

Approximation of the frame coefficients using finite-dimensional methods.

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ABSTRACT

A frame is a family $\{f_i\}_{i=1}^{\infty}$ of elements in a Hilbert space \mathcal{H} with the property that every element in \mathcal{H} can be written as a (infinite) linear combination of the frame elements. Frame theory describes how one can choose the corresponding coefficients, which are called frame coefficients. From the mathematical point of view this is gratifying, but for applications it is a problem that the calculation requires inversion of an operator on \mathcal{H} .

The projection method is introduced in order to avoid this problem. The basic idea is to consider finite subfamilies $\{f_i\}_{i=1}^n$ of the frame and the orthogonal projection P_n onto $\text{span}\{f_i\}_{i=1}^n$. For $f \in \mathcal{H}$, $P_n f$ has a representation as a linear combination of $f_i, i = 1, 2, \dots, n$, and the corresponding coefficients can be calculated using finite dimensional methods. We find conditions implying that those coefficients converge to the correct frame coefficients as $n \rightarrow \infty$, in which case we have avoided the inversion problem. In the same spirit we approximate the solution to a moment problem. It turns out, that the class of “well-behaving frames” are identical for the two problems we consider.

1. INTRODUCTION.

We begin with some definitions. Let \mathcal{H} be a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ linear in the first entry. All index sets are assumed to be countable.

Definitions: 1) A family $\{f_i\}_{i \in I} \subseteq \mathcal{H}$ is a *Riesz basis* for \mathcal{H} if $\{f_i\}_{i \in I}$ is total and there exist numbers $A, B > 0$ such that

$$A \sum_{i \in I} |c_i|^2 \leq \left\| \sum_{i \in I} c_i f_i \right\|^2 \leq B \sum_{i \in I} |c_i|^2$$

for all sequences $\{c_i\}_{i \in I} \in \ell^2(I)$.

2) A family $\{f_i\}_{i \in I} \subseteq \mathcal{H}$ is a *frame* for \mathcal{H} if

$$\exists A, B > 0 : A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}.$$

A, B are called *frame bounds*.

3) $\{f_i\}_{i \in I} \subseteq \mathcal{H}$ is a *frame sequence* if $\{f_i\}_{i \in I}$ is a frame for its closed span.

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We remind the reader that a frame $\{f_i\}_{i \in I}$ gives rise to a decomposition of the underlying space \mathcal{H} : if we define the *frame operator* by

$$S : \mathcal{H} \rightarrow \mathcal{H}, \quad Sf = \sum_{i \in I} \langle f, f_i \rangle f_i,$$

then S is bounded and invertible, and

$$(1) \quad f = SS^{-1}f = \sum_{i \in I} \langle f, S^{-1}f_i \rangle f_i, \quad \forall f \in \mathcal{H}.$$

$\{\langle f, S^{-1}f_i \rangle\}$ are called the *frame coefficients*.

(1) can be viewed as a generalization of the well known representation of f using an orthonormal basis $\{e_i\}_{i \in I}$:

$$(2) \quad f = \sum_{i \in I} \langle f, e_i \rangle e_i, \quad f \in \mathcal{H}.$$

The main difference is that the coefficients $\{\langle f, e_i \rangle\}$ in (2) are unique, while there might exist other coefficients $\{c_i\}_{i \in I}$ than the frame coefficients that satisfy $f = \sum_{i \in I} c_i f_i$ for a chosen element f . If $\{c_i\}_{i \in I}$ has this property, it is wellknown³ that

$$\sum_{i \in I} |c_i|^2 = \sum_{i \in I} |\langle f, S^{-1}f_i \rangle|^2 + \sum_{i \in I} |c_i - \langle f, S^{-1}f_i \rangle|^2,$$

i.e., the frame coefficients have minimal ℓ^2 -norm among the sequences that can be used to represent f .

A frame is a Riesz basis if and only if the elements f_i are ω -independent in the sense that

$$\{c_i\}_{i \in I} \in \ell^2(I), \quad \sum_{i \in I} c_i f_i = 0 \Rightarrow c_i = 0, \quad \forall i \in I.$$

By definition, every subfamily of a Riesz basis is a Riesz basis for its closed span. But in general, not every subfamily of a frame is a frame sequence. If every subfamily $\{f_i\}_{i \in J}$ of the frame $\{f_i\}_{i \in I}$ is a frame sequence with bounds which are common for all those frames, then we call $\{f_i\}_{i \in I}$ a *Riesz frame*. Clearly one only has to check the existence of a common lower bound, and it is easy to check that $\{f_i\}_{i \in I}$ is a Riesz frame if this condition is satisfied for all finite indexsets $J \subseteq I$.

