are everywhere-defined, bounded operators, each adjoint to the other.

The elements of a Bessel sequence are uniformly bounded above in norm, specifically, $\|f_i\|^2 \le B$ for each $i \in I$.

Comparing (1) and (3), we see that every frame is a Bessel sequence. However, a frame possesses additional useful properties. The *frame operator* $S = T^*T\colon H \to H$, given by $Sh = \sum_i \langle h, f_i\rangle\, f_i$, is a positive, continuous, invertible mapping of H onto itself, and satisfies $AI \le S \le BI$. The *canonical dual frame* is $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$ where $\tilde{f}_i = S^{-1}f_i$. If $\mathcal{F}$ has frame bounds A, B, then the canonical dual frame is a frame with frame bounds $\frac{1}{B}$, $\frac{1}{A}$. Furthermore, the frame expansions in (2) hold.

We say that a frame $\mathcal{F}$ is *tight* if it is possible to take $A = B$ in (1). It is *Parseval* if we can take $A = B = 1$. Parseval frames are also sometimes called *normalized tight frames*, but note that this term can be confusing since some authors define a normalized frame to be one where $\|f_i\| = 1$ for every $i \in I$. The frame operator for a tight frame is $S = AI$. In particular, if $\mathcal{F}$ is a Parseval frame, then $\tilde{\mathcal{F}} = \mathcal{F}$.

For a general frame $\mathcal{F}$, the frame operator S is a positive operator and therefore has a positive square root $S^{1/2}$. Further, $S^{-1/2}$ is a bounded, continuously invertible operator and $S^{-1/2}(\mathcal{F}) = \{S^{-1/2}f_i\}_{i\in I}$ is a Parseval frame for H. Thus every frame is equivalent in this sense to a Parseval frame.

A *Riesz sequence* is a sequence $\mathcal{F} = \{f_i\}_{i \in I}$ for which there exist A, $B > 0$ such that

$$\forall\, c \in \ell^2(I), \quad A \sum_{i \in I} |c_i|^2 \le \biggl\| \sum_{i \in I} c_i f_i \biggr\|^2 \le B \sum_{i \in I} |c_i|^2.$$

A Riesz sequence is a frame for its closed span in H. A complete Riesz sequence is called a *Riesz basis* for H. If $\mathcal{F}$ is a frame, then the frame expansion given in (2) is unique for each $h \in H$ if and only if $\mathcal{F}$ is a Riesz basis.

Definition 2.1. The *excess* of a sequence $\mathcal{F} = \{f_i\}_{i \in I}$ in a Hilbert space H is

$$e(\mathcal{F}) \;=\; \sup\bigl\{|\mathcal{G}| : \mathcal{G} \subset \mathcal{F} \text{ and } \overline{\mathrm{span}}(\mathcal{F} \setminus \mathcal{G}) = \overline{\mathrm{span}}(\mathcal{F})\bigr\}. \tag{4}$$

It was shown in Balan et al.[2] that the supremum in (4) is achieved, i.e., the excess is the greatest cardinal $e(\mathcal{F})$ such that there exists a subset $\mathcal{G} \subset \mathcal{F}$ of cardinality $e(\mathcal{F})$ so that $\mathcal{F} \setminus \mathcal{G}$ is complete in $\overline{\mathrm{span}}(\mathcal{F})$. However, this does not imply that $\mathcal{F} \setminus \mathcal{G}$ is a frame for $\overline{\mathrm{span}}(\mathcal{F})$, and in fact there exists a frame $\mathcal{F}$ with infinite excess such that there is no infinite subset $\mathcal{G}$ such that $\mathcal{F} \setminus \mathcal{G}$ is a frame for H.

Note that a Riesz sequence has zero excess. Further, if a frame has zero excess, then it is a Riesz basis for H.[10]

The following result connects the excess to the dimension of the kernel of the synthesis operator and to certain inner products of frame elements with corresponding dual frame elements.[2] Note in this result that $\langle f_i, \tilde{f}_i \rangle = \|S^{-1/2} f_i\|^2 \ge 0$ for each i. Further, each element of the Parseval frame $S^{-1/2}(\mathcal{F})$ can have norm at most 1, so $\langle f_i, \tilde{f}_i \rangle = \|S^{-1/2} f_i\|^2 \le 1$ for each i.

Lemma 2.2. *Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a Bessel sequence in H, and let $T \colon H \to \ell^2(I)$ be the associated analysis operator.*

(a) $e(\mathcal{F}) \ge \dim(\ker T^*)$.

