Chirps on finite cyclic groups

Peter G. Casazza[a] and Matthew Fickus[b]

[a]Department of Mathematics, University of Missouri, Columbia, Missouri 65211, USA;
[b]Department of Mathematics and Statistics, Air Force Institute of Technology,
Wright-Patterson Air Force Base, Ohio 45433, USA

ABSTRACT

Chirps arise in many signal processing applications, and have been extensively studied, especially in the case where chirps are regarded as functions of the real-line or of the integers. However, less attention has been paid to study of chirps over finite cyclic groups. We discuss the basic properties of such chirps, including a way in which they may be used to construct finite tight frames.

Keywords: chirps, discrete Fourier transform, Gauss sums

1. INTRODUCTION

A *linear chirp* is a function whose frequency changes linearly with time. For example, while a wave function of the form $\exp(2\pi\mathrm{i}xt)$ has constant frequency x, the chirp $\exp(2\pi\mathrm{i}(xt + yt^2/2))$ has frequency $x+yt$ at time $t\in\mathbb{R}$. Chirps often arise in nature as a consequence of the Doppler effect, the phenomenon by which the perceived frequency of a wave is altered whenever the wave is emanating from or reflecting off a moving body. As such, chirps have historically been of great interest in applications such as radar and sonar.

However, the study of chirps has mostly been confined to the real line and the integers, in the context of integral transforms and the chirp $\mathscr{Z}$-transform, respectively. Less attention has been paid to the study of chirps over finite cyclic groups, that is, to chirps over $\mathbb{Z}_a \equiv \mathbb{Z}/a\mathbb{Z} = \{0,\dots,a-1\}$, where a is a positive integer. This is in contrast to wave functions which, in the context of Fourier transforms, have been studied for many decades on arbitrary locally compact abelian groups.

At the same time, the concept of a finite chirp is by no means new. Xia has recently introduced a *discrete chirp-Fourier transform*,[8] and *chirplets* have been used in image processing for over a decade.[5] For that matter, discrete chirps, under a different name, were investigated by Gauss in his study of quadratic reciprocity. Indeed, the computation of *Gauss sums*[2] is equivalent to finding the Discrete Fourier Transform (DFT) of a finite chirp, and was a subject of great interest in the mid-nineteenth century. This connection between modern signal processing and classical number theory was noted by Auslander and Tolimieri,[1] who observed that the trace of the DFT matrix,

$$\mathrm{Tr}(\mathrm{F}_a) = \frac{1}{\sqrt{a}}\sum_{t=0}^{a-1} \mathrm{e}^{2\pi\mathrm{i}t^2/a},$$

is the canonical example of a Gauss sum. An independent derivation of this trace is given by McClellan and Parks,[6] giving an interesting example of when the concepts of applied signal processing may be used to prove results in pure mathematics.

Further author information: (Send correspondence to M.F.)
P.G.C.: E-mail: pete@math.missouri.edu
M.F.: E-mail: Matthew.Fickus@afit.edu

2. FINITE CHIRPS

Let $\mathbb{Z}^+$ be the set of all positive integers. For any $a \in \mathbb{Z}^+$, let $\mathbb{Z}_a \equiv \mathbb{Z}/a\mathbb{Z}$ be the finite group of integers in which addition is performed modulo a. Consider the space of all complex-valued functions over $\mathbb{Z}_a$, $\ell(\mathbb{Z}_a) = \{f : \mathbb{Z}_a \to \mathbb{C}\}$, in which function addition, scalar multiplication, and the inner product are defined in the usual fashion. We shall equivalently regard elements of $\ell(\mathbb{Z}_a)$ as a-periodic, complex-valued sequences over $\mathbb{Z}$, so as to consider $f(t)$ for any $t \in \mathbb{Z}$. The *Fourier basis* for $\ell(\mathbb{Z}_a)$ consists of the *wave functions* $\{\mathrm{m}_a^x\}_{x\in\mathbb{Z}_a}$,

