Uniform Tight Frames with Erasures

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Uniform tight frames have been shown to be useful for robust data transmission. The losses in the network are modeled as erasures of transmitted frame coefficients. We give the first systematic study of the general class of uniform tight frames and their properties. We search for efficient constructions of such frames. We show that the only uniform tight frames with the group structure and one or two generators are the generalized harmonic frames. Finally, we give a complete classification of frames in terms of their robustness to erasures.

1. INTRODUCTION

Frames are redundant sets of vectors in a Hilbert space which yield one natural representation for each vector in the space, but which may have infinitely many different representations for a given vector [4, 5, 6, 7, 8, 9, 10, 11, 16, 18, 20, 23]. Frames have been used in signal processing because of their resilience to additive noise [10], resilience to quantization [14], as well as their numerical stability of reconstruction [10], and greater freedom to capture signal characteristics [2, 3]. Recently, several new applications for (uniform tight) frames have been developed. The first, developed by Goyal, Kovačević and Vetterli [15, 25, 24, 26], uses the redundancy of a frame to mitigate the effect of losses in packet-based communication systems. Modern communication networks transport packets of data from a

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source to a recipient. These packets are sequences of information bits of a certain length surrounded by error-control, addressing, and timing information that assure that the packet is delivered without errors. This is accomplished by not delivering the packet if it contains errors. Failures here are due primarily to buffer overflows at intermediate nodes in the network. So to most users, the behavior of a packet network is not characterized by random loss, but by unpredictable transport time. This is due to a protocol, invisible to the user, that retransmits lost packets. Retransmission of packets takes much longer than the original transmission. In many applications, retransmission of lost packets is not feasible and the potential for large delay is unacceptable.

If a lost packet is independent of the other transmitted data, then the information is truly lost to the receiver. However, if there are dependencies between transmitted packets, one could have partial or complete recovery despite losses. This leads us naturally to use frames for encoding. The question, however, is: What are the best frames for this purpose? With an additive noise model for quantization, in [25], the authors show that a uniform frame minimizes mean-squared error if and only if it is tight. So it is this class of frames - the uniform normalized tight frames - which we seek to identify and study.

Another recent important application of uniform normalized tight frames is in multiple-antenna code design [17]. Much theoretical work has been done to show that communication systems which employ multiple antennas can have very high channel capacities [13, 21]. These methods rely on the assumption that the receiver knows the complex-valued Rayleigh fading coefficients. To remove this assumption, in [19] new classes of unitary space-time signals are proposed. If we have N transmitting antennas and we transmit in blocks of M time samples (over which the fading coefficients are approximately constant), then a *constellation of K unitary space-time signals* is a (weighted by $\sqrt{M}$) collection of $M \times N$ complex matrices $\{\Phi_k\}$ for which $\Phi_k^* \Phi_k = I$. The nth column of any Φ_k contains the signal transmitted on antenna n as a function of time. The only structure required in general is the time-orthogonality of the signals.

Originally it was believed that designing such constellations was a too cumbersome and difficult optimization problem for practice. However, in [19], it was shown that constellations arising in a “systematic” fashion can be done with relatively little effort. Systematic here means that we need to design high-rate space-time constellations with low encoding and decoding complexity. It is known that full transmitter diversity (that is, where the constellation is a set of unitary matrices whose differences have nonzero determinant) is a desirable property for good performance. In a tour-de-force, in [17], the authors used fixed-point-free groups and their representations to design high-rate constellations with full diversity. Moreover, they classified all full-diversity constellations that form a group, for all rates and numbers of transmitting antennas.

For these applications, and a host of other applications in signal processing, it has become important that we understand the class of uniform normalized tight frames. In this paper, we make the first systematic study of such frames. In Section 2 we review basic notions on frames. In particular, we state the Naimark's Theorem [1] which has been rediscovered several times in recent years [16, 12] although it has been used for several decades in operator theory. We give examples of uniform normalized tight frames such as harmonic and Gabor frames. For harmonic frames, we define a more general class – *general harmonic frames (GHF)* – and study when harmonic frames are equivalent to each other and general harmonic frames. In Proposition 2.4, we give a simple equivalence condition on general harmonic frames which states that the inner product between any two frame vectors φ_i and φ_j in a GHF is equal to the inner product between φ_{i+1} and φ_{j+1}. Section 3 concentrates on uniform tight frames (UTF). We start with a review in Section 3.1 and proceed with several ways of classifying UTFs including providing a correspondence between subspaces of the original space and the UTFs, obtaining UTFs as alternate dual frames for a given frame as well as finding UTFs through frames equivalent to them. In Section 4, we shift our attention to UTFs with group structure. We show that the UNTFs generated by the family $\{U^k\varphi_0\}_{k=0}^{M-1}$ (where U is unitary and $\varphi_0 \in \mathbb{H}$) are precisely the GHFs. We then extend our discussion to UNTFs generated by $\{U^kV^j\varphi_0\}$ and higher numbers of generators. Finally, Section 5 introduces erasures modeled as losses of transform coefficients $\langle f, \varphi_i\rangle$, where f is the signal to be transmitted and $\{\varphi_i\}_{i\in I}$ is a set of frames vectors corresponding to erased transform coefficients. We give a complete classification of frames with respect to their robustness to erasures. We study when we can obtain frames robust to a certain number of erasures as a projection from another frame with a different number of erasures.

2. FRAME REVIEW

A set of vectors $\Phi = \{\varphi_i\}_{i\in I}$ in a Hilbert space $\mathbb{H}$, is called a *frame* if

$$0 \;<\; A\|x\|^2 \;\leq\; \sum_{i\in I} |\langle x, \varphi_i\rangle|^2 \;<\; B\|x\|^2 \;<\; +\infty, \quad x \neq 0, \tag{1}$$

where I is the index set and the constants A, B are called *frame bounds*. Although many of our results hold in more general settings, in this paper, we concentrate mostly on the N-dimensional real or complex Hilbert spaces $\mathbb{R}^N$ and $\mathbb{C}^N$ (which we denote $\mathbb{H}_N$) with the usual Euclidean inner product. When results generalize to the infinite-dimensional setting, we will point it out.

When $A = B$ the frame is *tight* (TF). If $A = B = 1$, the frame is *normalized tight* (NTF). A frame is *uniform* (UF) if all its elements have the same norm c,[1] $\|\varphi_i\| = c$. For a uniform tight frame (UTF), the frame bound A gives the *redundancy ratio*. A UNTF of norm-1 vectors is an orthonormal basis (ONB).

The *analysis frame operator*[2] F maps the Hilbert space $\mathbb{H}$ into $\ell_2(I)$

$$(Fx)_i \;=\; \langle \varphi_i, x\rangle, \tag{2}$$

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[1] Often, uniform is used to mean $c = 1$; here, however, we use the more general definition.

[2] F is sometimes called just *frame operator*. Here, we use *analysis frame operator* for F, *synthesis frame operator* for F^* and *frame operator* for F^*F.

for $i \in I$. When $\mathbb{H} = \mathbb{H}_N$, the analysis frame operator is an $M \times N$ matrix whose rows are the transposed frame vectors φ_i^*:

$$F = \begin{pmatrix} \varphi_{11}^* & \cdots & \varphi_{1N}^* \\ \vdots & \ldots & \vdots \\ \varphi_{M1}^* & \cdots & \varphi_{MN}^* \end{pmatrix}. \tag{3}$$

We say that two frames $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ for $\mathbb{H}$ are *equivalent* if there is an invertible operator L on $\mathbb{H}$ for which $L\varphi_i = \psi_i$ for all $i \in I$, and they are *unitarily equivalent* if L can be chosen to be a unitary operator. A direct calculation shows that $\{S^{-1/2}\varphi_i\}$ is a normalized tight frame for any frame $\{\varphi_i\}$. Thus, every frame is *equivalent* to a normalized tight frame. If $\{\varphi_i\}_{i\in I}$ is a frame with frame bounds A, B, and if P is an orthogonal projection on $\mathbb{H}$, then by (1) we have that $\{P\varphi_i\}_{i\in I}$ is a frame for $P\mathbb{H}$ with frame bounds A and B. In particular, if $\{\varphi_i\}_{i\in I}$ is a normalized tight frame for $\mathbb{H}$ (for example, if it is an orthonormal basis for $\mathbb{H}$), then $\{P\varphi_i\}_{i\in I}$ is a normalized tight frame for $P\mathbb{H}$.

The following theorem tells us that every normalized tight frame can be realized as a projection of an orthonormal basis from a larger space. It serves as a converse to the observation above that orthogonal projections of normalized tight frames produce normalized tight frames for their span.

THEOREM 2.1 (Naimark [1], Han & Larson [16]). [3] *A family $\{\varphi_i\}_{i\in I}$ in a Hilbert space $\mathbb{H}$ is a normalized tight frame for $\mathbb{H}$ if and only if there is a larger Hilbert space $\mathbb{H} \subset \mathbb{K}$ and an orthonormal basis $\{e_i\}_{i\in I}$ for $\mathbb{K}$ so that the orthogonal projection P of $\mathbb{K}$ onto $\mathbb{H}$ satisfies: $Pe_i = \varphi_i$, for all $i \in I$.*

Let us now go through certain important frame notions. Using the analysis frame operator F, (1) can be rewritten as

$$AI \;\le\; F^*F \;\le\; BI. \tag{4}$$

We call $S = F^*F$ the *frame operator*. It follows that S is invertible (Lemma 3.2.2 in [10]), and furthermore

$$B^{-1}I \;\le\; S^{-1} \;\le\; A^{-1}I. \tag{5}$$

It follows that $\{\varphi_i\}_{i\in I}$ is a normalized tight frame if and only if F is an isometry. Also, S is a positive self-adjoint invertible operator on $\mathbb{H}$ and $S = AI$ if and only if the frame is tight. In finite dimensions, the *canonical dual frame* of Φ is a frame defined as $\tilde{\Phi} = \{\tilde{\varphi}_i\}_{i=1}^M = \{S^{-1}\varphi_i\}_{i=1}^M$, where

$$\tilde{\varphi}_i \;=\; S^{-1}\varphi_i, \tag{6}$$

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[3] This theorem has been rediscovered by several people in recent years: The first author first heard it from I. Daubechies in the mid-90's. Han and Larson rediscovered it in [16]; they came up with the idea that a frame could be obtained by compressing a basis in a larger space and that the process is reversible (the statement in this paper is due to Han and Larson). Finally, it was pointed out to the first author by E. Šoljanin [22] that this is, in fact, the Naimark's theorem, which has been widely known in operator theory and has been used in quantum theory.

for $k = 1, \ldots, M$. Now,

$$f = \sum_{i \in I} \langle f, S^{-1}\varphi_i \rangle \varphi_i, \quad \text{for all } f \in H.$$

So the canonical dual frame can be used to reconstruct the elements of $\mathbb{H}$ from the frame. However, there may be other sequences in $\mathbb{H}$ which give reconstruction. This formula points out both the strengths and weaknesses of frames. First, we see that every element $f \in \mathbb{H}$ has at least one natural series representation in terms of the frame elements. Also, this element may have infinitely many other representations. However, in order to find this natural representation of f, we need to invert the frame operator, which may be difficult or even impossible in practice. The best frames then are clearly the tight frames since in this case the frame operator becomes a multiple of the identity.

Noting that $\tilde{\varphi}_i^* = \varphi_i^* S^{-1}$ and stacking $\tilde{\varphi}_1^*$, $\tilde{\varphi}_2^*$, …, $\tilde{\varphi}_M^*$ in a matrix, the frame operator associated with $\tilde{\Phi}$ is

$$\tilde{F} = FS^{-1}. \tag{7}$$

Since $\tilde{F}^* \tilde{F} = S^{-1}$, (5) shows that B^{-1} and A^{-1} are frame bounds for $\tilde{\Phi}$.

Another important concept is that of a *pseudo-inverse* $F^\dagger$. It is the analysis frame operator associated with the dual frame,

$$F^\dagger = \tilde{F}^*. \tag{8}$$

Note that for any matrix F with rows φ_i^*

$$S = F^* F = \sum_{i=1}^{M} \varphi_i \varphi_i^*. \tag{9}$$

This identity will prove to be useful in many proofs.

2.1. The Role of Eigenvalues

The product $S = F^*F$ will appear everywhere and its eigenstructure will play an important role. Denote by λ_k's the eigenvalues of $S = F^*F$. We now summarize the important eigenvalue properties.