(b) *If $\mathcal{F}$ is a frame then $e(\mathcal{F}) = \dim(\ker T^*)$. Furthermore, if $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i \in I}$ is the canonical dual frame then*

$$e(\mathcal{F}) \;=\; \sum_{i \in I} \bigl(1 - \langle f_i, \tilde{f}_i \rangle\bigr).$$

Example 2.3. If $\mathcal{F}$ is a Bessel sequence that is not a frame, then it is possible that $e(\mathcal{F})$ can strictly exceed $\dim(\ker T^*)$. For example, let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for a Hilbert space H, and set $f = \sum_{n=1}^{\infty} e_n/n$. Then $\mathcal{F} = \{e_n/n\}_{n \in \mathbb{N}} \cup \{f\}$ is a Bessel sequence but is not a frame, and it can be shown that $e(\mathcal{F}) = 1$ while $\dim(\ker T^*) = 0$. It is similarly possible to construct Bessel sequences where $e(\mathcal{F})$ is any specified finite value or infinity yet $\dim(\ker T^*) = 0$.

The following example shows that there even exist Gabor systems $\mathcal{G}$ that are Bessel sequences which have positive but finite excess. In particular, this Gabor Bessel sequence satisfies $e(\mathcal{G}) = 1$ and $\dim(\ker T^*) = 0$.

Example 2.4. Consider the Gabor system $\mathcal{G} = \{e^{2\pi i m x}\, g(x-n)\}_{m,n \in \mathbb{Z}}$ generated by the Gaussian function $g(x) = e^{-x^2}$, with $\alpha = \beta = 1$. For simplicity, write $g_{m,n}(x) = e^{2\pi i m x} g(x-n)$. Because $\alpha = \beta = 1$, it can be shown that if this system was a frame for $L^2(\mathbb{R})$ then it would be a Riesz basis.[13] This would contradict the Balian–Low Theorem, so this system cannot be a frame. Let $Q = [0,1) \times [0,1)$. The Zak transform[9] is the isometric isomorphism $Z \colon L^2(\mathbb{R}) \to L^2(Q)$ defined by

$$Zf(x,\omega) \;=\; \sum_{k \in \mathbb{Z}} e^{2\pi i k \omega} f(x+k).$$

It can be shown that Zg is a continuous and bounded function on Q and has a single zero in Q. This implies that $\mathcal{G}$ is a Bessel sequence but is not a frame for $L^2(\mathbb{R})$.

The synthesis operator for $\mathcal{G}$ is the mapping $T^*\colon \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{R})$ defined by

$$T^*c \;=\; \sum_{m,n} c_{m,n} g_{m,n} \quad \text{for } c = (c_{m,n})_{m,n\in\mathbb{Z}} \in \ell^2\bigl(\mathbb{Z}^2\bigr).$$

Suppose that $T^*c = 0$ for some $c \in \ell^2(\mathbb{Z}^2)$. Then, using basic properties of the Zak transform,

$$0 \;=\; Z(T^*c) \;=\; \sum_{m,n} c_{m,n} Z g_{m,n} \;=\; \sum_{m,n} c_{m,n} e_{m,n} Zg,$$

where $e_{m,n}(x,\omega) = e^{2\pi i m x} e^{2\pi i n \omega}$. Since $c \in \ell^2(\mathbb{Z}^2)$ and $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(Q)$, we have that $H = \sum_{m,n} c_{m,n} e_{m,n}$ is a well-defined function in $L^2(Q)$. Therefore, since Zg is bounded we have that $0 = Z(T^*c) = H \cdot Zg$. However, Zg is nonzero a.e., so this implies that $H = 0$ a.e., and therefore $c = 0$. Thus $\ker T^* = \{0\}$.

A similar argument, using the fact that $1/Zg \notin L^2(Q)$, shows that $e(\mathcal{F}) = 1$. This was first proved by Perelomov.[17] Thus this Gabor system $\mathcal{G}$ is a Bessel sequence but not a frame and satisfies $\dim(\ker T^*) < e(\mathcal{F})$. This shows that even for Gabor systems, the inequality in Lemma 2.2(a) can be strict.