$$\mathrm{m}_a^x(t) = \mathrm{e}^{2\pi\mathrm{i}xt/a} = \mathrm{w}_a^{xt},$$

where $\mathrm{w}_a \equiv \mathrm{e}^{2\pi\mathrm{i}/a}$ is the "first" ath root of unity. When suitably scaled by a factor of $\sqrt{a}$, the Fourier basis is also an orthonormal basis for $\ell(\mathbb{Z}_a)$. As such, the corresponding *Fourier transform* on $\mathbb{Z}_a$, namely $\mathrm{F}_a : \ell(\mathbb{Z}_a) \to \ell(\mathbb{Z}_a)$,

$$(\mathrm{F}_a f)(x) = \frac{1}{\sqrt{a}} \sum_{t\in\mathbb{Z}_a} f(t)\mathrm{w}_a^{-xt},$$

is a unitary operator. Other unitary operators frequently discussed in the signal processing literature are the *translation* and *modulation* operators: for any $x \in \mathbb{Z}$, let $\mathrm{T}_a^x, \mathrm{M}_a^x : \ell(\mathbb{Z}_a) \to \ell(\mathbb{Z}_a)$,

$$(\mathrm{T}_a^x f)(t) = f(t-x), \quad (\mathrm{M}_a^x f)(t) = \mathrm{m}_a^x(t) f(t).$$

Perhaps the most difficult aspect of working with chirps in this setting is the problem of finding their proper definition. In Xia's work,[8] the lowest-order nontrivial finite chirp is defined to be $f \in \ell(\mathbb{Z}_a)$,

$$f(t) = \mathrm{w}_a^{t^2} = \mathrm{e}^{2\pi\mathrm{i}t^2/a}. \tag{1}$$

We note that this chirp is well-defined over $\mathbb{Z}_a$, that is,

$$f(t+a) = \mathrm{e}^{2\pi\mathrm{i}(t+a)^2/a} = \mathrm{e}^{2\pi\mathrm{i}t^2/a}\mathrm{e}^{2\pi\mathrm{i}(2t+a)} = \mathrm{e}^{2\pi\mathrm{i}t^2/a}1 = f(t).$$

However, in the case of continuous chirps, we note that the squared term in the exponent is usually accompanied by an additional factor of $1/2$. This is done to compensate for the fact that the frequency of a wave is obtained from the derivative of the wave function. From this perspective, the finite chirp should instead be defined as

$$g(t) = \mathrm{w}_a^{t^2/2} = \mathrm{e}^{\pi\mathrm{i}t^2/a}.$$

Indeed, much of the original work on Gauss sums[2] takes $g(t)$ as the "canonical" chirp, rather than $f(t)$. However, under this definition,

$$g(t+a) = \mathrm{e}^{\pi\mathrm{i}(t+a)^2/a} = \mathrm{e}^{\pi\mathrm{i}t^2/a}\mathrm{e}^{\pi\mathrm{i}(2t+a)} = \mathrm{e}^{\pi\mathrm{i}t^2/a}(-1)^a = (-1)^a g(t),$$

and thus g is not well-defined over $\mathbb{Z}_a$ when a is odd. When dealing with Gauss sums, this problem is usually avoided by simply summing over $\{0,\ldots,a-1\}$, rather than an arbitrary collection of coset representatives. This approach is inadequate from the point of view of harmonic analysis and signal processing, as one often needs to make translation-based changes of variables in various sums. We remedy the problem with $g(t)$ in another way: by replacing the quadratic t^2 with $t(t-a)$. That is, we consider $\mathrm{c}_a \in \ell(\mathbb{Z}_a)$,

$$\mathrm{c}_a(t) = \mathrm{w}_a^{t(t-a)/2} = \mathrm{e}^{\pi\mathrm{i}t(t-a)/a}.$$

This definition was originally proposed by N. Kaiblinger,[4] as was presented publicly at the conference "Harmonic Analysis and Applications" in honor of John J. Benedetto's 60th birthday. As Kaiblinger noted, this chirp is well-defined since

$$\mathrm{c}_a(t+a) = \mathrm{e}^{\pi\mathrm{i}(t+a)t/a} = \mathrm{e}^{\pi\mathrm{i}t^2/a}(-1)^t = \mathrm{e}^{\pi\mathrm{i}t^2/a}(-1)^{-t} = \mathrm{e}^{\pi\mathrm{i}t(t-a)/a} = \mathrm{c}_a(t).$$