General Frame. For any frame in $\mathbb{H}_N$, the sum of the eigenvalues of $S = F^*F$, equals the sum of the lengths of the frame vectors:

$$\sum_{k=1}^{N} \lambda_k = \sum_{i=1}^{M} \|\varphi_i\|^2. \tag{10}$$

Uniform Frame. For a uniform frame, that is, when $\|\varphi_i\| = c$, $i = 1, \ldots, M$,

$$\sum_{k=1}^{N} \lambda_k = \sum_{i=1}^{M} \|\varphi_i\|^2 = M \cdot c^2. \tag{11}$$

Tight Frame. Since tightness means $A = B$, for a TF, we have from (1)

$$\sum_{i=1}^{M} |\langle f, \varphi_i \rangle|^2 \;=\; A\|f\|^2, \tag{12}$$

for all $f \in \mathbb{H}_N$. Moreover, according to (5), a frame is a TF if and only if

$$F^*F \;=\; A \cdot I_N. \tag{13}$$

Thus, for a TF, all the eigenvalues of the frame operator $S = F^*F$ are equal to A. Then, using (10), the sum of the eigenvalues of $S = F^*F$ is as follows:

$$N \cdot A \;=\; \sum_{k=1}^{N} \lambda_k \;=\; \sum_{i=1}^{M} \|\varphi_i\|^2. \tag{14}$$

Normalized Tight Frame. If a frame is an NTF, that is, $A = B = 1$, then

$$\sum_{i=1}^{M} |\langle f, \varphi_i \rangle|^2 \;=\; \|f\|^2, \tag{15}$$

for all $f \in \mathbb{H}_N$. In operator notation, a frame is an NTF if and only if

$$S \;=\; F^*F \;=\; I_N. \tag{16}$$

For an NTF, all the eigenvalues of $S = F^*F$ are equal to 1.
Then, using (10), the sum of the eigenvalues of $S = F^*F$ is as follows:

$$N \;=\; \sum_{k=1}^{N} \lambda_k \;= \sum_{i=1}^{M} \|\varphi_i\|^2. \tag{17}$$

Uniform Tight Frame. From (11) and (14), we see that

$$N \cdot A \;=\; \sum_{k=1}^{N} \lambda_k \;=\; \sum_{i=1}^{M} \|\varphi_i\|^2 \;=\; M \cdot c^2. \tag{18}$$

Then, from (12) and (18),

$$\sum_{i=1}^{M} |\langle f, \varphi_i \rangle|^2 \;=\; \frac{M}{N} c^2 \|f\|^2, \tag{19}$$

for all $f \in \mathbb{H}_N$. The redundancy ratio is then

$$A \;=\; \frac{M}{N} \cdot c^2. \tag{20}$$

Since $S = F^*F = (M/N)I$, the following is obvious:

$$\sum_{i=1}^{M} |\varphi_{ik}|^2 \;=\; \frac{M}{N}. \tag{21}$$

Uniform Normalized Tight Frame. If a frame is a UNTF, that is, we also ask for $A = B = 1$, and if the frame vectors have norm c, then

$$N \;=\; \sum_{k=1}^{N} \lambda_k \;=\; \sum_{i=1}^{M} \|\varphi_i\|^2 \;=\; c^2 M.$$

Thus, a UNTF with norm $c = 1$ is an orthonormal basis.

2.2. Examples of Uniform Normalized Tight Frames

We start with a simple example of a frame; 3 vectors in 2 dimension. The particular frame we examine is termed *Mercedes-Benz (MB)* frame (for obvious reasons, just draw the vectors). The UNTF version of it is given by the analysis frame operator

$$F \;=\; \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 1 \\ -\sqrt{3}/2 & -1/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}. \tag{22}$$

This is obviously a UNTF since $(2/3)F^*F = I$. We could make this frame just a UTF with norm 1 (as in [25]) by having the analysis frame operator be simply F.

We now review two general classes of uniform tight frames which are commonly used. The (general) harmonic frames and the tight Gabor frames. More general classes of uniform tight frames are the full-diversity constellations that form a group given in [17].

2.3. Harmonic Frames

Harmonic tight frames (HTF) are obtained by keeping the first N coordinates of an $M \times M$ discrete Fourier transform basis as in Naimark's Theorem 2.1. They have been proven to be useful in applications [19].

An HTF is given by:

$$\varphi_k \;=\; (w_1^k, w_2^k \cdots, w_N^k), \tag{23}$$

for $k = 0, \ldots, M-1$, where w_i are distinct Mth roots of unity. This is a TF – $F^*F = MI$. A more general definition of the harmonic frame (general harmonic frame) is as follows:

Definition 2.1. Fix $M \geq N$, $|c| = 1$ and $\{b_i\}_{i=1}^N$ with $|b_i| = \frac{1}{\sqrt{M}}$. Let $\{c_i\}_{i=1}^N$ be distinct Mth roots of c, and for $0 \leq k \leq M-1$, let

$$\varphi_k \;=\; (c_1^k b_1, c_2^k b_2, \ldots, c_N^k b_N).$$

Then $\{\varphi_k\}_{k=0}^{M-1}$ is a uniform normalized tight frame for $\mathbb{H}_N$ called a *general harmonic frame (GHF).*

Note the difference between the HTFs and GHFs; the former ones are only tight, while the latter ones are uniform, normalized and tight. We will now examine the general harmonic frames in detail. Our goal is to determine when two such frames are unitarily equivalent. We start by showing that the harmonic frames (not GHFs) are unique up to a permutation of the orthonormal basis (that is, if and only if their columns are permutations of each other).

Proposition 2.1. *Let* $\{\varphi_k\}_{k=0}^{M-1}$ *and* $\{\psi_k\}_{k=0}^{M-1}$ *be harmonic frames on* $\mathbb{H}_N$*. Then* $\{\varphi_k\}_{k=0}^{M-1}$ *is equivalent to* $\{\psi_k\}_{k=0}^{M-1}$ *if and only if there is a permutation* σ *of* $\{1,2,\ldots,N\}$ *so that* $\varphi_{kj} = \psi_{k\sigma(j)}$*, for all* $0 \le k \le M-1$ *and all* $1 \le j \le N$*. Hence, two harmonic frames are equivalent if and only if they are unitarily equivalent.*

Proof (Proposition 2.1). Let $\{e_j\}_{j=1}^{N}$ be the natural orthonormal basis for $\mathbb{H}_N$ and

$$\varphi_k \;=\; (w_1^k, w_2^k \cdots, w_N^k) \;=\; \sum_{j=1}^{N} w_j^k e_j,$$

and

$$\psi_k \;=\; (v_1^k, v_2^k, \cdots, v_N^k) \;=\; \sum_{j=1}^{N} v_j^k e_j,$$

where w_j, v_j are sets of distinct Mth roots of unity. If

$$\{w_j | 1 \le j \le N\} \;\ne\; \{v_j | 1 \le j \le N\},$$

then, without loss of generality, we may assume that $v_1 \notin \{w_j | 1 \le j \le N\}$. Therefore,

$$\sum_{k=0}^{M-1} \overline{v}_1^k \varphi_k \;=\; \sum_{j=1}^{N} \left(\sum_{k=0}^{M-1} (\overline{v}_1 w_j)^k \right) e_j \;=\; \sum_{j=1}^{N} 0 \cdot e_j \;=\; 0.$$

On the other hand,

$$\sum_{k=0}^{M-1} \overline{v}_1^k \psi_k(1) \;=\; \sum_{k=0}^{M-1} |v_1^k|^2 \;=\; M.$$

It follows that $\{\varphi_k\}_{k=0}^{M-1}$ is not equivalent to $\{\psi_k\}_{k=0}^{M-1}$. The other direction is immediate. ■

We now show that every general harmonic frame is unitarily equivalent to a simple variation of a harmonic frame.

Proposition 2.2. *Every general harmonic frame is unitarily equivalent to a frame of the form* $\{c^k \psi_k\}_{k=0}^{M-1}$*, where* $|c| = 1$ *and* $\{\psi_k\}_{k=0}^{M-1}$ *is a harmonic frame.*

Proof (Proposition 2.2). Let $M \ge N$, $|c| = 1$ and $|b_k| = 1/\sqrt{M}$, for all $1 \le k \le N$. Let $\{c_j\}_{j=1}^{N}$ be distinct Mth roots of c and let the GHF be

$$\varphi_k \;=\; (c_1^k b_1, c_2^k b_2, \ldots, c_N^k b_N),$$

for all $0 \le k \le M-1$. If $c = e^{i\theta}$, let $d = e^{i\theta/M}$. Then there exist distinct Mth roots of unity $w_1, w_2 \ldots, w_N$ with $c_j = e^{i\theta/M} w_j = d w_j$. For all sets of complex numbers $\{a_k\}_{k=0}^{M-1}$ we have:

$$\| \sum_{k=0}^{M-1} a_k \varphi_k \|^2 \;=\; \sum_{j=1}^{N} | \sum_{k=0}^{M-1} a_k c_j^k b_j |^2 \;=\; \frac{1}{M} \sum_{j=1}^{N} | \sum_{k=0}^{M-1} a_k c_j^k |^2 \;=$$

$$\frac{1}{M}\sum_{j=1}^{N}|\sum_{k=0}^{M-1} a_k e^{ik\theta/M} w_j^k|^2 \;=\; \frac{1}{M}\|\sum_{k=0}^{M-1} a_k d^k \psi_k\|^2,$$

where $\psi_k = (w_1^k, w_2^k, \ldots, w_N^k)$. Thus, the GHF $\{\varphi_k\}_{k=0}^{M-1}$ is unitarily equivalent to $\frac{1}{\sqrt{M}}\{\psi_k\}_{k=0}^{M-1}$, with $\{\psi_k\}_{k=0}^{M-1}$ an HTF. ■

Finally, we show that the frames given in Proposition 2.2 are not unitarily equivalent to each other or to harmonic frames except in the trivial case.

PROPOSITION 2.3. *Let $\{\varphi_k\}_{k=0}^{M-1}$ and $\{\psi_k\}_{k=0}^{M-1}$ be harmonic frames and let $|c| = 1$. Then $\{c^k\varphi_k\}_{k=0}^{M-1}$ is equivalent to $\{\psi_k\}_{k=0}^{M-1}$ if and only if c is an Mth root of unity and there is a permutation σ of $\{1, 2, \ldots, N\}$ so that $\varphi_{kj} = \psi_{k\sigma(j)}$, for all $0 \le k \le M-1$ and all $1 \le j \le N$. In particular, a general harmonic frame is equivalent to a harmonic tight frame if and only if it equals a harmonic tight frame.*

Proof (Proposition 2.3). Let $\varphi_k = \sum_{j=1}^{N} w_j^k e_j$, for all $0 \le k \le M-1$. If $c = w_j^{-1}$, for some j, we are done. Otherwise, since $\sum_{k=0}^{M-1}\psi_k = 0$, if $\{c^k\varphi_k\}$ is equivalent to $\{\psi_k\}$ then also $\sum_{k=0}^{M-1} c^k\varphi_k = 0$. Hence, for all $1 \le j \le N$ we have

$$\sum_{k=0}^{M-1} c^k w_j^k \;=\; \sum_{k=0}^{M-1}(cw_j)^k \;=\; \frac{1-(cw_j)^M}{1-cw_j} \;=\; 0.$$

Hence, $(cw_j)^M = c^M w_j^M = c^M = 1$. The proposition now follows from Proposition 2.1. ■

We now have immediately,

COROLLARY 2.1. *Let $\{\varphi_k\}_{k=0}^{M-1}$ and $\{\psi_k\}_{k=0}^{M-1}$ be harmonic frames and let $|c| = |d| = 1$. The frames $\{c^i\varphi_k\}_{k=0}^{M-1}$ and $\{d^i\psi_k\}_{k=0}^{M-1}$ are equivalent if and only if $c = d$ and there is a permutation σ of $\{1, 2, \ldots, N\}$ so that $\varphi_{kj} = \psi_{k\sigma(j)}$, for all $0 \le k \le M-1$.*

The next proposition gives a classification of GHFs.

PROPOSITION 2.4. *A family $\{\varphi_i\}_{i=0}^{M-1}$ in $\mathbb{H}_N$ is a GHF if and only if for all $0 \le i, j \le M-1$ we have*

$$\langle\varphi_i, \varphi_j\rangle \;=\; \langle\varphi_{i+1}, \varphi_{j+1}\rangle,$$

where $\varphi_M = \varphi_0$.

Proof. If $\{\varphi_i\}_{i=0}^{M-1}$ in $\mathbb{H}_N$ is a generalized harmonic frame then there is a unitary operator U on $\mathbb{H}_N$ so that $U^i\varphi_0 = \varphi_i$, for all $0 \le i \le M-1$. Hence, for all $0 \le i, j \le M-1$ we have

$$\langle\varphi_{i+1}, \varphi_{j+1}\rangle \;=\; \langle U^{i+1}\varphi_0, U^{j+1}\varphi_0\rangle \;=\; \langle U^i\varphi_0, U^j\varphi_0\rangle \;=\; \langle\varphi_i, \varphi_j\rangle.$$

Conversely, given our equality, for any sequence of scalars $\{a_i\}_{i=0}^{M-1}$,

$$\begin{aligned} \|\sum_{i=0}^{M-1} a_i\varphi_{i+1}\|^2 &= \langle \sum_{i=0}^{M-1} a_i\varphi_{i+1}, \sum_{i=0}^{M-1} a_i\varphi_{i+1}\rangle \\ &= \sum_{i,j=0}^{M-1} a_i\overline{a_j}\langle\varphi_{i+1},\varphi_{j+1}\rangle = \sum_{i,j=0}^{M-1} a_i\overline{a_j}\langle\varphi_i,\varphi_j\rangle \\ &= \langle \sum_{i=0}^{M-1} a_i\varphi_i, \sum_{i=0}^{M-1} a_i\varphi_i\rangle = \|\sum_{i=0}^{M-1} a_i\varphi_i\|^2. \end{aligned}$$

It follows that $U\varphi_i = \varphi_{i+1}$ is a unitary operator and $U^i\varphi_0 = \varphi_i$, for all $0 \leq i \leq M-1$. ■

As we will see in the next section, UTFs with $M = N + 1$ are all unitarily equivalent. Thus, as a direct consequence of Theorem 3.3, any UTF with $M = N+1$ is unitarily equivalent to the HTF with $M = N + 1$. This is a very useful result since we have HTFs for any N and M; thus, for $M = N + 1$, we always have an expression for all UTFs. For example, this means that the MB frame we introduced earlier is equivalent to the HTF with $M = 3$ and $N = 2$.