The excess in this example is exactly 1. In particular, $\mathcal{G} \setminus \{g\} = \{g_{m,n}\}_{(m,n)\neq(0,0)}$ is complete, but no proper subset of $\mathcal{G} \setminus \{g\}$ is complete. However, $\mathcal{G} \setminus \{g\}$ is not a Riesz basis (or even just a Schauder basis) for $L^2(\mathbb{R})$.[11]

3. EXCESS OF FRAMES

It was shown in Balan et al.[2] that if $\mathcal{F}$ is a frame that has infinite excess, then there exists an infinite subset $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{F} \setminus \mathcal{G}$ is complete. However, the following example[2] shows that it is possible that there may be no way to choose $\mathcal{G}$ so that $\mathcal{F} \setminus \mathcal{G}$ is a frame. This same frame is an example of a Parseval frame which contains no subset that is a Riesz basis.[6]

Example 3.1. Index an orthonormal basis for a Hilbert space H as $\{e^n_j\}_{n\in\mathbb{N},\, j=1,\dots,n}$. Set $H_n = \operatorname{span}\{e^n_1, \dots, e^n_n\}$. Define

$$f^n_j \;=\; e^n_j - \frac{1}{n} \sum_{i=1}^{n} e^n_i, \qquad j = 1, \dots, n,$$

$$f^n_{n+1} \;=\; \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e^n_i.$$

Then $\mathcal{F}_n = \{f^n_1, \dots, f^n_{n+1}\}$ is a Parseval frame for H_n.[6] Since H_n is n-dimensional, at most one element can be removed from $\mathcal{F}_n$ if the remaining elements are to span H_n. Moreover f^n_{n+1} is orthogonal to $f^n_1, \dots, f^n_n$, so f^n_{n+1} cannot be removed. If one of the other elements is removed, say f^n_1, then since

$$\sum_{j=2}^{n+1} \bigl|\langle e^n_1, f^n_j\rangle\bigr|^2 \;=\; \left(\sum_{j=2}^{n} \frac{1}{n^2}\right) + \frac{1}{n} \;=\; \frac{2}{n} - \frac{1}{n^2},$$

the lower frame bound for $\mathcal{F}_n \setminus \{f^n_1\}$ as a frame for H_n is at most $2/n - 1/n^2$.

Now consider the fact that $H \cong \bigl(\sum_{n=1}^{\infty} H_n\bigr)_{\ell^2}$ with the H_n mutually orthogonal. The sequence $\mathcal{F} = \{f^n_j\}_{n\in\mathbb{N},\, j=1,\dots,n+1}$ is a Parseval frame for H with infinite excess. Suppose that $\mathcal{G}$ is any infinite subset of $\mathcal{F}$ such that $\mathcal{F} \setminus \mathcal{G}$ is complete. Then $\mathcal{G}$ cannot contain any elements of the form f^n_{n+1}. Hence $\mathcal{G} = \{f^{n_k}_{j_k}\}_{k\in\mathbb{N}}$ with $n_1 < n_2 < \cdots$ and $j_k \le n_k$ for every k. But then the lower frame bound for $\mathcal{F} \setminus \mathcal{G}$ can be at most $2/n_k - 1/n_k^2$ for every k, which implies that $\mathcal{F} \setminus \mathcal{G}$ cannot have a positive lower frame bound and therefore is not a frame.

Note that in this example, if we fix a particular k then the subsequence $\mathcal{F} \setminus \{f_{j_k}^{n_k}\}$ formed by deleting the single element $f_{j_k}^{n_k}$ from $\mathcal{F}$ is a frame for H. However, there is no single positive number that can serve as a common lower frame bound for all of the subframes $\mathcal{F} \setminus \{f_{j_k}^{n_k}\}$.

Suppose that $\mathcal{F}$ was a frame such that there did exist an infinite subsequence $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ so that $\mathcal{F} \setminus \mathcal{G}$ was a frame for H, say with lower frame bound L. Then for each fixed n, since $\mathcal{F} \setminus \mathcal{G} \subset \mathcal{F} \setminus \{g_n\} \subset \mathcal{F}$, we have that $\mathcal{F} \setminus \{g_n\}$ is a frame for H with lower frame bound L. Hence the existence of such a sequence $\{g_n\}_{n\in\mathbb{N}}$ with uniform lower frame bound for each $\mathcal{F} \setminus \{g_n\}$ is a necessary condition in order to be able to delete infinitely many elements from a frame and still leave a frame. We will show that this condition is sufficient as well as necessary. Specifically, we will show below that if such g_n exist, then there exists an infinite subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ such that $\mathcal{F} \setminus \{g_{n_k}\}_{k\in\mathbb{N}}$ is a frame. First, however, we quote the following result,[2] which gives an equivalent condition for the existence of such elements g_n.