A similar technique may be used to produce discrete versions of $\mathrm{e}^{2\pi\mathrm{i}t^n/n}$ for any $n \in \mathbb{Z}^+$. For example, when $n = 3$ the function $\mathrm{e}^{2\pi\mathrm{i}t(t-a)(t-2a)/3a}$ is well-defined over $\mathbb{Z}_a$. Returning to the quadratic, we have that for any $x \in \mathbb{Z}$, we may raise c_a to the xth power in a pointwise fashion in order to obtain a well-defined *xth-order chirp on* $\mathbb{Z}_a$,

$$\mathrm{c}_a^x(t) = \mathrm{w}_a^{xt(t-a)/2} = (-1)^{xt}\mathrm{w}_a^{xt^2/2},$$

and with it, a *chirp-modulation operator* $\mathrm{C}_a^x : \ell(\mathbb{Z}_a) \to \ell(\mathbb{Z}_a)$,

$$(\mathrm{C}_a^x f)(t) = \mathrm{c}_a^x(t) f(t).$$

We note that for even integers x, c_a^x reduces to a power of the finite chirp given in equation (1), and thus our definition is in fact an extension of that given by Xia.[8] For $a = 7$, the graphs of the functions $\mathrm{c}_a^x(t)$, $x = 0, \ldots, a-1$ are given below in Figure 1. Though these functions are formally defined only over the discrete group $\mathbb{Z}_a$, they have been plotted along with their continuous analogues over the real-variable interval $[-\frac{1}{2}, a-\frac{1}{2})$. Unlike the discrete functions, these continuous chirps $\mathrm{c}_a^x : \mathbb{R} \to \mathbb{C}$ are not a-periodic. In particular, we have

$$\frac{\mathrm{c}_a^x(t+a)}{\mathrm{c}_a^x(t)} = \frac{\mathrm{e}^{\pi \mathrm{i} x(t+a)t/a}}{\mathrm{e}^{\pi \mathrm{i} xt(t-a)/a}} = \mathrm{e}^{\pi \mathrm{i} xt[(t+a)-(t-a)]/a} = \mathrm{e}^{2\pi \mathrm{i} xt}$$

for any $x \in \mathbb{Z}$ and any $t \in \mathbb{R}$. Thus, $\mathrm{c}_a^x(t+a) = \mathrm{c}_a^x(t)$ if and only if $xt \in \mathbb{Z}$. We now state and prove some elementary properties of the finite chirps $\mathrm{c}_a^x : \mathbb{Z}_a \to \mathbb{C}$:

Proposition 2.1. *For $a \in \mathbb{Z}^+$, and any $t, x, y \in \mathbb{Z}$,*

1. $\overline{\mathrm{c}_a^x} = \mathrm{c}_a^{-x}$,
2. $\mathrm{c}_a^{x+a} = \mathrm{c}_a^x$ *when a is odd,*
3. $\mathrm{c}_a^{x+a} = \mathrm{m}_a^{a/2}\mathrm{c}_a^x$ *when a is even,*
4. $\mathrm{c}_a^x(yt) = \mathrm{c}_a^{xy^2}(t)$,
5. $\mathrm{c}_a^x(-t) = \mathrm{c}_a^x(t)$,
6. $\mathrm{c}_a^y(t-x) = \mathrm{c}_a^y(t)\mathrm{m}_a^{-xy}(t)\mathrm{c}_a^y(x)$,
7. $\mathrm{T}_a^x\mathrm{C}_a^y = \mathrm{c}_a^y(x)\mathrm{M}_a^{-xy}\mathrm{C}_a^y\mathrm{T}_a^x$,
8. $\mathrm{M}_a^x\mathrm{C}_a^y = \mathrm{C}_a^y\mathrm{M}_a^x$.

Proof. (1) This follows immediately from the fact that $\overline{\mathrm{w}_a} = \mathrm{w}_a^{-1}$. (2,3) Since $t \in \mathbb{Z}$ is even if and only if t^2 is even,

$$\mathrm{w}_a^{at^2/2} = \mathrm{e}^{\pi \mathrm{i} at^2/a} = (-1)^{t^2} = (-1)^t.$$

Therefore, for any $a \in \mathbb{Z}$ we have,

$$\mathrm{c}_a^{x+a}(t) = (-1)^{(x+a)t}\mathrm{w}_a^{(x+a)t^2/2} = (-1)^{xt}\mathrm{w}_a^{xt^2/2}(-1)^{at}\mathrm{w}_a^{at^2/2} = \mathrm{c}_a^x(t)(-1)^{at}(-1)^t = (-1)^{(a+1)t}\mathrm{c}_a^x(t).$$