Another interesting property of an HTF is that it is the only UNTF such that its elements are generated by a group of unitary operators with one generator, as we will see in Section 4. That is, $\Phi = \{\varphi_i\}_{i=1}^{M} = \{U^i\varphi_0\}_{i=1}^{M}$, where U is a unitary operator.

Moreover, HTFs have a very convenient property when it comes to erasures. We can erase any $e \leq (M - N)$ elements from the original frame and what is left is still a frame (Theorem 4.2 from [25]). We provide a complete classification of frames in terms of their robustness to erasures in Section 5.

2.4. Gabor or Weyl-Heisenberg Frames

For the other general class of frames, we introduce two special operators on $L_2(\mathbb{R})$. Fix $0 < a, b$ and for $f \in L_2(\mathbb{R})$ define *translation by a* as

$$T_a f(t) = f(t-a),$$

and *modulation by b* as

$$E_b f(t) = e^{2\pi imbt} f(t).$$

Now, fix $g \in L_2(\mathbb{R})$. If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L_2(\mathbb{R})$, we call it a *Gabor frame* (or a *Weyl-Heisenberg frame*). It is clear that in this class the frames are uniform. Also, since the frame operator S for a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ must commute with translation and modulation, each Gabor frame is equivalent to the (uniform) normalized tight Gabor frame $\{E_{mb}T_{na}S^{-1/2}g\}_{m,n\in\mathbb{Z}}$. For an introduction to Gabor frames we refer the reader to [7] and [18].

3. UNIFORM TIGHT FRAMES

3.1. What is Known about UTFs?

As we have mentioned in the introduction, UNTFs have become popular in applications. In particular, in [25], the authors attack the problem of robust transmission over the Internet by using frames. Being redundant sets of vectors, they provide robustness to losses, which are modeled as erasures of certain frame coefficients. It is further shown in [25] that, assuming a particular quantization model, a uniform frame with quantized coefficients, minimizes mean-squared error if and only if it is tight. Moreover, the same is true when we consider one erasure and look at both the average- and worst-case MSE.

As a result, our aim is to construct useful families of frames for such applications. To that end, it is important to classify uniform normalized tight frames in order to facilitate the search for useful families.

Our notion of usefulness includes any families which would be computationally efficient; for example, those that can be obtained from one or more generating vectors or those with a simple structure, for example, such that any of the frame operators could be expressed as a product of sparse matrices.

Finally, we are interested in the robustness of our frames to coefficient (frame vector) erasures. These and other issues will be explored in the rest of the paper.

3.1.1. Construction of UTFs

There is a general well known method for getting finite UNTFs. We state this result here in its standard form and will give a new proof of the result in Section 5. In Section 5 we will also extend this result to uniform frames.

Theorem 3.1. *There is a unique way to get UNTFs with M elements in $\mathbb{H}_N$. Take any orthonormal set $\{\phi_k\}_{k=1}^N$ in $\mathbb{H}_M$ which has the property*

$$\sum_{k=1}^{N} |\phi_{ki}|^2 \;=\; c, \quad \textit{for all } i.$$

Thinking of the ϕ_k as row vectors, switch to the M column vectors and divide by $\sqrt{c}$. This family is a uniform tight frame for $\mathbb{H}_M$ with M elements, and all UNTFs for $\mathbb{H}_N$ with M elements are obtained in this way.

Corollary 3.1. *If $\{\varphi_i\}_{i=1}^M$ is a uniform normalized tight frame for $\mathbb{H}_N$ then*

$$\|\varphi_i\|^2 \;=\; \frac{M}{N}.$$

There is a detailed discussion concerning the uniform normalized tight frames for $\mathbb{R}^2$ in [25]. In [17], there is a deep classification of groups of unitary operators which generate uniform normalized tight frames. The simplest case of this is the harmonic frames. In Section 4 we do not assume that we have a “group” of unitaries, but instead conclude that our family of unitaries must be a group.

The picture becomes much more complicated if the uniform normalized tight frame is generated by a group of unitaries with more than one generator (see [17]) or worse, if the uniform tight frame comes from a subset of the elements of such a group.

3.1.2. Classification of UNTFs

Although we would like to classify all uniform tight frames, especially those which can be obtained by reasonable algorithms, this is an impossible task in general. The following theorem shows that every finite family of norm-1 vectors in a Hilbert space can be extended to become a uniform tight frame.

THEOREM 3.2. *If* $\{\varphi_i\}_{i=1}^M$ *is a family of norm-*1 *vectors in a Hilbert space* $\mathbb{H}$, *then there is a uniform tight frame for* $\mathbb{H}$ *which contains the family* $\{\varphi_i\}_{i=1}^M$.

Proof (Theorem 3.2). For each $1 \le i \le M$ choose an orthonormal basis $\{e_{ij}\}_{j\in J}$ for H which contains the vector φ_i. Now the family $\{e_{ij}\}_{i=1,j\in J}^M$ is made up of norm-1 vectors and for any $f \in \mathbb{H}$ we have

$$\sum_{i=1}^M \sum_{j\in J} |\langle f, e_{ij}\rangle|^2 \;=\; \sum_{i=1}^M \|f\|^2 \;=\; M\|f\|^2.$$

■

In the above construction we get as a tight frame bound the number of elements M in the family $\{\varphi_i\}_{i=1}^M$. In general, this is the best possible. For example, just let $\varphi_i = \varphi_j$ be norm-1 vectors for all $1 \le i, j \le M$. Then

$$\sum_{j=1}^M |\langle \varphi_i, \varphi_j\rangle|^2 \;=\; M.$$

Hence, any tight frame containing the family $\{\varphi_i\}$ has the tight frame bound at least M.

There is another general class of uniform tight frames (see [25]). The result below tells us that all UNTFs with $M = N + 1$ are unitarily equivalent. It is a direct consequence of Theorem 2.6 from [25] where it is stated for UTFs with norm 1.

THEOREM 3.3 (Goyal, Kovačević and Kelner [25]). *A family* $\{\varphi_i\}_{i=1}^{N+1}$ *is a uniform normalized tight frame for* $\mathbb{H}_N$ *if and only if* $\{\varphi_i\}_{i=1}^{N+1}$ *is unitarily equivalent to the frame* $\{Pe_i\}_{i=1}^{N+1}$ *where* $\{e_i\}_{i=1}^{N+1}$ *is an orthonormal basis for* $\mathbb{H}_{N+1}$ *and* P *is the orthogonal projection of* $\mathbb{H}_{N+1}$ *onto the orthogonal complement of the one-dimensional subspace of* $\mathbb{H}_{N+1}$ *spanned by* $\sum_{i=1}^{N+1} e_i$.

Since there are HTFs with $N + 1$-elements in $\mathbb{H}_N$, it follows that every UNTF for $\mathbb{H}_N$ with $N + 1$-elements is unitarily equivalent to a HTF.

3.2. Normalized and Uniform Tight Frames and Subspaces of the Hilbert Space

Here, we begin to classify NTFs and UNTFs by providing a correspondence with subspaces of the original Hilbert space. We give two results, one for NTFs and another for UNTFs with the correspondence being one-to-one. The material in this section grew out of conversations between the first author and V. Paulsen.

As we saw in Theorem 2.1, there is a unique way to get normalized tight frames in $\mathbb{H}_N$ with M elements. Namely, we take an orthonormal basis $\{e_i\}_{i=1}^M$ for $\mathbb{H}_M$ and take the orthogonal projection $P_{\mathbb{H}_N}$ of $\mathbb{H}_M$ onto $\mathbb{H}_N$. Then $\{P_{\mathbb{H}_N} e_i\}_{i=1}^M$ is a normalized tight frame for $\mathbb{H}_N$ with M elements. In particular, there is a natural one-to-one correspondence between the normalized tight frames for $\mathbb{H}_N$ with M elements and the orthonormal bases for $\mathbb{H}_M$. Then, the uniform normalized tight frames for $\mathbb{H}_N$ are the ones for which $\|Pe_i\| = \|Pe_j\|$, for all $1 \leq i, j \leq M$. We use this to exhibit a natural correspondence between these families and certain subspaces of $\mathbb{H}_M$. Here we treat two frames as the same if they are unitarily equivalent.

Theorem 3.4. *Let P be a rank-N orthogonal projection on $\mathbb{H}_M$ and let $\{e_i\}_{i=1}^M$ be an orthonormal basis for $\mathbb{H}_M$. There is a natural one-to-one correspondence between the normalized tight frames for $P\mathbb{H}_M$ with M elements and the family of all N-dimensional subspaces of $\mathbb{H}_M$.*

Proof (Theorem 3.4). If $\{\varphi_i\}_{i=1}^M$ is any normalized tight frame for $P\mathbb{H}_M$, then by Naimark's Theorem, there is an orthonormal basis $\{e_i^{'}\}_{i=1}^M$ for $\mathbb{H}_M$ so that $Pe_i^{'} = \varphi_i$. Define a unitary operator U on $\mathbb{H}_M$ by $Ue_i = e_i^{'}$. Now, $\varphi_i = PUe_i$ which is unitarily equivalent to $\varphi_i^{'} = U^*PUe_i$. So we will associate our normalized tight frame $\{\varphi_i\}$ with the subspace $U^*PU\mathbb{H}_M$. Now we need to check that this correspondence is one-to-one. Let $\{\psi_i\}$ be another normalized tight frame for $P\mathbb{H}_M$ which is associated with the same subspace of $\mathbb{H}_M$, namely $U^*PU\mathbb{H}_M$. Then there is an orthonormal basis $\{e_i^{''}\}$ and a unitary operator $Ve_i = e_i^{''}$ with $V^*PV = U^*PU$ and $PVe_i = e_i^{''}$. Hence, $V^*PVe_i = V^*\psi_i = U^*PUe_i = U^*\varphi_i$. This implies that the normalized tight frames $\{\varphi_i\}$ and $\{\psi_i\}$ are unitarily equivalent (and hence the same). Finally, we need to see that this correspondence covers all subspaces. If W is any subspace of $\mathbb{H}_M$ of dimension N, we can define a unitary operator U on $\mathbb{H}_M$ so that $UW = P\mathbb{H}_M$. Then $U^*PU = P_W$ while $\{PUe_i\}_{i=1}^M$ is a normalized tight frame for $P\mathbb{H}_M$ (which under our association corresponds to W). ■

One of the consequences of the above result is that if we have *one* normalized tight frame for $\mathbb{H}_N$, then all the others can be obtained from it by this process. We now turn our attention to uniform frames:

Theorem 3.5. *Fix an orthonormal basis $\{e_i\}_{i=1}^M$ for $\mathbb{H}_M$ so that if P is the orthogonal projection of $\mathbb{H}_M$ onto $\mathbb{H}_N$ then $\{Pe_i\}_{i=1}^M$ is a uniform normalized tight frame for $\mathbb{H}_N$. Then there is a natural one-to-one correspondence between the uniform normalized tight frames for $\mathbb{H}_N$ with M elements and the subspaces W of $\mathbb{H}_M$ for which $\|P_We_i\|^2 = M/N$, for all $1 \leq i \leq M$.*

Proof (Theorem 3.5). We already have our classification of the normalized tight frames in terms of all subspaces. Now we need to see which of these subspaces correspond to the uniform normalized tight frames. Suppose U is any unitary operator on H so that $\{PUe_i\}$ is a uniform normalized tight frame for $P\mathbb{H}_M$. Then this frame is associated to the subspace $W = U^*P\mathbb{H}_M$ and $P_W = U^*PU$. Hence, $PUe_i = UP_We_i$, for all i. But, $\{PUe_i\}$ is a uniform normalized tight frame for

$P\mathbb{H}_M$ and so $\|PUe_i\| = N/M$, for all i. Hence,

$$\|PUe_i\| \;=\; \frac{N}{M} \;=\; \|UP_We_i\| \;=\; \|P_We_i\|.$$

So our association between normalized tight frames and subspaces given in Theorem 3.4 identifies the subspaces W of $\mathbb{H}_M$ for which $\|P_We_i\| = N/M$, for all i. ■

3.3. Uniform Dual Frames

Another method for classifying UTFs uses a back door: get UTFs as alternate dual frames for a given frame. To do this, we need the definition of alternate dual frames:

Definition 3.1. Let $\{\varphi_i\}_{i\in I}$ be a frame for a Hilbert space $\mathbb{H}$. A family $\{\psi_i\}_{i\in I}$ is called an *alternate dual frame* for $\{\varphi_i\}_{i\in I}$ if

$$f \;=\; \sum_{i\in I}\langle f,\psi_i\rangle\varphi_i, \quad \text{for all } f\in\mathbb{H}.$$

In Definition 3.1, if we let F_1 (respectively, F_2) be the analysis frame operators for $\{\varphi_i\}$ (respectively $\{\psi_i\}$), then we see that $F_2^*F_1 = I$.