It turns out that the problems we consider here have very satisfying solutions for Riesz frames, so it is interesting to notice that they were introduced in a very different context. One of the main problems in⁶ was to find conditions implying that a frame $\{f_i\}_{i \in I}$ contains a Riesz basis, and it was shown that every Riesz frame has this property. Later it was discovered that the same is true under the weaker condition “every subfamily of $\{f_i\}_{i \in I}$ is a frame sequence”¹.

2. APPROXIMATION OF THE FRAME COEFFICIENTS

From the point of view of applications the problem with calculation of the frame coefficients is to invert S . An idea to overcome this difficulty is to “truncate” the problem. So let $\{I_n\}_{n=1}^{\infty}$ be a family of finite subsets of I such that

$$I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \nearrow I.$$

Corresponding to a subfamily $\{f_i\}_{i \in I_n}$ we define the space $\mathcal{H}_n := \text{span}\{f_i\}_{i \in I_n}$ and the frame operator

$$S_n : \mathcal{H}_n \rightarrow \mathcal{H}_n, \quad S_n f = \sum_{i \in I_n} \langle f, f_i \rangle f_i.$$

The orthogonal projection of $f \in \mathcal{H}$ onto \mathcal{H}_n is given by

$$P_n f = \sum_{i \in I_n} \langle f, S_n^{-1}f_i \rangle f_i.$$

Since $P_n f \rightarrow f$ as $n \rightarrow \infty$, one can hope that the coefficients $\langle f, S_n^{-1} f_i \rangle$ converges to the frame coefficients for f , i.e., that

$$(3) \quad \langle f, S_n^{-1} f_i \rangle \rightarrow \langle f, S^{-1} f_i \rangle \text{ as } n \rightarrow \infty, \quad \forall i \in I, \forall f \in \mathcal{H}.$$

If (3) is satisfied we say that the *projection method* works. In this case the frame coefficients can be approximated as close as we want using finite dimensional methods, i.e., linear algebra, since S_n is an operator on the finite dimensional space \mathcal{H}_n . This is a very important property for applications: for example, it makes it possible to use computers to approximate the frame coefficients.

If the projection method works the next natural question is how fast the convergence in (3) is. For example, one might wish that the set of coefficients $\{\langle f, S_n^{-1} f_i \rangle\}_{i \in I_n}$ converges to the set of frame coefficients in ℓ^2 -sense, i.e., that

$$(4) \quad \sum_{i \in I_n} |\langle f, S_n^{-1} f_i \rangle - \langle f, S^{-1} f_i \rangle|^2 + \sum_{i \in I - I_n} |\langle f, S^{-1} f_i \rangle|^2 \rightarrow 0, \quad \forall f \in \mathcal{H}.$$

We say that the *strong projection method* works if (4) is satisfied. Note that this condition depends on the indexing of the elements. The second term $\sum_{i \in I - I_n} |\langle f, S^{-1} f_i \rangle|^2 \rightarrow 0$ for every frame, $\forall f \in \mathcal{H}$, so we only need to show that $\sum_{i \in I_n} |\langle f, S_n^{-1} f_i \rangle - \langle f, S^{-1} f_i \rangle|^2 \rightarrow 0$, $\forall f \in \mathcal{H}$.

The projection method was introduced in the paper ³. The original presentation is slightly more general than here, since it is formulated for a family $\{f_i\}_{i \in I}$ which does not need to be a frame. In the special case of a frame, the paper ³ contains an equivalent characterization of the projection method:

Theorem 2.1: *Let $\{f_i\}_{i \in I}$ be a frame. Then the projection method works if and only if for any $j \in I$ there exists a constant c_j such that*

$$\|S_n^{-1} f_j\| \leq c_j, \quad \forall n \text{ such that } j \in I_n.$$

The paper ³ also contains an example of a frame, where the projection method does not work: if $\{e_i\}_{i=1}^\infty$ is an orthonormal basis and we define the family $\{f_i\}_{i=1}^\infty := \{e_1\} \cup \{e_{i-1} + \frac{1}{i} e_i\}_{i=2}^\infty$, then $\{f_i\}_{i=1}^\infty$ is a frame, but

$$\|S_n^{-1} f_1\|^2 = \sum_{i=1}^n (i!)^2 \rightarrow \infty \text{ for } n \rightarrow \infty.$$