Thus, when a is odd, $\mathrm{c}_a^{x+a}(t) = \mathrm{c}_a^x(t)$. If, on the other hand, a is even, then $a/2 \in \mathbb{Z}$ and

$$\mathrm{c}_a^{x+a}(t) = (-1)^t\mathrm{c}_a^x(t) = \mathrm{e}^{2\pi \mathrm{i}(a/2)t/a}\mathrm{c}_a^x(t) = \mathrm{m}_a^{a/2}\mathrm{c}_a^x.$$

(4) As before, note that $(-1)^y = (-1)^{y^2}$. Thus,

$$\mathrm{c}_a^x(yt) = (-1)^{xyt}\mathrm{w}_a^{x(yt)^2/2} = (-1)^{xy^2t}\mathrm{w}_a^{xy^2t^2/2} = \mathrm{c}_a^{xy^2}(t).$$

(5) Apply the previous result to $y = -1$. (6) Simply note,

$$\mathrm{c}_a^y(t-x) = (-1)^{y(t+x)}\mathrm{w}_a^{y(t-x)^2/2} = (-1)^{yt}(-1)^{yx}\mathrm{w}_a^{yt^2/2}\mathrm{w}_a^{-yxt}\mathrm{w}_a^{yx^2/2} = \mathrm{c}_a^y(t)\mathrm{m}_a^{-xy}(t)\mathrm{c}_a^y(x).$$

(7) This follows immediately from the previous result. (8) This follows immediately from the fact that both M_a^x and C_a^y are multiplicative operators. □