There are many alternate dual frames for a given frame. In fact, a frame has a unique alternate dual frame (the canonical dual frame) if and only if it is a Riesz basis [16]. Moreover, no two distinct alternate dual frames for a given frame are equivalent [16]. For a normalized tight frame, its canonical dual frame is the frame itself since $\tilde{F} = FS^{-1} = F\cdot I = F$. Moreover,

Proposition 3.1. *If $\{\varphi_i\}_{i\in I}$ is a normalized tight frame for $\mathbb{H}_N$, then the only normalized tight alternate dual frame for $\{\varphi_i\}_{i\in I}$ is $\{\varphi_i\}_{i\in I}$ itself.*

Proof (Proposition 3.1). Let F_1 be the analysis frame operator for $\{\varphi_i\}_{i\in I}$. Let $\{\psi_i\}_{i\in I}$ be any normalized tight alternate dual frame for $\{\varphi_i\}_{i\in I}$ with analysis frame operator F_2. It follows that F_1, F_2 are isometries. Now,

$$F_2^*(F_1-F_2) \;=\; F_2^*F_1 - F_2^*F_2 \;=\; I-I \;=\; 0.$$

Hence, for every $f,g\in\mathbb{H}$ we have

$$\langle F_2^*g,(F_1-F_2)f\rangle \;=\; \langle g,F_2^*(F_1-F_2)f\rangle \;=\; \langle g,0\rangle \;=\; 0.$$

Since $\{\varphi_i\}_{i\in I}$ is an NTF, for every $f\in\mathbb{H}$ we have

$$\|f\|^2 \;=\; \sum_{i=1}^{M}|\langle f,\varphi_i\rangle|^2 \;=\; \|F_1f\|^2. \tag{24}$$

We can further expand this as

$$\begin{aligned}\|F_1f\|^2 \;&=\; \|(F_1\pm F_2)f\|^2\\ &=\; \|F_2f\|^2+\|(F_1-F_2)f\|^2+\underbrace{\langle F_2^*(F_1-F_2)f,f\rangle}_{0}+\underbrace{\langle F_2^*(F_1-F_2)f,f\rangle^*}_{0}.\end{aligned}$$

However, since F_2 corresponds to an NTF as well,

$$\|F_2 f\|^2 + \|(F_1 - F_2)f\|^2 \; = \; \|f\|^2 + \|(F_1 - F_2)f\|^2. \tag{25}$$

Equating (24) and (25), we get

$$\|(F_1 - F_2)f\|^2 \; = \; 0.$$

Hence, $F_1 = F_2$ and so $\varphi_i = \psi_i$, for all $i \in I$. ■

If we ask for the dual frame just to be tight, but not normalized, then:

PROPOSITION 3.2. *If $\{\varphi_i\}_{i=1}^M$ is a normalized tight frame for $\mathbb{H}_N$ and $M < 2N$, then the only tight dual frame for $\{\varphi_i\}_{i=1}^M$ is $\{\varphi_i\}_{i=1}^M$ itself. If $M \geq 2N$ then there are infinitely many (nonequivalent) tight alternate dual frames for $\{\varphi_i\}_{i=1}^M$.*

Proof (Proposition 3.2). With the notation of the proof of Proposition 3.1, the only change is that, since the dual frame is only tight and not normalized tight, there is a frame bound $A > 0$ so that

$$\|F_2\|^2 \; = \; A\|f\|^2, \;\; \text{for all} \;\; f \in \mathbb{H}.$$

Hence, as in the proof of Proposition 3.1 we have

$$\|f\|^2 \; = \; \|F_1 f\|^2 \; = \; \|F_2 f\|^2 + \|(F_1 - F_2)f\|^2 \; = \; A\|f\|^2 + \|(F_1 - F_2\|^2.$$

Hence, $\|(F_1 - F_2)f\|^2 = (1 - A)\|f\|^2$ for all $f \in \mathbb{H}$. Hence, either $F_1 - F_2 = 0$ and $A = 1$ which gives us the NTF as in the previous proposition, (and so $\varphi_i = \psi_i$ for all $1 \leq i \leq M$) or $(1/\sqrt{1-A})(F_2 - F_1)$ is an isometry on $\mathbb{H}_N$. In the latter case it follows that dim $(F_1 \mathbb{H}_N)^{\perp} \geq N$, and so $M \geq 2N$. Thus, if $M < 2N$, the only tight dual frame is the frame itself.

On the other hand, if $M \geq 2N$ given any $F : \mathbb{H}_N \to \ell_2^M$ which is a constant times an isometry, and with $F_1 \mathbb{H}_N \perp F\mathbb{H}_N$, $F_1 + F$ defines a tight frame which is an alternate dual frame for $\{\varphi_i\}$. ■

However, since it is really the uniform case we are interested in,

PROPOSITION 3.3. *If $\{\varphi_i\}_{i=1}^M$ is a uniform normalized tight frame for $\mathbb{H}_N$ and $M = 2N$, then there are infinitely many uniform tight alternate dual frames for $\{\varphi_i\}_{i=1}^{2N}$.*

Proof (Proposition 3.3). Again we use the notation of the proof of Proposition 3.1. Choose any $F_2 : \mathbb{H}_N \to \ell_2^M$ which is a constant multiple of an isometry and with $F_2 \mathbb{H}_N = (F_1 \mathbb{H}_N)^{\perp}$. Then $F_1 + F_2$ defines a tight alternate dual frame for $\{\varphi_i\}_{i=1}^M$, say $\psi_i = (F_1^* + F_2^*)e_i$, for all $1 \leq i \leq M$. We just need to check that this frame is uniform. Let $P : \ell_2^M \to F_1 \mathbb{H}_N$ be the orthogonal projection, so that $Pe_i = F_1 \varphi_i$, for all $1 \leq i \leq M$. It follows that $\|Pe_i\|^2 = N/M$ and $\|(I - P)e_i\|^2 = 1 - N/M$, for all $1 \leq i \leq M$. Now,

$$\psi_i = F_1^* e_i + F_2^* e_i = \varphi_i + F_2^*(I - P)e_i.$$

Also, there is an $a > 0$ so that for every $1 \leq i \leq M$ we have

$$\|\psi_i\|^2 = \|\psi_i\|^2 + \|F_2^*(I-P)e_i\|^2 = \frac{N}{M} + a\|(I-P)e_i\|^2$$

$$= \frac{N}{M} + a\left(1 - \frac{N}{M}\right) = a + (1-a)\frac{N}{M}.$$

Hence, $\{\psi_i\}_{i=1}^M$ is a uniform tight frame. ■

It can be shown that the results of this section actually classify when NTFs or UNTFs have tight (respectively uniform tight) alternate dual frames, that is, all tight alternate dual frames for $\{\varphi_i\}$ are obtained by the methods of this section. We do not know in general which frames (not tight) have tight (respectively uniform tight) alternate dual frames.

3.4. Frames Equivalent to UTFs

Another approach to the classification of UTFs looks into families of frames obtained from a UTF by equivalence relations. For example, we know that every frame is equivalent to a normalized tight frame. That is, given any frame $\{\varphi_i\}_{i\in I}$ with the frame operator S, the frame $\{S^{-1/2}\varphi_i\}_{i\in I}$ is a normalized tight frame which is equivalent to $\{\varphi_i\}_{i\in I}$. Therefore, it is natural to try to find ways to turn frames into uniform normalized tight frames. As it turns out, this is not possible in most cases:

THEOREM 3.6. *If a frame $\{\varphi_i\}_{i\in I}$ with the frame operator S is equivalent to a UTF, then $\{S^{-1/2}\varphi_i\}_{i\in I}$ is a UNTF. In particular, a tight frame which is not uniform cannot be equivalent to any UNTF.*

Proof (Theorem 3.6). It is known that $\{S^{-1/2}\varphi_k\}$ is a normalized tight frame which is equivalent to $\{\varphi_k\}$. So if $\{\varphi_k\}$ is equivalent to a uniform tight frame, say $\{\psi_k\}$, and $\|\psi_k\| = c$, for all k, then $\{S^{-1/2}\varphi_k\}$ is equivalent to $\{\psi_k\}$. That is, there is an invertible operator T on $\mathbb{H}$ so that $TS^{-1/2}\varphi_k = \psi_k$.

Now we show that $T/\sqrt{A}$ is a unitary operator. Let A be the tight frame constant of $\{TS^{-1/2}\varphi_k\}$. Then for all $f \in \mathbb{H}$, the tightness of $\{TS^{-1/2}\varphi_k\}$ implies

$$A\|f\|^2 \;=\; \sum_k |\langle f, TS^{-1/2}\varphi_k\rangle|^2 \;=\; \sum_k |\langle T^*f, S^{-1/2}\varphi_k\rangle|^2. \tag{26}$$

However, since $\{S^{-1/2}\varphi_k\}$ is an NTF, then this leads us to

$$\sum_k |\langle T^*f, S^{-1/2}\varphi_k\rangle|^2 \;=\; \|T^*f\|^2. \tag{27}$$

Equating (26) and (27), we get that

$$A\|f\|^2 \;=\; \|T^*f\|^2,$$

that is, $T/\sqrt{A}$ is unitary. Hence, the frame

$$S^{-1/2}\varphi_k \;=\; T^{-1}\psi_k,$$

is a UNTF since

$$\|T^{-1}\psi_k\| = \frac{1}{\sqrt{A}}\|\psi_k\| = \frac{1}{\sqrt{A}}c, \qquad \text{for all } k = 1, \ldots, M.$$

■

4. UNIFORM TIGHT FRAMES WITH GROUP STRUCTURE

In this section, we examine frames which are generated by a group of unitary operators applied to a fixed vector in the Hilbert space. These classes are especially important in applications. The traditional use of one of these classes is in the Gabor frames used in signal/image processing. These are groups with two generators. There is also a class of wavelet frames which has two generators [9]. Recently, several new applications have arisen. In [25], the authors proposed using the redundancy of frames to mitigate the losses in packet-based communication systems such as the Internet. In [17], frames are used for multiple antenna coding/decoding (see the discussion Section 4.3 for further explanation). In [22, 12], connections between quantum mechanics and tight frames are established. Although each of these applications requires a different class of UNTFs, they all have one important common constraint. Namely, calculations for the frame must be easily implementable on the computer. Since UNTFs generated by a group of unitary operators satisfy this constraint, this class is one of the most important to understand.

In this section we will show that the UNTFs generated by the family $\{U^k\varphi_0\}_{k=0}^{M-1}$ (where U is unitary and $\varphi_0 \in \mathbb{H}$) are precisely the GHFs. We then extend our discussion to UNTFs generated by $\{U^kV^j\varphi_0\}$ and higher numbers of generators. In the discussion Section 4.3 we will relate the results of this section to the literature.

4.1. UTFs with a Single Generator

Here, we give a complete classification of UNTFs of the form $\{U^k\varphi_0\}_{k=0}^{M-1}$ where U is a unitary operator on $\mathbb{H}_N$. We will, in fact, find that in this case $\{U^k\}_{k=0}^{M-1}$ must be a group. Moreover, we will see the the GHFs will be that special class.

Since the proofs of the following results are long, the reader can find them in the Appendix. Our first result classifies GHFs as those frames generated by powers of a special class of unitary operators applied to a fixed vector in $\mathbb{H}$.

PROPOSITION 4.1. *$\{\varphi_k\}_{k=0}^{M-1}$ is a general harmonic frame for $\mathbb{H}_N$ if and only if there is a vector $\varphi_0 \in \mathbb{H}_N$ with $\|\varphi_0\|^2 = \frac{N}{M}$, an orthonormal basis $\{e_i\}_{i=1}^N$ for $\mathbb{H}_N$ and a unitary operator U on $\mathbb{H}_N$ with $Ue_i = c_ie_i$, with $\{c_i\}_{i=1}^N$ distinct Mth roots of some $|c| = 1$ so that $\varphi_k = U^k\varphi_0$, for all $0 \leq k \leq M-1$.*

Proof. See Appendix A.1. ■

We now show that the only UNTFs with group structure and a single generator are GHFs. This result is an excellent result and a disappointment at the same time. On the one hand, the fact that GHFs are the only ones with such a group structure proves once more why they are so universally used. It also spares us the trouble of looking for other such UNTFs. On the other hand, we cannot find any other useful families via this route as we were hoping for. Another important property of

GHFs (Theorem 4.2 from [25]) is that GHFs are robust to the maximum number of possible erasures, that is, a GHF with M elements in $\mathbb{H}_N$ is robust to e erasures for any $e \leq (M-N)$.