As the first theoretical results concerning the strong projection method we have

Theorem 2.2: *The strong projection method works for every Riesz frame.*

Proof: Let A, B denote common bounds for the Riesz frame $\{f_i\}_{i \in I}$. By ⁵, $\|S_n^{-1}\| \leq \frac{1}{A}$ for all $n \in N$. Now fix $f \in \mathcal{H}$. Then

$$\begin{aligned} \sum_{i \in I_n} |\langle f, S_n^{-1} f_i \rangle - \langle f, S^{-1} f_i \rangle|^2 &= \sum_{i \in I_n} |\langle P_n f, S_n^{-1} f_i \rangle - \langle f, S^{-1} f_i \rangle|^2 \\ &= \sum_{i \in I_n} |\langle S_n^{-1} P_n f - S^{-1} f, f_i \rangle|^2 \leq B \|S^{-1} f - S_n^{-1} P_n f\|^2 \\ &\leq B (\|S^{-1} f - P_n S^{-1} f\| + \|P_n S^{-1} f - S_n^{-1} P_n f\|)^2 \\ &= B (\|S^{-1} f - P_n S^{-1} f\| + \|\sum_{i \in I_n} \langle S^{-1} f, f_i \rangle S_n^{-1} f_i - S_n^{-1} P_n f\|)^2 \\ &\leq B (\|S^{-1} f - P_n S^{-1} f\| + \|S_n^{-1}\| \cdot \|\sum_{i \in I_n} \langle S^{-1} f, f_i \rangle f_i - P_n f\|)^2 \\ &\leq B (\|S^{-1} f - P_n S^{-1} f\| + \frac{1}{A} \|\sum_{i \in I_n} \langle S^{-1} f, f_i \rangle f_i - P_n f\|)^2 \rightarrow 0 \text{ for } n \rightarrow \infty \end{aligned}$$

since $\sum_{i \in I_n} \langle S^{-1}f, f_i \rangle f_i \rightarrow f$ for $n \rightarrow \infty$ **Q.E.D.**

The readers who have checked the proof might have observed that the proof does not use that *all* the sequences $\{f_i\}_{i \in J}, J \subseteq I$, are frame sequences with common bounds: Theorem 2.2 holds if $\{f_i\}_{i \in I}$ is a frame and $\{I_n\}_{n=1}^\infty$ is a family of finite index sets such that the frame sequences $\{f_i\}_{i \in I_n}, n \in N$, have a common lower bound, or equivalently,

$$\exists A > 0 \forall n : \sum_{i \in I_n} |\langle f, f_i \rangle|^2 \geq A \|f\|^2, \quad \forall f \in \mathcal{H}_n.$$

Let us say that $\{f_i\}_{i \in I}$ is a *conditional Riesz frame* (with respect to $\{I_n\}_{n=1}^\infty$) if the frame sequences $\{f_i\}_{i \in I_n}, n \in N$, have common frame bounds. In the case of a frame indexed by the natural numbers we always use the convention $I_n = \{1, 2, \dots, n\}$.

This is indeed a weaker condition. For example, if $\{e_i\}_{i=1}^\infty$ is an orthonormal basis for \mathcal{H} , then

$$\{f_i\}_{i \in I} := \{e_2, e_2 + \frac{1}{2}e_1, e_3, e_3 + \frac{1}{4}e_1, \dots\} = \{e_i, e_i + \frac{1}{2^i}e_1\}_{i=2}^\infty$$

is a conditional Riesz frame, but not a Riesz frame. The reason for formulating Theorem 2.1 for Riesz frames is that Riesz frames give a good intuitive feeling for the problems. Furthermore Riesz frames have the advantage, that one do not need to care about indexing the frame elements: the conclusions holds for *every* arrangement of the frame elements.

In terms of the operators S_n^{-1} , a frame $\{f_i\}_{i \in I}$ is a conditional Riesz frame if and only if $\sup_n \|S_n^{-1}\| < \infty$. Observe that the notion depends on the indexing of the frame elements. For example, if $\{e_i\}_{i=1}^\infty$ is an orthonormal basis then $\{e_i, \frac{1}{i}e_i\}_{i=1}^\infty$ is a conditional Riesz frame, but $\{\frac{1}{i}e_i, e_i\}_{i=1}^\infty$ is not.