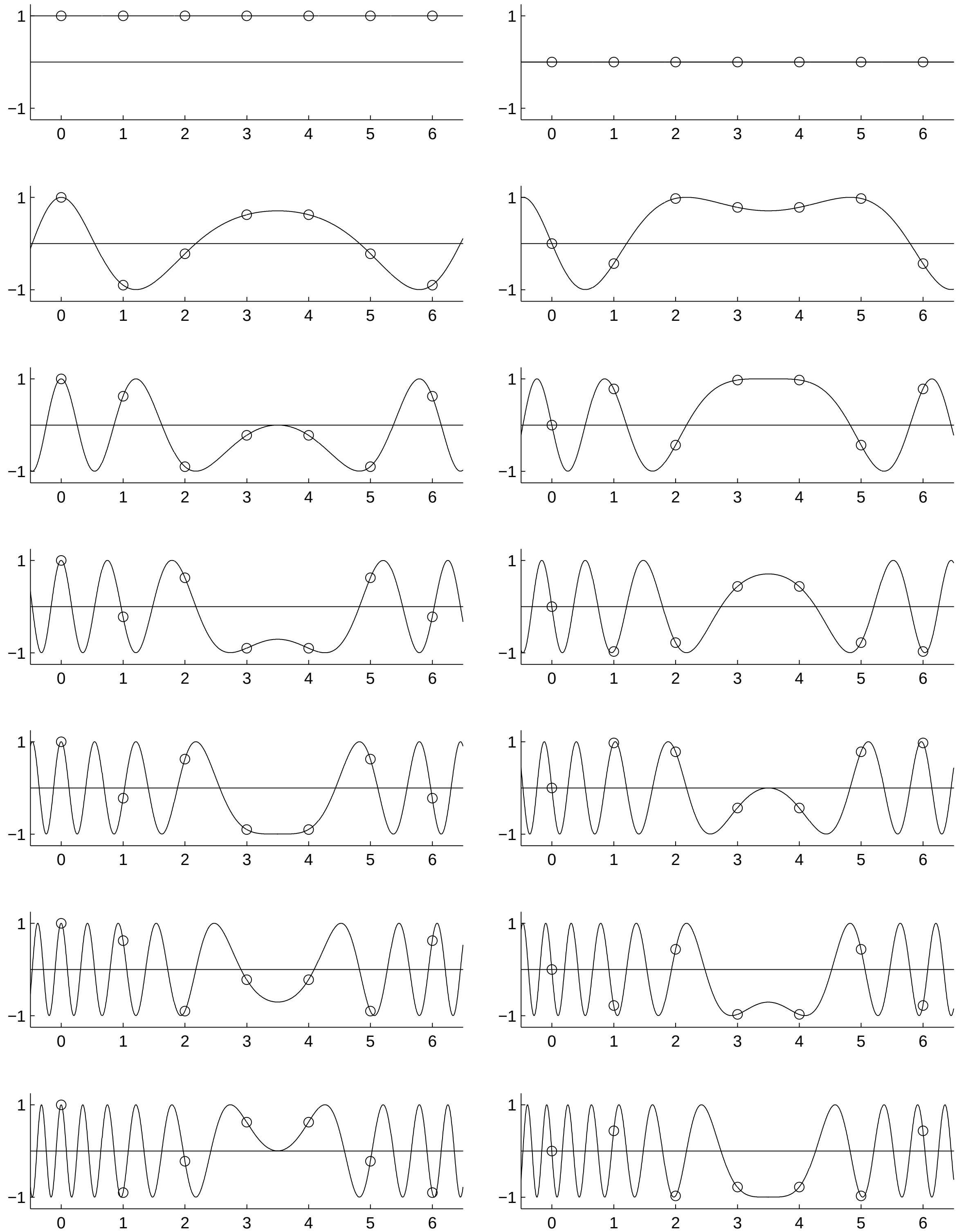

Figure 1. The real and imaginary parts of the functions $c_7^x(t) = e^{\pi i x t(t-7)/7} = \cos(\frac{\pi x t(t-7)}{7}) + i\sin(\frac{\pi x t(t-7)}{7})$ for $x = 0, \dots, 6$.

3. FRAMES OF FINITE CHIRPS

We now discuss a simple way in which our finite chirps may be used to provide overcomplete decompositions of elements of $\ell(\mathbb{Z}_a)$. In particular, we consider the collection of a^2 functions obtained by taking every modulation of $\{\mathrm{c}_a^y\}_{y=0}^{a-1}$. That is, we consider the collection:

$$\{\mathrm{m}_a^x \mathrm{c}_a^y\}_{y=0,x\in\mathbb{Z}_a}^{a-1}.$$

Note that by the previous result, this collection is "complete," namely that it contains every possible modulation of every possible chirp. Furthermore, this collection is effectively translation invariant, as the translation of any element is a unit-scalar multiple of another element.

The next result shows that this collection is a finite *tight frame* for $\ell(\mathbb{Z}_a)$, and thus possesses an exceptionally efficient means by which any $f\in\ell(\mathbb{Z}_a)$ may be decomposed into a linear combination of the collection's elements. The argument used is essentially the same idea used to show that any union of orthonormal bases is a tight frame.

Proposition 3.1. *For any $a\in\mathbb{Z}^+$,*

$$f=\frac{1}{a^2}\sum_{y=0}^{a-1}\sum_{x\in\mathbb{Z}_a}\langle f,\mathrm{m}_a^x\mathrm{c}_a^y\rangle\mathrm{m}_a^x\mathrm{c}_a^y$$

for all $f\in\ell(\mathbb{Z}_a)$.

Proof. We begin by recalling that the normalized Fourier basis $\{(1/\sqrt{a})\mathrm{m}_a^x\}_{x\in\mathbb{Z}_a}$ is an orthonormal basis for $\ell(\mathbb{Z}_a)$. For any $y\in\mathbb{Z}$, the fact that $|\mathrm{c}_a^y(t)|=1$ for all $t\in\mathbb{Z}_a$ automatically implies that $\{(1/\sqrt{a})\mathrm{m}_a^x\mathrm{c}_a^y\}_{x\in\mathbb{Z}_a}$ is also an orthonormal basis for $\ell(\mathbb{Z}_a)$. Thus, for any $y\in\mathbb{Z}$, Parseval's identity gives

$$af=\sum_{x\in\mathbb{Z}_a}\langle f,\mathrm{m}_a^x\mathrm{c}_a^y\rangle\mathrm{m}_a^x\mathrm{c}_a^y$$

for all $f\in\ell(\mathbb{Z}_a)$. Summing these orthonormal basis decompositions over all $y=0,\dots,a-1$ gives