Our next theorem completes the classification of GHFs as being those UNTFs of the form $\{U^k\varphi_0\}_{k=0}^{M-1}$ where U is a unitary operator on $\mathbb{H}$. As a consequence, we discover that, in this case, $\{U^k\}_{k=0}^{M-1}$ is a group.

Theorem 4.1. *Let U be a unitary operator on $\mathbb{H}_N$, $\varphi_0 \in \mathbb{H}_N$ and assume $\{U^k\varphi_0\}_{k=0}^{M-1}$ is a UNTF for $\mathbb{H}_N$. Then $U^M = cI$ for some $|c| = 1$ and $\{U^k\varphi_0\}_{k=0}^{M-1}$ is a general harmonic frame. That is, the GHFs are the only UNTFs generated by a group of unitary operators with a single generator.*

Proof. See Appendix A.2. ■

4.2. UTFs with Two or More Generators

Having completely characterized UTFs which come from a group of unitary operators with one generator, we now turn our attention to those with more than one. The fundamental examples of frames with two generators are the Gabor frames $\{E_{mb}T_{ma}g\}_{m,n\in\mathbb{Z}}$ (or, in our case, the finite discrete Gabor frames). Each of these frames is equivalent to the UNTF Gabor frame $\{E_{mb}T_{na}S^{-1}g\}_{m,n\in\mathbb{Z}}$ where S is the frame operator for $\{E_{mb}T_{ma}g\}_{m,n\in\mathbb{Z}}$.

We will classify the UNTFs of the form $\{U^iV^j\varphi_0\}_{i=0,j=0}^{L-1,M-1}$ where $\{V^j\varphi_0\}_{j=0}^{M-1}$ is a UNTF for its span. In the process of proving these results, we introduce a general method for using special families of finite-rank projections to produce frames for a space. If we associate a frame $\{\varphi_i\}_{i=1}^{M}$ with the family of rank-1 orthogonal projections P_i taking $\mathbb{H}$ onto span φ_i, then we can view the results of this section as a generalization of the very notation of a frame to the case where the P_i have arbitrary rank.

Our first result states that given a GHF generated by a unitary operator V, a unitary operator U and an integer M, we can always find a corresponding UNTF generated by U, V.

Proposition 4.2. *Let V be a unitary operator on $\mathbb{H}_N$, let $m \in \mathbb{N}$ and $\psi_0 \in H$ so that $\{V^i\psi_0\}_{i=0}^{m-1}$ is a UNTF for $\mathbb{H}_N$. Then for every unitary operator U on $\mathbb{H}_N$ and every $M \in \mathbb{N}$, there is a vector $\varphi_0 \in \mathbb{H}_N$ such that $\{U^kV^j\varphi_0\}_{k=0,j=0}^{M-1,m-1}$ is a uniform, normalized tight frame for $\mathbb{H}_N$.*

Proof (Proposition 4.2). Let $\varphi_0 = \psi_0/\sqrt{M}$. Now, for every $f \in \mathbb{H}_N$ we have

$$\sum_{k=0}^{M-1}\sum_{j=0}^{m-1} |\langle f, U^kV^j\varphi_0\rangle|^2 = \sum_{k=0}^{M-1}\sum_{j=0}^{m-1} |\langle U^{-k}f, V^j\varphi_0\rangle|^2 = \frac{1}{M}\sum_{k=0}^{M-1}\sum_{j=0}^{m-1} |\langle U^{-k}f, V^j\psi_0\rangle|^2.$$

Since $\{V^j\psi_0\}_{j=0}^{m-1}$ is a UNTF, the previous sum equals

$$\frac{1}{M}\sum_{k=0}^{M-1} \|U^{-k}f\|^2 = \frac{1}{M}\sum_{k=0}^{M-1} \|f\|^2 = \|f\|^2.$$

■

It is easily seen that an invertible operator on $\mathbb{H}$ maps a frame for $\mathbb{H}$ to another frame for $\mathbb{H}$. We next observe that there is a natural relationship between the frame operators of these equivalent frames.

PROPOSITION 4.3. *If $\{\varphi_k\}$ is a frame for $\mathbb{H}$ with the frame operator S, and T is an invertible operator on $\mathbb{H}$, then TST^* is the frame operator for the frame $\{T\varphi_k\}$. In particular, if $\{\varphi_k\}$ is a UNTF then every frame unitarily equivalent to $\{\varphi_k\}$ is also a UNTF.*

Proof (Proposition 4.3). Let L be the frame operator for $\{T\varphi_k\}$. For any $f \in H$ we have:

$$Lf \;=\; \sum_k \langle f, T\varphi_k\rangle T\varphi_k \;=\; T\left(\sum_k \langle T^* f, \varphi_k\rangle \varphi_k\right) \;=\; T[S(T^*)f].$$

■

As a consequence of Proposition 4.3, we can apply a unitary operator to any UNTF and get a new UNTF. If this unitary operator is part of the generating set for our UNTF, then our new UNTF has a special form as given in the next proposition.

PROPOSITION 4.4. *If U is unitary and $\{U^k V^j \varphi_0\}$ is a UNTF for $\mathbb{H}_N$, then for every $M \in \mathbb{Z}$ and for every $f \in \mathbb{H}_N$ we have,*

$$f \;=\; \sum_{k,j} \langle f, U^{k+M} V^j \varphi_0\rangle U^{k+M} V^j \varphi_0,$$

that is, $\{U^{k+M} V^j \varphi_0\}_{k,j}$ is a UNTF as well.

Proof (Proposition 4.4). Since U^M is unitary and $\{U^k V^j \varphi_0\}_{k,j}$ is a UNTF for $\mathbb{H}_N$, it follows that $\{U^M U^k V^j \varphi_0\}_{k,j}$ is also a UNTF for $\mathbb{H}_N$. ■

To explain where we are going from here, let us revisit the notion of a UNTF and look at it from a slightly different point of view. Suppose $\{\varphi_i\}_{i\in I}$ is a UNTF for a (finite or infinite-dimensional) Hilbert space $\mathbb{H}$ with $\|\varphi_i\| = c$, for all $i \in I$. For each $i \in I$, let P_i be the orthogonal projection of $\mathbb{H}$ onto the one-dimensional subspace of $\mathbb{H}$ spanned by φ_i. Then, for all $f \in \mathbb{H}$

$$\begin{aligned}\sum_{i\in I} \|P_i f\|^2 \;&=\; \sum_{i\in I} \|\langle f, \varphi_i\rangle \varphi_i\|^2 \\ &=\; \sum_{i\in I} |\langle f, \varphi_i\rangle|^2 \|\varphi_i\|^2 \;=\; c^2 \sum_{i\in I} |\langle f, \varphi_i\rangle|^2 = c^2\|f\|^2.\end{aligned}$$

Conversely, let $\{P_i\}_{i\in I}$ be rank-1 orthogonal projections on $\mathbb{H}$ satisfying

$$\sum_{i\in I} \|P_i f\|^2 = a\|f\|^2, \quad \text{for all} \quad f \in \mathbb{H}.$$

Then $\{\varphi_i\}_{i\in I}$ is a UNTF for $\mathbb{H}$ where $\varphi_i \in P_i\mathbb{H}$ and $\|\varphi_i\|^2 = 1/a$. That is, if $f \in \mathbb{H}$ then

$$\sum_{i\in I} |\langle f, \varphi_i\rangle|^2 = \sum_{i\in I} |\langle P_i f, \varphi_i\rangle|^2 \sum_{i\in I} \|P_i f\|^2 \|\varphi_i\|^2$$

$$= \frac{1}{a} \sum_{i\in I} \|P_i f\|^2 = \frac{1}{a} a \|f\|^2 = \|f\|^2.$$

It follows that the condition on $\{\varphi_i\}$ which guarantees that we have a UNTF is really a condition on the rank-1 orthogonal projections of $\mathbb{H}$ onto the span of φ_i. It is this which we now generalize to find our next general class of UNTFs. Since this material is also new for general frames, we start here.

PROPOSITION 4.5. *Let $\{P_i\}_{i\in I}$ be a family of projections (of any rank) on a (finite or infinite-dimensional) Hilbert space $\mathbb{H}$ and assume there are constants $0 < A, B$ satisfying,*

$$A\|f\|^2 \ \le\ \sum_{i\in I} \|P_i f\|^2 \ \le\ B\|f\|^2, \quad \textit{for all } f \in \mathbb{H}.$$

Let $\{\varphi_{ij}\}_{j=1}^{M_i}$ be a frame for $P_i\mathbb{H}$ with frame bounds A_i, B_i, for all $i \in I$. Then $\{\varphi_{ij}\}_{j=1,i\in I}^{M_i}$ has frame bounds,

$$A \ \textit{inf}_{i\in I} A_i, \quad B \ \textit{sup}_{i\in I} B_i.$$

Proof. For any $f \in H$ we have

$$\begin{aligned} \sum_{j=1,i\in I}^{M_i} |\langle f, \varphi_{ij}\rangle|^2 \ &=\ \sum_{i\in I} \sum_{j=1}^{M_i} |\langle f, \varphi_{ij}\rangle|^2 \\ &=\ \sum_{i\in I} \sum_{i=1}^{M_i} |\langle P_i f, \varphi_{ij}\rangle|^2 \ \le\ \sum_{i\in I} B_i \|P_i f\|^2 \\ &\le\ \sup_{i\in I} B_i \sum_{i\in I} \|P_i f\|^2 \ \le\ B\left(\sup_{i\in I} B_i\right) \|f\|^2. \end{aligned}$$

The lower frame bound is computed similarly. ∎

The cases of interest to us are the next two corollaries: the first for NTFs and the second for UNTFs.

COROLLARY 4.1. *Let $\{P_i\}_{i\in I}$ be a family of projections on a Hilbert space $\mathbb{H}$ satisfying:*

$$\sum_{i\in I} \|P_i f\|^2 \ =\ a^2 \|f\|^2, \quad \textit{for all } f \in \mathbb{H}.$$

Then $a \ge 1$, and if $\{\varphi_{ij}\}_{j=1}^{M_i}$ is an NTF for $P_i\mathbb{H}$, for all $i \in I$, then $\{\frac{1}{a}\varphi_{ij}\}_{j=1,i\in I}^{M_i}$ is an NTF for $\mathbb{H}$.

In the next corollary we assume that the projections P_i all have the same rank and the frames for $P_i\mathbb{H}$ all have the same number of elements. This assumption is necessary to guarantee that our frame is uniform. That is, as we observed in Section 2.1, if dim $P_i\mathbb{H} = N$ and $\{\varphi_{i,j}\}_{j\in J}$ is a UNTF for $P_i\mathbb{H}$ then

$$\|\varphi_{ij}\|^2 \;=\; \frac{N}{|J|}.$$

Corollary 4.2. *Let $\{P_i\}_{i\in I}$ be a sequence of projections (all with the same rank) on a Hilbert space $\mathbb{H}$ satisfying:*

$$\sum_{i\in I}\|P_i f\|^2 \;=\; a^2\|f\|^2, \quad \textit{for all } f \in \mathbb{H}.$$

Then $a \geq 1$, and if $\{\varphi_{ij}\}_{j\in J}$ is a UNTF for $P_i\mathbb{H}$, for all $i \in I$, then $\{\frac{1}{a}\varphi_{ij}\}_{j\in J, i\in I}$ is a UNTF for $\mathbb{H}$.

We are now ready to consider UNTFs with two generators.

Theorem 4.2. *Let U be a unitary operator on $\mathbb{H}$, and let $\{V^j\phi_0\}_{j=0}^{M-1}$ be a UTF (with tight frame bound $1/a$) for its closed linear span in $\mathbb{H}$. Let P_0 be the orthogonal projection of $\mathbb{H}$ onto this span. Assume that $\{U^iV^j\varphi_0\}_{i=0,j=0}^{L-1,M-1}$ is a UNTF for $\mathbb{H}$. If P_i is the orthogonal projection of $\mathbb{H}$ onto the span of $U^iP_0\mathbb{H}$, for all $1 \leq i \leq L-1$, then*

$$\sum_{i=0}^{L-1}\|P_i f\|^2 \;=\; a\|f\|^2, \quad \textit{for all } f \in \mathbb{H}.$$

Proof. Note that for all $1 \leq i \leq K-1$ we have $P_i = U^iP_0U^{-i}$. Now, for all $f \in \mathbb{H}$ we compute,

$$\begin{aligned}
\sum_{i=0}^{L-1}\|P_i f\|^2 &= \sum_{i=0}^{L-1}\|U^iP_0U^{-i}f\|^2 \\
&= \sum_{i=0}^{L-1}\|P_0U^{-i}f\|^2 \;=\; \sum_{i=0}^{L-1} a \sum_{j=0}^{M-1} |\langle U^{-i}f, V^j\varphi_0\rangle|^2 \\
&= a\sum_{i=0}^{K-1}\sum_{j=0}^{M-1}|\langle f, U^iV^j\varphi_0\rangle|^2 \;=\; a\|f\|^2.
\end{aligned}$$

■

The next theorem is the converse of the above.