LEMMA 3.2. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a frame in a Hilbert space H with canonical dual $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$. Let $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ be a subsequence of $\mathcal{F}$. Then the following two statements are equivalent.*

(a) *There exists a constant $L > 0$ such that for each $n \in \mathbb{N}$, $\mathcal{F} \setminus \{g_n\}$ is a frame for H with lower frame bound L.*

(b) $\sup_{n\in\mathbb{N}} \langle g_n, \tilde{g}_n\rangle < 1.$

We will also need two standard results. The following lemma appears in the article by Casazza.[5]

LEMMA 3.3. *If $\mathcal{F} = \{f_i\}_{i\in I}$ is a Parseval frame for an n-dimensional Hilbert space, then*

$$\sum_{i\in I} \|f_i\|^2 \; = \; n.$$

The next result is a perturbation theorem of Christensen.[7]

LEMMA 3.4. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a frame for a Hilbert space H with frame bounds A, B. If $\mathcal{G} = \{g_i\}_{i\in I}$ is a sequence of elements of H such that*

$$R \; = \; \sum_{i\in I} \|f_i - g_i\| \; < \; A,$$

then $\mathcal{G}$ is a frame for H with frame bounds $A(1 - \sqrt{R/A})^2$, $B(1 + \sqrt{R/A})^2$.

Now we will prove our main result on the excess of Parseval frames, and then extend to general frames in a corollary. For Parseval frames, we can construct the infinite subset to be removed in such a way that the frame bounds of the resulting set change by an arbitrarily small amount. A different proof of this result was given in Balan et al.[2]

THEOREM 3.5. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a Parseval frame for a Hilbert space H, and assume that there exists a subsequence $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}$ and a constant $L > 0$ such that for each $n \in \mathbb{N}$, $\mathcal{F} \setminus \{g_n\}$ is a frame for H with lower frame bound L. Then for every $0 < \varepsilon < L$ there exists an infinite subsequence $\mathcal{G}_\varepsilon$ of $\mathcal{G}$ such that $\mathcal{F} \setminus \mathcal{G}_\varepsilon$ is a frame for H with lower frame bound $L - \varepsilon$.*

Proof. For simplicity of notation, let us take $I = \mathbb{N}$. Note that $L \le 1$ since the lower frame bound of $\mathcal{F}$ is 1.

Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for H, and let $P_n f = \sum_{k=1}^{n} \langle f, e_k\rangle\, e_k$ be the orthogonal projection of H onto $H_n = \operatorname{span}\{e_1, \dots, e_n\}$. Set $P_0 = 0$. Since

$$\forall\, f \in H_n, \quad \sum_{i=1}^{\infty} |\langle f, P_n f_i\rangle|^2 \; = \; \sum_{i=1}^{\infty} |\langle P_n f, f_i\rangle|^2 \; = \; \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \; = \; \|f\|^2,$$

we see that $\{P_n f_i\}_{i\in\mathbb{N}}$ is a Parseval frame for H_n. Hence, by Lemma 3.3, we have $\sum_i \|P_n f_i\|^2 = n$. In particular, since $\mathcal{G} = \{g_j\}_{j\in\mathbb{N}}$ is an infinite subset of $\mathcal{F}$, we must have

$$\forall\, n \in \mathbb{N}, \quad \lim_{j\to\infty} \|P_n g_j\| \;=\; 0. \tag{5}$$

Let $0 < \varepsilon < L$ be fixed, and set $\eta = \varepsilon^2/16$. Let $m_0 = 0$ and $k_1 = 1$. Choose $m_1 > m_0$ such that

$$\|(I - P_{m_1}) g_{k_1}\| \;<\; \frac{\eta}{2^2}.$$

By (5), there exists $k_2 > k_1$ such that

$$\|P_{m_1} g_{k_2}\| \;<\; \frac{\eta}{2^3}.$$

Now choose $m_2 > m_1$ so that

$$\|(I - P_{m_2}) g_{k_2}\| \;<\; \frac{\eta}{2^3}.$$

Continuing by induction, we can find $k_1 < k_2 < \cdots$ and $m_0 < m_1 < m_2 < \cdots$ so that for every $j \in \mathbb{N}$,