Actually, the strong projection method works if and only if $\{f_i\}_{i \in I}$ is a conditional Riesz frame. This is the content of the following result, which we prove in ²:

Theorem 2.3: *Let $\{f_i\}_{i \in I}$ be a frame. Then the following are equivalent:*

- (a) *The strong projection method works.*
- (b) *$\{f_i\}_{i \in I}$ is a conditional Riesz frame.*
- (c) *$S_n^{-1}P_n f \rightarrow S^{-1}f$ for all $f \in \mathcal{H}$.*
- (d) *$\langle S_n^{-1}P_n f, g \rangle \rightarrow \langle S^{-1}f, g \rangle$ for all $f, g \in \mathcal{H}$*
- (e) *$\lim_{n \rightarrow \infty} \sum_{i=n+1}^\infty |\langle S_n^{-1}P_n f, f_i \rangle|^2 = 0, \forall f \in \mathcal{H}$.*

Usually (c) is formulated by saying that $S_n^{-1}P_n \rightarrow S^{-1}$ in the strong operator topology, and (d) by saying that $S_n^{-1}P_n \rightarrow S^{-1}$ in the weak operator topology. That (c) and (d) are equivalent is a consequence of the fact that the involved operators are positive.

Also, observe that (d) shows that the condition for the strong projection method to work can be formulated in a very similar way as the condition for the projection method to work, cf. (3).

3. APPROXIMATION OF THE SOLUTION TO A MOMENT PROBLEM.

We now consider a related question. Let again $\{f_i\}_{i \in I}$ be a frame for \mathcal{H} and let $\{a_i\}_{i \in I} \subseteq \ell^2(I)$. We ask whether there exists $f \in \mathcal{H}$ such that

$$\langle f, f_i \rangle = a_i, \quad \forall i \in I.$$

A problem of this type is called a *moment problem*. For the general theory we refer to ⁸. It is easy to construct examples where there is no solution f , but as shown in ⁴ there always exists a unique element in \mathcal{H} minimizing $\sum_{i \in I} |a_i - \langle f, f_i \rangle|^2$; this element is $f = \sum_{i \in I} a_i S^{-1}f_i$. We call $\sum_{i \in I} a_i S^{-1}f_i$ the *b.a.s. (best approximation solution) to the moment problem*.

Corresponding to a subset $\{a_i\}_{i \in I_n}$ the unique element in \mathcal{H}_n minimizing $\sum_{i \in I_n} |a_i - \langle f, f_i \rangle|^2$ is $\sum_{i \in I_n} a_i S_n^{-1} f_i$. We say that $\sum_{i \in I_n} a_i S_n^{-1} f_i$ is the *b.a.s. to the truncated moment problem*. In analogy with above we would like to find conditions implying that

$$(5) \quad \sum_{i \in I_n} a_i S_n^{-1} f_i \rightarrow \sum_{i \in I} a_i S^{-1} f_i \text{ for } n \rightarrow \infty, \forall \{a_i\}_{i \in I} \in \ell^2(I).$$

Observe the connection to the projection method: if (5) is satisfied (or just for all sequences with 1 in one entry, otherwise 0), then $S_n^{-1} f_i \rightarrow S^{-1} f_i$ for $n \rightarrow \infty, \forall i$, so the projection method works.

Zwaan⁹ has shown that (5) is satisfied if $\{f_i\}_{i \in I}$ is a Riesz basis. Here we prove that (5) is satisfied if and only if $\{f_i\}_{i \in I}$ is a conditional Riesz frames. We need the operator (sometimes called the *pre-frame operator*)

$$T : \ell^2(I) \rightarrow \mathcal{H}, \quad T\{c_i\}_{i \in I} = \sum_{i \in I} c_i f_i.$$

We denote the kernel of T by N_T . T is bounded and the adjoint operator $T^* : \mathcal{H} \rightarrow \ell^2(I)$ is given by $T^* f = \{\langle f, f_i \rangle\}_{i \in I}$.