Theorem 4.3. *Let $\{P_i\}_{i=0}^{L-1}$ be a family of orthogonal projections on $\mathbb{H}$ satisfying*

$$\sum_{i=0}^{L-1}\|P_i f\|^2 \;=\; a^2\|f\|^2, \quad \textit{for all } f \in \mathbb{H}.$$

Assume that U is a unitary operator on $\mathbb{H}$ so that $U^iP_0\mathbb{H} = P_i\mathbb{H}$. Let $\varphi_0 \in P_0\mathbb{H}$ and let V be a unitary operator on $P_0\mathbb{H}$ such that $\{V^j\varphi_0\}_{j=0}^{M-1}$ is a UNTF for $P_0\mathbb{H}$. Then $\{\frac{1}{a}U^iV^j\varphi_0\}_{i=0,j=0}^{L-1,M-1}$ is a UNTF for $\mathbb{H}$.

Proof. For all $f \in \mathbb{H}$ we have,

$$\sum_{i=0}^{L-1}\sum_{j=0}^{M-1} |\langle f, \frac{1}{a} U^i V^j \varphi_0\rangle|^2 \;=\; \frac{1}{a^2}\sum_{i=0}^{L-1}\sum_{j=0}^{M-1} |\langle U^{-i} f, V^j \varphi_0\rangle|^2 \;=$$

$$\frac{1}{a^2}\sum_{i=0}^{L-1} \|P_0 U^{-i} f\|^2 \;=\; \frac{1}{a^2}\sum_{i=0}^{L-1} \|U^i P_0 U^{-i} f\|^2 \;=\; \frac{1}{a^2}\sum_{i=0}^{L-1} \|P_i f\|^2 \;=\; \|f\|^2.$$

■

4.3. Discussion

Some remarks are in order:

1. The assumptions in Theorems 4.2 and 4.3 that $\{V^j\varphi_0\}_{j=0}^{M-1}$ is a UNTF for its span is sufficient for the conclusion of the theorems but not necessary. For example, it is known to Gabor frame specialists that for $ab < 1$, there are Gabor frames $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for which neither of $\{E_{mb}T_{na}g\}_{m\in\mathbb{Z}}$ nor $\{E_{mb}T_{na}g\}_{n\in\mathbb{Z}}$ is a frame for its closed linear span. The equivalent UNTF Gabor frame $\{E_{mb}T_{na}S^{-1/2}g\}_{m,n\in\mathbb{Z}}$ fails the hypotheses of Theorems 4.2 and 4.3 but satisfies the conclusion.

2. The results of Section 4.2 generalize to three or more generators. For example, for three generators, Theorem 4.3 becomes:

THEOREM 4.4. *Let U, V be unitary operators with '$\{U^iV^j\varphi_0\}_{i=0,j=0}^{L-1,M-1}$ a UNTF for its span. Let P_0 be the projection of the Hilbert space H onto this span. Let W be a unitary operator on H so that*

$$\sum_{k=0}^{K-1} \|W^k P_0 W^{-k} f\|^2 \;=\; a^2\|f\|^2, \quad \textit{for all } f \in \mathbb{H}.$$

Then $\{\frac{1}{a}W^kU^iV^j\varphi_0\}_{k=0,i=0,j=0}^{K-1,L-1,M-1}$ is a UNTF for $\mathbb{H}$.

3. It would be very interesting to classify when a finite group of unitaries G generates a UNTF for $\mathbb{H}$ of the form $\{U\varphi_0\}_{U\in G}$. Since finite Abelian groups are isomorphic to a direct product of cyclic groups, the results of this section give sufficient conditions for $\{U\varphi_0\}_{U\in G}$ to generate a UNTF for $\mathbb{H}$. Frames generated by Abelian groups of unitaries were studied in [4] where they are called *geometrically uniform frames* (GU frames). It is shown there that the canonical dual frame of a GU frame is also a GU frame and that the equivalent NTF frame is also GU. Since GU frames have strong symmetry properties, they are particularly useful in applications. In [4], it is further shown that the frame bounds resulting from removing a single vector of a GU frame are the same regardless of the particular vector removed. This result for HTFs was observed in [25].

4. Frames generated by possibly noncommutative groups of unitaries are important in multiple antenna coding and decoding [17]. Here, one needs classes of

unitary space-time signals (that is, frames generated by classes of unitary operators) called *constellations*. It is also desirable in this setting to have *full transmitter diversity*, meaning that

$$\det\,(I-U)\neq 0, \quad \text{for all} \quad I\neq U\in G.$$

Groups with this property are called *fixed-point free groups*. In a tour-de-force, the authors in [17] classify all full-diversity constellations that form a group, for all rates and numbers of transmitting antennas. Along the way they correct some errors in the classification theory of fixed point free groups.

5. CLASSIFICATION OF UNIFORM TIGHT FRAMES WITH ERASURES

As mentioned in the introduction, one of the main applications that motivated us to examine uniform tight frames is that of robust data transmission. This means that at some point, the system experiences losses. These losses are modeled as erasures of transmitted frame coefficients. Since at the receiver side, this looks like the original frame was the one without vectors corresponding to erased coefficients, we examine the structure of our frames after losses.

The first question is whether after erasures what we have is still a frame? For the MB frame, if we erase any one element, the remaining two are enough for reconstruction. However, if we erase any two, we have lost one subspace and reconstruction is not possible. We can provide more robustness by adding more vectors. For example, take a uniform tight frame in $\mathbb{R}^2$ made up of two orthonormal bases $(1,0),(0,1),(-1,0),(0,-1)$. This frame is robust to any *one* erasure; what is left contains at least one of the two bases. However, if two elements are erased, the situation is not that clear anymore. We could erase coefficients corresponding to one of the bases and still have a basis able to reconstruct. If, on the other hand, we lose coefficients corresponding to two collinear vectors, we are stuck; we have lost one entire subspace and cannot reconstruct. We could, however, take another UNTF with $M=4$ vectors in $\mathbb{R}^2$ – the HTF with the analysis frame operator

$$F \;=\; \begin{pmatrix} 1 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 1 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}. \tag{28}$$

This frame is also made up of two orthonormal bases. However, this frame is robust to *any two* erasures. It is thus more robust than the MB frame; we pay for this added benefit with more redundancy (more vectors). It is also more robust than the previous frame with 4 vectors.

These simple examples demonstrate the types of questions we will be asking in this section. One might think that starting with a tight frame might provide some resilience to losses (as opposed to starting with a general frame). As we have seen in our example, this is not the case. Thus, we are searching for frames *robust* to a certain number of erasures. That is, frames which remain frames after erasures. There are frames on $\mathbb{H}_N$ with M elements which are robust to $M-N$ erasures, namely the HTFs (Theorem 4.2 from [25]). Such frames are the best. This is clearly optimal since any more erasures would not leave enough elements to span $\mathbb{H}_N$. In this section, we will characterize frames based on their robustness to erasures.

5.1. UTFs and UNTFs Robust to One Erasure

In this section we will consider UNTFs which are robust to one erasure. This contains the basic idea for the general case.

We start with the notion of a frame robust to k erasures, or a k-robust frame.

Definition 5.1. A frame $\{\varphi_i\}_{i=1}^M$ is said to be robust to k erasures if $\{\varphi_i\}_{i\in I^c}$ is still a frame, for I any index set of k erasures, $I \subset \{1, 2, \dots M\}$ and $|I| = k$.

The following property tells us we do not destroy the robustness of the frame by projection. This observation lead us to the idea that we could classify frames by starting from a large space and "step down"using projections. For example, we could start with a frame robust to one erasure and step down to frames (hopefully) robust to two erasures, then once more to frames robust to three erasures and so on. Although this will not be the case, the idea of stepping down lead us to the results in this section.

Note that it is clear from the definition of a frame that if we apply an orthogonal projection P on $\mathbb{H}$ to a frame we will get a frame for $P\mathbb{H}$ with the same frame bounds.

Proposition 5.1. *Let $\{\varphi_i\}_{i=1}^M$ be a frame in $\mathbb{H}_N$ robust to k erasures and let P be an orthogonal projection on $\mathbb{H}_N$. Then $\{P\varphi_i\}_{i=1}^M$ is a frame for $P\mathbb{H}_N$ robust to k erasures.*

Proof. Let I be the index set of erasures, $I \subset \{1, 2, \dots M\}$ with $|I| = k$. Now, $\{\varphi_i\}_{i\in I^c}$ spans $\mathbb{H}_N$ and so $\{P\varphi_i\}_{i\in I^c}$ is a frame for $P\mathbb{H}_N$ with the same frame bounds. ■

The main ingredient for classifying frames robust to one erasure is contained in the next proposition.

Proposition 5.2. *Let $\{\varphi_i\}_{i=1}^M$ be a set of vectors in $\mathbb{H}_N$. The following are equivalent:*

1. *$\{\varphi_i\}_{i=1}^M$ is a frame robust to one erasure.*
2. *There are nonzero scalars $a_i \neq 0$, for $1 \leq i \leq M$ so that*

$$\sum_{i=1}^{M} a_i \varphi_i \;=\; 0.$$

Proof. (1) $\Rightarrow$ (2): Choose $I \subset \{1, 2, \dots M\}$ maximal for which there are nonzero a_i's, $i \in I$ and

$$g \;=\; \sum_{i\in I} a_i \varphi_i \;=\; 0.$$

We claim that $|I| = M$. We proceed by contradiction. If $I \neq \{1, 2, \ldots M\}$, choose $m \in I^c$. Since $\{\varphi_i\}_{i=1}^M$ is robust to one erasure, there are scalars $\{b_i\}$, not all zero, so that if φ_m is erased, it can be recovered from the rest as

$$\varphi_m \; = \; \sum_{i \neq m} b_i \varphi_i,$$

or

$$h \; = \; \varphi_m - \sum_{i \in I} b_i \varphi_i - \sum_{m \neq i \in I^c} b_i \varphi_i \; = \; 0.$$

We have two cases:

Case I: Assume that $b_i = 0$ for all $i \in I$.

Then, $h = \varphi_m - \sum_{m \neq i \in I^c} b_i \varphi_i = 0$. In this case, $g + h = 0$ and has nonzero coefficients on every φ_i, $i \in I$, plus a nonzero coefficient on φ_m contradicting the maximality of I.

Case II: At least one $b_i \neq 0$ for some $i \in I$.

Since $a_i \neq 0$ for all $i \in I$, we can choose an $\epsilon > 0$ so that

$$\epsilon \neq a_i / b_i, \quad \text{for all} \;\; i \in I.$$

Now, $g + \epsilon h = 0$ and has nonzero coordinates on φ_i, for all $i \in I$, as well as ϵ for a coordinate on φ_m, again contradicting the maximality of I.

(2) ⇒ (1): Assume $a_i \neq 0$, for all $1 \leq i \leq M$ and

$$\sum_{i \in I} a_i \varphi_i \; = \; 0.$$

Then for each $m \in I$ we have:

$$\varphi_m \; = \; - \sum_{m \neq i \in I} \frac{a_i}{a_m} \varphi_i,$$

that is, any vector lost can be recovered using the rest and so $\{\varphi_i\}_{i=1}^M$ is robust to the erasure of φ_m, for an arbitrary $m \in I$. ■

Note that to construct a frame guaranteed not to be robust to one erasure, it is enough to put one vector, say φ_M, orthogonal to the span of the rest. Erasing that vector destroys the frame property. Thus, we cannot find a nonzero coefficient a_M such that $\sum_{i=1}^M a_i \varphi_i = 0$.

For example, we know that the MB frame $\{\varphi_i\}_{i=1}^3$ is robust to one erasure (Theorem 4.1 from [25]). Hence, we can find $a_1 = a_2 = a_3 = 1$ such that $\sum_{i=1}^3 a_i \varphi_i = 0$.

The next results will characterize frames robust to one erasure. Since we mentioned the idea of stepping down, we will start from an orthonormal basis $\{e_i\}_{i=1}^M$ for $\mathbb{H}_M$ and project it using a projection operator P (**or** $I - P$). We thus look into a few facts connecting P and $(I - P)$.

1. If P is an orthogonal projection, then $(I - P)$ is an orthogonal projection as well.

2. The subspaces P and $(I-P)$ project onto are orthogonal. For any $f \in \mathbb{H}$,

$$\langle (I-P)f, Pf\rangle \;=\; \langle P^*(I-P)f, f\rangle \;=\; \langle (P-P)f, f\rangle \;=\; 0.$$

3. $\{(I-P)e_i\}_{i=1}^M$ is uniform if and only if $\{Pe_i\}_{i=1}^M$ is uniform. To see this we compute, for all i, j

$$\begin{aligned}
\langle (I-P)e_i, (I-P)e_i\rangle &= \langle (I-P)e_j, (I-P)e_j\rangle \\
\langle (I-P^*)(I-P)e_i, e_i\rangle &= \langle (I-P^*)(I-P)e_j, e_j\rangle \\
\langle (I-P)e_i, e_i\rangle &= \langle (I-P)e_j, e_j\rangle \\
1 - \langle Pe_i, e_i\rangle &= 1 - \langle Pe_j, e_j\rangle \\
\langle P^*Pe_i, e_i\rangle &= \langle P^*Pe_j, e_j\rangle \\
\langle Pe_i, Pe_i\rangle &= \langle Pe_j, Pe_j\rangle.
\end{aligned}$$

Here we used the fact that P is an orthogonal projection and thus $P = P^2 = P^*P$.