$$\|P_{m_{j-1}} g_{k_j}\| \;<\; \frac{\eta}{2^{j+1}} \tag{6}$$

and

$$\|(I - P_{m_j}) g_{k_j}\| \;<\; \frac{\eta}{2^{j+1}}. \tag{7}$$

Define

$$h_j \;=\; (P_{m_j} - P_{m_{j-1}}) g_{k_j}.$$

Then, by (6) and (7),

$$\sum_{j=1}^{\infty} \|g_{k_j} - h_j\| \;\le\; \sum_{j=1}^{\infty} \|(I - P_{m_j}) g_{k_j}\| \;+\; \sum_{j=1}^{\infty} \|P_{m_{j-1}} g_{k_j}\| \;\le\; \sum_{j=1}^{\infty} \frac{\eta}{2^{j+1}} \;+\; \sum_{j=1}^{\infty} \frac{\eta}{2^{j+1}} \;=\; \eta.$$

Let $\mathcal{G}_\varepsilon = \{g_{k_j}\}_{j\in\mathbb{N}}$. Since $\mathcal{F}$ is a Parseval frame and $\eta < 1$, it follows from Lemma 3.4 that $\mathcal{F} \setminus \mathcal{G}_\varepsilon \cup \{h_j\}_{j\in\mathbb{N}}$ is a frame for H with frame bounds $(1 - \sqrt{\eta})^2$, $(1 + \sqrt{\eta})^2$.

We claim now that

$$\forall\, j \in \mathbb{N}, \quad \forall\, f \in H, \quad |\langle f, h_j\rangle|^2 \;\le\; \big((1+\eta)^2 - L\big)\, \|f\|^2. \tag{8}$$

To see this, fix $j \in \mathbb{N}$ and $f \in H$ and recall that $\mathcal{F} \setminus \{g_{k_j}\}$ is a frame for H with lower frame bound L. Therefore,

$$L\, \|f\|^2 \;\le\; \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \;-\; |\langle f, g_{k_j}\rangle|^2 \;=\; \|f\|^2 \;-\; |\langle f, g_{k_j}\rangle|^2,$$

the final equality following from the fact that $\mathcal{F}$ is a Parseval frame. Since the upper frame bound for $\mathcal{F}$ is 1 and since g_{k_j} is an element of $\mathcal{F}$, we have $\|g_{k_j}\| \le 1$. Therefore,

$$\begin{aligned}
L\, \|f\|^2 + |\langle f, h_j\rangle|^2 \;&\le\; \|f\|^2 - |\langle f, g_{k_j}\rangle|^2 + |\langle f, h_j\rangle|^2 \\
&=\; \|f\|^2 + |\langle f, h_j - g_{k_j} + g_{k_j}\rangle|^2 - |\langle f, g_{k_j}\rangle|^2 \\
&\le\; \|f\|^2 + \big(|\langle f, h_j - g_{k_j}\rangle| + |\langle f, g_{k_j}\rangle|\big)^2 - |\langle f, g_{k_j}\rangle|^2 \\
&=\; \|f\|^2 + |\langle f, h_j - g_{k_j}\rangle|^2 + 2\, |\langle f, h_j - g_{k_j}\rangle|\, |\langle f, g_{k_j}\rangle| \\
&\le\; \|f\|^2 + \|f\|^2\, \|h_j - g_{k_j}\|^2 + 2\, \|f\|\, \|h_j - g_{k_j}\|\, \|f\|\, \|g_{k_j}\| \\
&\le\; \|f\|^2 + \eta^2\, \|f\|^2 + 2\eta\, \|f\|^2 \\
&=\; (1+\eta)^2\, \|f\|^2,
\end{aligned}$$

from which (8) follows.

Let $J = \{i : f_i = g_{k_j} \text{ for some } j\}$. We will show now that $\mathcal{F} \setminus \mathcal{G}_\varepsilon = \{f_i\}_{i\in\mathbb{N}\setminus J}$ is a frame for H. Since $\mathcal{F} \setminus \mathcal{G}_\varepsilon$ is a subset of $\mathcal{F}$, it has an upper frame bound of 1. Therefore we need only compute its lower frame bound. Fix $f \in H$, and define

$$p_j \;=\; (P_{m_j} - P_{m_{j-1}})f, \qquad j \in \mathbb{N}.$$

Then

$$\langle f, h_j\rangle \;=\; \langle p_j, h_j\rangle \qquad \text{and} \qquad \sum_{j=1}^{\infty} \|p_j\|^2 \;=\; \|f\|^2.$$