Theorem 3.1: *Let $\{f_i\}_{i \in I}$ be a frame. Then the following are equivalent:*

- (a) $\sum_{i \in I_n} a_i S_n^{-1} f_i \rightarrow \sum_{i \in I} a_i S^{-1} f_i$ for $n \rightarrow \infty, \forall \{a_i\}_{i \in I} \in \ell^2(I)$.
- (b) $S_n^{-1} \sum_{i \in I_n} b_i f_i \rightarrow 0$ for $n \rightarrow \infty$ for all $\{b_i\}_{i \in I} \in \ell^2(I)$ such that $\sum_{i \in I} b_i f_i = 0$.
- (c) $\{f_i\}_{i \in I}$ is a conditional Riesz frame.

Proof:

(a) \Leftrightarrow (b): Since $\ell^2(I)$ is the orthogonal sum of the range of T^* and the kernel of T , we can write any sequence $\{a_i\}_{i \in I} \in \ell^2(I)$ as

$$\{a_i\}_{i \in I} = \{\langle g, f_i \rangle\}_{i \in I} + \{b_i\}_{i \in I}$$

for some $g \in \mathcal{H}, \{b_i\}_{i \in I} \in N_T$. Now, the b.a.s. to the truncated moment problem is

$$\sum_{i \in I_n} a_i S_n^{-1} f_i = \sum_{i \in I_n} \langle g, f_i \rangle S_n^{-1} f_i + \sum_{i \in I_n} b_i S_n^{-1} f_i = P_n g + S_n^{-1} \sum_{i \in I_n} b_i f_i.$$

The b.a.s. to the moment problem is

$$\sum_{i \in I} a_i S^{-1} f_i = \sum_{i \in I} \langle g, f_i \rangle S^{-1} f_i + S^{-1} \sum_{i \in I} b_i f_i = g,$$

from which the result follows.

(a) \Rightarrow (c): Define $Q : \ell^2(I) \rightarrow \mathcal{H}$ by,

$$Q\{a_i\}_{i \in I} = \sum_{i \in I} a_i S^{-1} f_i,$$

and define $Q_n : \ell^2(I) \rightarrow \mathcal{H}$ by,

$$Q_n\{a_i\}_{i \in I} = \sum_{i \in I_n} a_i S_n^{-1} f_i.$$

(a) states that $Q_n \rightarrow Q$ in the strong operator topology. Hence $\sup_n \|Q_n\| < \infty$. But, $Q_n^* f = \{\langle f, S_n^{-1} f_i \rangle\}_{i \in I_n}$, and $P_n f = \sum_{i \in I_n} \langle f, S_n^{-1} f_i \rangle f_i$, implies

$$S_n^{-1} P_n f = \sum_{i \in I_n} \langle f, S_n^{-1} f_i \rangle S_n^{-1} f_i = Q_n Q_n^* f.$$

Hence,

$$\sup_n \|S_n^{-1}\| = \sup_n \|S_n^{-1}P_n\| = \sup_n \|Q_n Q_n^*\| \leq \sup_n \|Q_n\|^2 < \infty.$$

(c) \Rightarrow (b) : Follows from

$$\|S_n^{-1} \sum_{i \in I_n} b_i f_i\| \leq \|S_n^{-1}\| \cdot \left\| \sum_{i \in I_n} b_i f_i \right\|.$$

Q.E.D.

4. CONDITIONAL RIESZ FRAMES.

By introducing the distance between an element $f \in \mathcal{H}$ and a subspace $V \subseteq \mathcal{H}$ we can find a condition implying that $\{f_i\}_{i=1}^\infty$ is a conditional Riesz frame. Let

$$\text{dist}(f, V) = \inf_{g \in V} \|f - g\|.$$

Proposition 4: *Let $\{f_i\}_{i=1}^\infty \subseteq \mathcal{H}$ satisfy the upper frame condition with bound B and suppose that there exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$,*

$$|\langle f, f_{n+1} \rangle| \geq \epsilon \cdot \text{dist}(f, \mathcal{H}_n), \quad \forall f \in \mathcal{H}_{n+1}.$$

Choose ϵ so small that $\|f_1\| \geq \epsilon$. Then $\{f_i\}_{i=1}^\infty$ is a conditional Riesz frame with bounds ϵ^2, B .