4. Both $\{Pe_i\}_{i=1}^M$ and $\{(I-P)e_i\}_{i=1}^M$ are NTFs.

By Theorem 2.1, all NTFs are of the form $\{Pe_i\}_{i=1}^M$, where $\{e_i\}_{i=1}^M$ is an orthonormal basis for $\mathbb{H}$ and P is an orthogonal projection on $\mathbb{H}$. The UNTFs form a subclass of these frames. In the rest of this paper we will identify this subclass. So we will be working with frames of the form above. We will discover that the subspace $P\mathbb{H}$ determines when the frame is a UNTF. From our earlier discussion, if P is an orthogonal projection on H, we may work either with $\{Pe_i\}$ or $\{(I-P)e_i\}$ to classify UNTFs. We will freely switch between these classes because each of them has certain advantages for specific results. The class $\{Pe_i\}$ works well for classifications when the dimension of $P\mathbb{H}$ is "large" relative to the dimension of $\mathbb{H}$. On the other hand, $\{(I-P)e_i\}$ is easier to work with when the dimension of $P\mathbb{H}$ is "small" relative to the dimension of $\mathbb{H}$. Also, as we will see, important information about $\{Pe_i\}$ is often contained in $(I-P)\mathbb{H}$ (and vice-versa). So we will need to bring both of these into our discussion.

Now we give the general classification for frames robust to one erasure.

Corollary 5.1. *Let $\{e_i\}_{i=1}^M$ be an orthonormal basis for $\mathbb{H}_M$ and let P be an orthogonal projection on $\mathbb{H}_M$. The following are equivalent:*

1. *$\{(I-P)e_i\}_{i=1}^M$ is a frame robust to one erasure.*
2. *There are nonzero scalars $a_i \neq 0$, for $1 \leq i \leq M$ so that*

$$\sum_{i=1}^M a_i(I-P)e_i \;=\; 0.$$

3. *There is an $f \in span_{1\leq i\leq M} Pe_i$ so that*

$$\langle f, e_i\rangle \neq 0, \quad \textit{for all} \;\; 1 \leq i \leq M.$$

Proof. The equivalence of (1) and (2) comes from Proposition 5.2. Let us check the equivalence of (2) and (3).

(2) ⇒ (3): Given (2), choose f to be

$$f = \sum_{i=1}^{M} a_i e_i .$$

By our assumption, $\sum_{i=1}^{M} a_i e_i = \sum_{i=1}^{M} a_i P e_i$ and thus

$$f = \sum_{i=1}^{M} a_i P e_i \in P\mathbb{H}_M .$$

Moreover, $\langle f, e_i\rangle = a_i \neq 0$, for all $1 \leq i \leq M$.

(3) ⇒ (2): Choose $f \in P\mathbb{H}_M$ as in (3) and call $a_i = \langle f, e_i\rangle \neq 0$, for all $1 \leq i \leq M$. Now

$$\sum_{i=1}^{M} a_i (I-P)e_i = \sum_{i=1}^{M} \langle f, e_i\rangle (I-P)e_i = \sum_{i=1}^{M} \langle f, e_i\rangle e_i - \sum_{i=1}^{M} \langle Pf, e_i\rangle e_i = 0,$$

since $f \in P\mathbb{H}_M$, and thus $Pf = f$. ■

Corollary 5.1 classifies frames which are robust to one erasure. We now extend this to a classification of UNTFs robust to one erasure.

COROLLARY 5.2. *Let* $\{e_i\}_{i=1}^{M}$ *be an orthonormal basis for* $\mathbb{H}_M$ *and let* P *be an orthogonal projection on* $\mathbb{H}_M$. *The following are equivalent:*

1. $\{(I-P)e_i\}_{i=1}^{M}$ *is a frame robust to one erasure.*
2. *There is a* $g = \sum_{i=1}^{M} a_i e_i$ *with nonzero scalars* $a_i \neq 0$, *for all* $1 \leq i \leq M$ *such that* $Pf = \langle f, g\rangle g$, *for all* $f \in \mathbb{H}_M$, *and* $\|g\| = 1$.

Moreover, if P *is rank-1,* $\{(I-P)e_i\}_{i=1}^{M}$ *is a uniform frame if and only if* $|a_i| = 1/\sqrt{M}$, *for all* $1 \leq i \leq M$.

Proof. (1) ⇔ (2) is immediate from Corollary 5.1.

For the moreover part, we know that $\{(I-P)e_i\}_{i=1}^{M}$ is uniform if and only if $\{Pe_i\}_{i=1}^{M}$ is uniform. Also, since P is rank-1, for all $1 \leq i \leq M$ and using the expression expression $Pf = \langle f, g\rangle g$ in Part 2. with $f = e_i$ we have,

$$Pe_i = a_i g.$$

Hence, $\|Pe_i\| = \|Pe_j\|$ is true if and only if $|a_i| = |a_j|$, for all $1 \leq i, j \leq M$. That is, $\{Pe_i\}_{i=1}^{M}$ is uniform (and hence $\{(I-P)e_i\}_{i=1}^{M}$ is uniform) if and only if $|a_i| = |a_j|$, for all $1 \leq i, j \leq M$. Finally, $\|Pe_i\| = |a_i|\|g\| = |a_i| = 1/\sqrt{M}$. ■

5.2. UTFs Robust to More than One Erasure

We now try to apply the same ideas from the previous section to classify frames robust to more than one erasure. The classification of UNTFs with k erasures in this section is useful if we have M vectors in an N-dimensional space and $M-N$ is "small". In the next section we will give another classification of this family which works best when $M-N$ is "large".

In what follows, $I \subset \{1,2,\ldots,M\}$ will denote the erasure index set.

Proposition 5.3. *Let* $\{e_i\}_{i=1}^M$ *be an orthonormal basis for* $\mathbb{H}_M$*. Let* P *be an orthogonal projection of* $\mathbb{H}_M$ *onto an* L*-dimensional subspace* $\mathbb{H}_L$*. Fix* $I \subset \{1,2,\ldots,M\}$ *with* $|I| = k \le L$ *and let* $\mathbb{K} = span_{i\in I^c} e_i$*. If* $\{\varphi_j\}_{j=1}^L$ *is any orthonormal basis for* $\mathbb{H}_L$ *and we have the following* $L \times M$ *matrix*

$$A = \left(\langle \varphi_j, e_i\rangle\right)_{j=1,i=1}^{L\ \ M},$$

then

$$dim\ [ker\ (I-P)|_{\mathbb{K}}] \ = \ L - [\text{row rank of the } k \text{ columns of } A \text{ indexed by } I].$$

Proof. We have

$$\ker\ (I-P)|_{\mathbb{K}} \ = \ \{f \in \mathbb{K} | (I-P)f = 0\} \ = \ [\ker\ (I-P)] \cap \mathbb{K}.$$

Now, apply row reduction (relative to $\{e_i\}_{i=1}^M$) to $\{\varphi_i\}_{i=1}^L$ using the elements of the columns in I. This gives a linearly independent set $\{g_j\}_{j=1}^s \cup \{g_j\}_{j=s+1}^L$ spanning $\mathbb{K}$ with

$$s \ = \ \text{row rank of } \left(\langle \varphi_j, e_i\rangle\right)_{j=1,i\in I}^L,$$

$g_j \in \mathbb{K}$ for $s+1 \le j \le L$ and

$$(\text{span}_{1\le j\le s} g_j) \cap \mathbb{K} = 0.$$

Hence,

$$\ker\ (I-P) \cap \mathbb{K} = \text{span}_{s+1\le j\le L} g_j.$$

Therefore,

$$\dim\ (\ker\ (I-P)) \cap \mathbb{K} = \dim\ (\ker\ (I-P)|_{\mathbb{K}} = L - (s+1) + 1 = L - s.$$

■

We are looking for the NTFs which are robust to k erasures. Proposition 5.3 actually yields a stronger result. Namely, it gives necessary and sufficient conditions for $\{Pe_i\}$ to be robust to one particular choice of k erasures. We state this stronger result in two different forms in the next two propositions.

Proposition 5.4. *With the notation of Proposition 5.3, the following are equivalent:*

1. $\{(I-P)e_i\}_{i=1}^M$ *is robust to the erasure of the elements* $\{(I-P)e_i\}_{i\in I}$*.*
2. *We have rank* $(I-P)|_{\mathbb{K}} = M - L$*.*
3. *We have dim* $[ker\ (I-P)|_{\mathbb{K}}] = L - k$*.*

Proof. (1) $\Leftrightarrow$ (2): $\{(I-P)e_i\}_{i=1}^{M}$ is robust to the erasure of the elements $\{(I-P)e_i\}_{i\in I}$ if and only if $(I-P)|_{\mathbb{K}}$ is full rank, which must be rank $(I-P) = M-L$.

(2) $\Leftrightarrow$ (3): By Proposition 5.3,

$$\dim\,(\ker\,(I-P)|_{\mathbb{K}} = L - [\text{row rank of the } k \text{ columns of } A \text{ indexed by } I] \geq L-k.$$

On the other hand, $(I-P)|_{\mathbb{K}}$ is full rank if and only if $(I-P)(I-P_i)$ is full rank, where P_i is the orthogonal projection of $\mathbb{H}_M$ onto $\text{span}_{i\in I} e_i$. Hence,

$$\dim\,(\ker\,(I-P)(I-P_i)) \leq L.$$

But, $e_i \in \ker\,(I-P_i)$, for all $i \in I$. Hence,

$$\dim\,(\ker\,(I-P)(I-P_i) = \dim\,(\ker\,(I-P) + k \leq L.$$

Therefore,

$$\dim\,(\ker\,(I-P) \leq L-k.$$

■

Proposition 5.4 gives precise information when a frame is robust to the erasure of one particular choice of k elements in terms of the coefficients of $\{\varphi_i\}$ relative to the unit vector basis of $\mathbb{H}_L$. The following corollary is a statement for a fixed choice of erasures as well.

Corollary 5.3. *With the notation of Proposition 5.3, the following are equivalent:*

1. $\{(I-P)e_i\}_{i=1}^{M}$ *is robust to the erasure of the elements* $\{(I-P)e_i\}_{i\in I}$.
2. *The row rank of* $(\langle\varphi_j, e_i\rangle)_{j=1,i\in I}^{L} = k$.

Applying the above result to every choice of erasures results in Theorem 5.1 and a classification of when our frame is robust to any k erasures. We are dealing here with general frames. We will deal with uniform frames right afterwards.

Theorem 5.1. *With the notation of Proposition 5.3, the following are equivalent:*

1. $\{(I-P)e_i\}_{i=1}^{M}$ *is robust to* k *erasures.*
2. *For every* $I \subset \{1,2,\dots M\}$ *with* $|I| = k$*, the row rank of* $A = (\langle\varphi_j, e_i\rangle)_{j=1,i\in I}^{L}$ *equals* k.

Note the crucial role played by erased elements in the theorem, since in the statement, the indices are from I.

Next we see what it takes for such frames to be uniform. Here, we call the "angle" between two vectors the inner product when this is really the cosine of the angle. Note that this result classifies when $\{(I-P)e_i\}$ is a UNTF without any reference to erasures.