Therefore, applying equation (8) to each of the functions p_j, we see that

$$\sum_{j=1}^{\infty} |\langle f, h_j\rangle|^2 \;=\; \sum_{j=1}^{\infty} |\langle p_j, h_j\rangle|^2 \;\le\; \sum_{j=1}^{\infty} \big((1+\eta)^2 - L\big)\, \|p_j\|^2 \;=\; \big((1+\eta)^2 - L\big)\, \|f\|^2.$$

Using this and the fact that $\mathcal{F} \setminus \mathcal{G}_\varepsilon \cup \{h_j\}_{j\in\mathbb{N}}$ is a frame with lower frame bound $(1-\sqrt{\eta})^2$, we have

$$(1-\sqrt{\eta})^2\, \|f\|^2 \;\le\; \sum_{i\in\mathbb{N}\setminus J} |\langle f, f_i\rangle|^2 \;+\; \sum_{j=1}^{\infty} |\langle f, h_j\rangle|^2 \;\le\; \sum_{i\in\mathbb{N}\setminus J} |\langle f, f_i\rangle|^2 \;+\; \big((1+\eta)^2 - L\big)\, \|f\|^2.$$

Hence

$$\sum_{i\in\mathbb{N}\setminus J} |\langle f, f_i\rangle|^2 \;\ge\; \big((1-\sqrt{\eta})^2 - (1+\eta)^2 + L\big)\, \|f\|^2 \;\ge\; (L-\varepsilon)\, \|f\|^2.$$

Thus $\mathcal{F} \setminus \mathcal{G}_\varepsilon$ is a frame with lower frame bound $L - \varepsilon$. ☐

We can use the following lemma[12] to extend Theorem 3.5 to general frames.

Lemma 3.6. *Let $\mathcal{F}$ be a frame for a Hilbert space H with frame bounds A, B. If $U\colon H \to H$ is a continuous, invertible mapping, then $U(\mathcal{F})$ is a frame for H with frame bounds $A\|U^{-1}\|^{-2}$, $B\|U\|^2$.*

Corollary 3.7. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a frame for a Hilbert space H, and assume that there exists a subsequence $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}$ and a constant $L > 0$ such that for each $n \in \mathbb{N}$, $\mathcal{F} \setminus \{g_n\}$ is a frame for H with lower frame bound L. Then for every $0 < \varepsilon < L$ there exists an infinite subsequence $\mathcal{G}_\varepsilon$ of $\mathcal{G}$ such that $\mathcal{F} \setminus \mathcal{G}_\varepsilon$ is a frame for H with lower frame bound $L(A/B) - \varepsilon$.*

Proof. Let $S = T^*T$ be the frame operator for $\mathcal{F}$. Then $AI \le S \le BI$, and $B^{-1/2}I \le S^{-1/2} \le A^{-1/2}I$, so in particular we have $\|S^{1/2}\| \le B^{1/2}$ and $\|S^{-1/2}\| \le A^{-1/2}$. Since $\mathcal{F} \setminus \{g_n\}$ is a frame with lower frame bound L, by applying Lemma 3.6 with $U = S^{-1/2}$ we see that $S^{-1/2}(\mathcal{F} \setminus \{g_n\}) = S^{-1/2}(\mathcal{F}) \setminus \{S^{-1/2}(g_n)\}$ is a frame with lower frame bound L/B. Theorem 3.5 therefore implies that there exists an infinite subsequence $\mathcal{G}_\varepsilon$ of $\mathcal{G}$ such that $S^{-1/2}(\mathcal{F}) \setminus S^{-1/2}(\mathcal{G}_\varepsilon) = S^{-1/2}(\mathcal{F} \setminus \mathcal{G}_\varepsilon)$ is a frame with lower frame bound $L/B - \varepsilon/A$. Applying Lemma 3.6 again with $U = S^{1/2}$, we conclude that $\mathcal{F} \setminus \mathcal{G}_\varepsilon$ is a frame with lower frame bound $LA/B - \varepsilon$. ☐

An improvement to Corollary 3.7 was proved in Balan et al.,[2] namely that it is possible to construct $\mathcal{G}_\varepsilon$ so that $\mathcal{F} \setminus \mathcal{G}_\varepsilon$ has lower frame bound $L - \varepsilon$.

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