$$a^2 f=\sum_{y=0}^{a-1}\sum_{x\in\mathbb{Z}_a}\langle f,\mathrm{m}_a^x\mathrm{c}_a^y\rangle\mathrm{m}_a^x\mathrm{c}_a^y$$

for all $f\in\ell(\mathbb{Z}_a)$. □

However, not all tight frames are equally useful. As discussed in detail by Strohmer and Heath,[7] tight frames tend to perform better in applications when the frame elements are designed to be as *uncorrelated* as possible. In our context, the degree to which our frame elements are uncorrelated is measured by the quantity:

$$\max_{(x_1,y_1)\neq(x_2,y_2)}|\langle\mathrm{m}_a^{x_1}\mathrm{c}_a^{y_1},\mathrm{m}_a^{x_2}\mathrm{c}_a^{y_2}\rangle|.$$

As demonstrated elsewhere,[3] this quantity may be explicitly computed by first finding the magnitudes of the Discrete Fourier Transforms of an arbitrary finite chirp. We summarize the results here. In particular, let (a,b) denote the greatest common divisor of $a,b\in\mathbb{Z}$. We say that two integers a and b *have a common power of two* if there exists odd integers α and β such that $a=2^m\alpha$ and $b=2^m\beta$ for some positive integer m. That is, a and b have a common power of two if the exponents of two in their respective prime factorizations are equal.

Proposition 3.2. *Let $a\in\mathbb{Z}^+$, $b\in\mathbb{Z}$ have greatest common divisor (a,b). If a and b do not have a common power of two,*

$$|(\mathrm{F}_a\mathrm{c}_a^b)(x)|^2=\begin{cases}(a,b) & x\in(a,b)\mathbb{Z}\\ 0 & x\notin(a,b)\mathbb{Z}\end{cases}.$$

Alternatively, if a and b do have a common power of two,

$$|(\mathrm{F}_a\mathrm{c}_a^b)(x)|^2=\begin{cases}(a,b) & x\in(a,b)(\mathbb{Z}+\frac{1}{2})\\ 0 & x\notin(a,b)(\mathbb{Z}+\frac{1}{2})\end{cases}.$$

The previous result is proven using the finite version of the Poisson Summation Formula, and gives the following near-immediate corollary:

Corollary 3.3. *For any* $a \in \mathbb{Z}^+$ *and any* $x_1, x_2, y_1, y_2 \in \mathbb{Z}$*, we have* $|\langle \mathrm{m}_a^{x_1} \mathrm{c}_a^{y_1}, \mathrm{m}_a^{x_2} \mathrm{c}_a^{y_2} \rangle|^2 = a(a, y_1 - y_2)$ *when either:*

- *a and* $y_1 - y_2$ *do not have a common power of two and* $x_1 - x_2$ *is an integer multiple of* $(a, y_1 - y_2)$*,*
- *a and* $y_1 - y_2$ *have a common power of two and* $x_1 - x_2$ *is a half-integer multiple of* $(a, y_1 - y_2)$*,*

and we have $\langle \mathrm{m}_a^{x_1} \mathrm{c}_a^{y_1}, \mathrm{m}_a^{x_2} \mathrm{c}_a^{y_2} \rangle = 0$ *otherwise. In particular, when* $a = p$ *is prime,*

$$|\langle \mathrm{m}_p^{x_1} \mathrm{c}_p^{y_1}, \mathrm{m}_p^{x_2} \mathrm{c}_p^{y_2} \rangle|^2 = \begin{cases} p^2 & \text{if } y_1 = y_2,\, x_1 = x_2 \\ p & \text{if } y_1 \neq y_2 \\ 0 & \text{if } y_1 = y_2,\, x_1 \neq x_2 \end{cases},$$

where all equivalences between either x_1 *and* x_2*, or* y_1 *and* y_2*, are taken mod* p*.*

Thus, the level of "uncorrelation" in our tight frames is:

$$\max_{(x_1,y_1) \neq (x_2,y_2)} |\langle \mathrm{m}_a^{x_1} \mathrm{c}_a^{y_1}, \mathrm{m}_a^{x_2} \mathrm{c}_a^{y_2} \rangle| = \sqrt{ab},$$

where b is the greatest proper divisor of a. As such, the "best" modulated chirp frames are obtained when a is prime, a fact which was originally noted by Xia.[8]

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