Proof: We only need to care about the lower bound. First, observe that $\text{dist}(f, \mathcal{H}_n) = \|f - P_n f\|$. Let us now prove the existence of a common lower bound by induction. The assumption $\|f_1\| \geq \epsilon$ immediately implies that $\{f_1\}$ is a frame for $\text{span}\{f_1\}$ with lower bound ϵ^2 , so assume that $\{f_i\}_{i=1}^k$ has the same lower bound for some $k > 1$. For $f \in \mathcal{H}_{k+1}$ we then have

$$\begin{aligned} \sum_{i=1}^{k+1} |\langle f, f_i \rangle|^2 &= \sum_{i=1}^k |\langle P_k f, f_i \rangle|^2 + |\langle f, f_{k+1} \rangle|^2 \\ &\geq \epsilon^2 \|P_k f\|^2 + \epsilon^2 \text{dist}(f, \mathcal{H}_k) = \epsilon^2 \|P_k f\|^2 + \epsilon^2 \|(I - P_k)f\|^2 = \epsilon^2 \|f\|^2 \end{aligned}$$

as desired. Now we finish by showing that ϵ^2 also is a lower bound for $\{f_i\}_{i=1}^\infty$. But if $f \in \mathcal{H}$, then

$$\epsilon^2 \|P_n f\|^2 \leq \sum_{i=1}^n |\langle P_n f, f_i \rangle|^2 \leq \sum_{i=1}^\infty |\langle f, f_i \rangle|^2,$$

from which the result follows by letting $n \rightarrow \infty$. **Q.E.D.**

5. EXAMPLES.

The most important frames are Weyl-Heisenberg frames and wavelet frames. A Weyl-Heisenberg frame with lattice parameters $a, b > 0$ is a family $\{f_{m,n}(x)\}_{(m,n) \in \mathbb{Z}^2}$ of functions of the form

$$f_{m,n}(x) := e^{i2\pi m b x} f(x - na),$$

where $f \in L^2(\mathbb{R})$ is fixed. A wavelet frame consists of scaled and translated versions of a single function $f \in L^2(\mathbb{R})$; here one choose parameters $a > 1, b > 0$ and define

$$\{f_{m,n}(x)\}_{(m,n) \in \mathbb{Z}^2} = \left\{ \frac{1}{|a|^{n/2}} f\left(\frac{x}{a^n} - mb\right) \right\}_{(m,n) \in \mathbb{Z}^2}.$$

There exist easy verifiable conditions implying that such families are frames ⁷. The approximation questions considered here are important for such frames, so it would be interesting to answer the question

When is a wavelet frame or a Weyl-Heisenberg frame a (conditional) Riesz frame?

This question seems to be difficult. At present, the authors only know the answer in some special cases, namely

- 1) It is well known that a Weyl-Heisenberg frame is in fact a Riesz basis if $ab = 1$. Therefore we have a Riesz frame in this case.
- 2) Consider the gaussian, $g(x) = e^{-x^2}$. The Weyl-Heisenberg family corresponding to this function and the lattice parameters $a = 1, b = 1/2$ is a frame. But the subfamily $\{g_{m,n}\}_{m \in 2Z, n \in Z}$ is not a frame. So $\{g_{m,n}\}_{(m,n) \in Z^2}$ is not a Riesz frame.

6. REFERENCES:

- [1] P. G. Casazza and O. Christensen: "Hilbert space frames containing a Riesz basis and Banach spaces which have no subspace isomorphic to c_0 ." *J. Math. Anal. Appl.* **202** (1996) p. 940- 950.
- [2] P. G. Casazza and O. Christensen: "Riesz frames and approximation of the frame coefficients." Submitted, july 1996.
- [3] O. Christensen: "Frames and the projection method." *Appl. Comp. Harm. Anal.* **1** (1993) p. 50-53.
- [4] O. Christensen: "Moment problems and stability results for frames with applications to irregular sampling and Gabor frames." *Appl. Comp. Harm. Anal.* **3** (1996) p.82-86.
- [5] O. Christensen: "Frames and pseudo-inverses." *J. Math. Anal. Appl.* **195** (1995) p. 401-414.
- [6] O. Christensen: "Frames containing a Riesz basis and approximation of the frame coefficients using finite dimensional methods." *J. Math. Anal. Appl.* **199** (1996) p. 256-270.
- [7] C. Heil and D. Walnut: "Continuous and discrete wavelet transforms." *SIAM Rev.* **31** (1989) p.628-666.
- [8] R. M. Young: *An introduction to nonharmonic Fourier series*. Academic Press, New York, 1980.
- [9] M. Zwaan: "Approximation of the solution to the moment problem in a Hilbert space." *Numer. Funct. Anal. Optim.* **11** (1990) p.601-608.