Theorem 5.2. *With the notation of Proposition 5.3, the following are equivalent:*

1.$\{Pe_i\}_{i=1}^M$ *is uniform (and hence* $\{(I-P)e_i\}_{i=1}^M$ *is uniform).*
2.*For every* $1 \le i \le M$ *we have,*

$$(A^*A)_{ii} = \sum_{j=1}^{L} |\langle \varphi_j, e_i \rangle|^2 = \frac{L}{M}.$$

That is, the columns of $A = \{\varphi_j\}_{j=1}^L$ *all have the same square sums.*

Moreover, the angle between $(I-P)e_i$ *and* $(I-P)e_j$ *is given by the inner product of the* i*th and* j*th columns of* $\{\varphi_j\}_{j=1}^L$*.*

Proof. (1) $\Leftrightarrow$ (2): We know that $\{(I-P)e_i\}_{i=1}^M$ is uniform if and only if $\{Pe_i\}_{i=1}^M$ which, in turn, is uniform if and only if the diagonal elements of the matrix for P relative to $\{e_i\}_{i=1}^M$ have constant modulus. We have,

$$Pe_i = \sum_{n=1}^{L} \langle e_i, \varphi_n \rangle \varphi_n = \sum_{m=1}^{M} \left(\sum_{n=1}^{L} \langle e_i, \varphi_n \rangle \langle \varphi_n, e_m \rangle \right) e_m.$$

The diagonal element of Pe_i is:

$$\sum_{n=1}^{L} \langle e_i, \varphi_n \rangle \langle \varphi_n, e_i \rangle = \sum_{n=1}^{L} |\langle \varphi_n, e_i \rangle|^2.$$

So $\{Pe_i\}_{i=1}^M$ is a uniform tight frame if and only if for all $1 \le i, j \le M$ we have

$$\sum_{n=1}^{L} |\langle \varphi_n, e_i \rangle|^2 = \sum_{n=1}^{L} |\langle \varphi_n, e_j \rangle|^2.$$

For the moreover part, we compute for $1 \le i \ne j \le M$:

$$\langle (I-P)e_i, (I-P)e_j \rangle = \langle e_i, e_j \rangle - \langle Pe_i, e_j \rangle - \langle e_i, Pe_j \rangle + \langle Pe_i, Pe_j \rangle$$

$$= \langle Pe_i, Pe_j \rangle = \langle e_i, Pe_j \rangle$$

$$\langle e_i, \sum_{n=1}^{L} \langle e_j, \varphi_n \rangle \varphi_n \rangle = \langle e_i, \sum_{m=1}^{M} \left(\sum_{n=1}^{L} \langle e_j, \varphi_n \rangle \langle \varphi_n, e_m \rangle \right) e_m \rangle = \sum_{n=1}^{L} \langle e_j, \varphi_n \rangle \langle \varphi_n, e_i \rangle.$$

The right-hand side of the above equality is precisely the inner product of the ith and jth columns of $\{\varphi_j\}_{j=1}^L$. ■

The following consequence of the above is surprising at first. It says that we can almost never get (in the real case) frames robust to k erasures by stepping down from frames robust to one erasure, then to frames robust to two erasures etc.

COROLLARY 5.4. *In the real case, if* $M \ge 3$*, there do not exist rank-*1 *orthogonal projections* P_1, P_2 *with* $P_1P_2 = 0$ *so that* $\{(I-P_1)e_i\}_{i=1}^M$ *is uniform and* $\{(I-P_2)(I-P_1)e_i\}_{i=1}^M$ *is uniform and robust to two erasures.*

Proof. Assume such P_1, P_2 exist. Since $\{(I - P_1)e_i\}_{i=1}^M$ is uniform, by Theorem 5.2, there is a vector $\varphi_1 = \sum_{i=1}^M a_i e_i \in \mathbb{H}_M$ with $P_1 f = \langle f, \varphi_1\rangle \varphi_1$, for all $f \in \mathbb{H}_M$ and $|a_i| = |a_j|$, for all $1 \le i, j \le M$. Now choose $\varphi_2 = \sum_{i=1}^M b_i e_i \in \mathbb{H}_M$ so the $P_2 f = \langle f, \varphi_2\rangle \varphi_2$, for all $f \in \mathbb{H}_M$. Since P_1, P_2 are rank-1, $P_1 P_2 = 0$ implies $P_2 P_1 = 0$. That is, $\langle \varphi_1, \varphi_2\rangle = 0$. Let $P = P_1 + P_2$. Then $(I - P_2)(I - P_1) = I - P$. Since $\{(I - P)\}_{i=1}^M$ is uniform, by Theorem 5.2 we have

$$a_i^2 + b_i^2 \;=\; a_j^2 + b_j^2, \quad \text{for all} \quad a \le i, j \le M.$$

Hence, $b_i^2 = b_j^2$, for all $1 \le i, j \le M$. Since $\|\varphi_1\| = \|\varphi_2\| = 1$, it follows that

$$a_i^2 \;=\; b_i^2 \;=\; \frac{1}{M}, \quad \text{for all} \quad 1 \le i, j \le M.$$

Since $\{(I - P)e_i\}_{i=1}^M$ is robust to two erasures, by Theorem 5.2

$$a_i b_j - a_j b_i \;\neq\; 0, \quad \text{for all} \quad 1 \le i, j \le M. \tag{29}$$

However, for all i, j we have $a_i = \pm a_j = \pm b_i$. Hence, there is some i, j with either $a_i = a_j$ and $b_i = b_j$ or $a_i = -a_j$ and $b_i = -b_j$. In either case (29) fails. ■

Corollary 5.4 is heavily dependent on having a real Hilbert space. In the complex case, we will show in Corollary 5.6 that the GHFs have the property that we can step down to the frame by successively applying rank-1 projections to the orthonormal basis. Corollary 5.4 shows that in general we cannot step down one dimension at a time to construct frames robust to k erasures. It would be interesting to extend this to a classification of when we can step down from k_1 erasures to k_2 erasures and so on. One reason is that this has implications for *entangled states* in quantum detection theory [12, 22].

This raises the question of whether we can step down one dimension at a time as in Corollary 5.4 if we only ask for each level to be uniform. The answer is *no* as the next example shows.

Example 5.1. Let $\{e_i\}_{i=1}^4$ be an orthonormal basis for $\mathbb{H}_4$ and let

$$\varphi_1 \;=\; \frac{1}{\sqrt{2}} e_1 + \frac{1}{2} e_3 + \frac{1}{2} e_4,$$

and

$$\varphi_2 \;=\; \frac{1}{\sqrt{2}} e_2 + \frac{1}{2} e_3 + \frac{-1}{2} e_4.$$

Then $\langle \varphi_1, \varphi_2\rangle = 0$, and if P is the orthogonal projection of $\mathbb{H}_4$ onto the span of $\{\varphi_1, \varphi_2\}$, then by Theorem 5.2, $\{(I - P)e_i\}_{i=1}^4$ is a uniform normalized tight frame. However, there does not exist a rank-1 orthogonal projection P_1 of $\mathbb{H}_4$ into range of P so that $\{(I - P_1)e_i\}_{i=1}^4$ is a uniform frame.

Proof. By Theorem 5.2, in order for such a P_1 to exist, there must exist a vector in $P\mathbb{H}_4$ of the form $\sum_{i=1}^4 b_i e_i$ with $|b_i| = |b_j|$, for all $1 \le i, j \le 4$. For any a_1, a_2,

$$a_1\varphi_1 + a_2\varphi_2 \;=\; \frac{a_1}{\sqrt{2}}e_1 + \frac{a_2}{\sqrt{2}}e_2 + \frac{a_1+a_2}{2}e_3 + \frac{a_1-a_2}{2}e_4.$$

If $(|a_1 + a_2|)/2 = (|a_1 - a_2|)/2$, then one of $|a_1|, |a_2|$ equals 0 so $|a_1| \neq |a_2|$. ■

The next question is whether we can step down one dimension at a time as in Corollary 5.4 if all we want is for our frame to be robust to j erasures at the jth level but are willing to give up uniformity at each level. Surprisingly, the answer here is *yes*.

Proposition 5.5. *Let P be an orthogonal projection of $\mathbb{H}_M$ onto an L-dimensional subspace $\mathbb{H}_L$. Let $\{e_i\}_{i=1}^M$ be an orthonormal basis for $\mathbb{H}_M$. If $\{(I-P)e_i\}_{i=1}^M$ is robust to k erasures ($k \le L$), then there are mutually orthogonal rank-1 projections $\{P_i\}_{i=1}^L$ taking $\mathbb{H}_M$ into $\mathbb{H}_L$ so that $\{(I-P_j)(I-P_{j-1})\cdots(I-P_1)e_i\}_{i=1}^M$ is robust to j erasures, for all $1 \le j \le k$, and robust to k erasures for all $k \le j \le L$.*

Proof. See Appendix A.3. ■

5.3. UTFs Robust to More than One Erasure: An Alternative Approach

In the previous sections we used the orthogonal projection P on $\mathbb{H}_M$ to classify when $\{(I-P)e_i\}_{i=1}^M$ is robust to k erasures and when it is uniform. However, if the dimension of $P\mathbb{H}_M$ is large (that is, close to the dimension of $\mathbb{H}_M$) these results become difficult to implement since they require knowledge about the orthonormal bases for the large-dimensional space $P\mathbb{H}_M$. In this case, $\{(I-P)e_i\}_{i=1}^M$ is a *large* number of vectors in a *small*-dimensional space $(I-P)\mathbb{H}_M$. So it is easier in this case to work directly with $(I-P)\mathbb{H}$ and $\{(I-P)e_i\}$. Or, equivalently, with $P\mathbb{H}$ and $\{Pe_i\}$, where dim $P\mathbb{H}$ is small.

Proposition 5.6. *Let $\{\varphi_i\}_{i=1}^N$ be an orthonormal sequence in $\mathbb{H}_M$, let $\{e_i\}_{i=1}^M$ be an orthonormal basis for $\mathbb{H}_M$ and let P be the orthogonal projection of $\mathbb{H}_M$ onto the span of $\{\varphi_i\}_{i=1}^N$. Then the normalized tight frame $\{Pe_i\}_{i=1}^M$ is unitarily equivalent to $\{g_i\}_{i=1}^M$ where for $1 \le j \le M$ we have*

$$g_j \;=\; \sum_{i=1}^N \langle \varphi_i, e_j\rangle e_i.$$

That is, $\{Pe_i\}_{i=1}^M$ is the frame obtained by turning the columns of $\{\varphi_i\}_{i=1}^N$ into row vectors in $\mathbb{H}_N$.

Proof. For any $1 \le i \le M$ we have:

$$Pe_i \;=\; \sum_{n=1}^L \langle e_i, \varphi_n\rangle \varphi_n.$$

Since $\{\varphi_n\}_{n=1}^{L}$ is an orthonormal sequence, the operator $T\varphi_n = e_n$, for $1 \le n \le L$ is a unitary operator which takes Pe_i to $\{g_j\}_{j=1}^{M}$ where for $1 \le j \le M$ and we have

$$g_j \;=\; \sum_{i=1}^{N} \langle \varphi_i, e_j \rangle e_i .$$

■

Corollary 5.5. *Given the conditions in Proposition 5.6 we have:*

1.$\{g_j\}_{j=1}^{M}$ *is a uniform frame for* $\mathbb{H}_N$ *if and only if*

$$\sum_{i=1}^{N} |\langle \varphi_I, e_j \rangle|^2 \;=\; \frac{N}{M}, \quad \textit{for all} \;\; 1 \le j \le M.$$

2.*The following are equivalent:*

(a) $\{g_j\}_{j=1}^{M}$ *is robust to the erasure of* $\{g_j\}_{j\in I}$ *for some* $I \subset \{1, 2, \ldots M\}$.

(b) We have that

$$(\langle \varphi_i, e_j \rangle)_{i=1, j\in I^c}^{N}$$

has row rank N.

Hence, $\{g_j\}_{j=1}^{M}$ *is robust to any choice of* k *erasures,* $k \le M - N$, *if and only if every set of* $M - k$ *columns of the matrix*

$$A = (\langle \varphi_i, e_j \rangle)_{i=1, j=1}^{N \;\; M}$$

has row rank N.

Finally, the angle between g_i *and* g_j *is given by the inner product of the* i*th and* j*th columns of* A.

It would be interesting and useful to classify the GHFs in the format of this section. That is, precisely when is $\{(I-P)e_i\}_{i=1}^{M}$ a GHF? One important property of GHFs is contained in the following result which says they have the step-down property of Section 5.1.

Corollary 5.6. *Let* P *be an orthogonal projection of* $\mathbb{H}_M$ *onto an* L*-dimensional subspace* $\mathbb{H}_L$ *and let* $\{e_i\}_{i=0}^{M-1}$ *be an orthonormal basis for* $\mathbb{H}_M$. *If* $\{(I-P)e_i\}_{i=0}^{M-1}$ *is a GHF then there is an orthogonal sequence of rank-*1 *projections* $\{P_i\}_{i=1}^{L}$ *of* $\mathbb{H}_M$ *into* $\mathbb{H}_L$ *so that for all* $1 \le j \le L$ *and for all permutations* σ *of* $\{1, 2, \ldots, L\}$ *we have that*

$$\{\prod_{m=0}^{j-1} (I - P_{\sigma(m)}) e_i\}_{i=0}^{M-1},$$

is a GHF and hence is a UNTF which is robust to j *erasures.*

Proof. We will do the proof for HTFs since the GHF case requires only notational changes but obscures the basic ideas of the proof. By Proposition 5.6, there is a unique way to get HTFs. Namely, let $\{e_i\}_{i=0}^{M-1}$ be the natural orthonormal basis for $\mathbb{H}_M$. Let $\{w_i\}_{i=0}^{M-1}$ be distinct Mth roots of unity and consider the orthonormal basis $\{\varphi_i\}_{i=0}^{M-1}$ for $\mathbb{H}_M$ given by:

$$\varphi_i = \sum_{j=0}^{M-1} w_i^j e_j.$$

Without loss of generality, we may as well assume that $\mathbb{H}_L = \text{span } \{\varphi_i\}_{i=0}^{L}$. Now, turning the row vectors of $\{\varphi_i\}_{i=L}^{M-1}$ into column vectors gives a HTF for $\mathbb{H}^{M-L}$. Moreover, again by Proposition 5.6, this HTF is unitarily equivalent to $\{(I-P)e_i\}_{i=0}^{M-1}$ where $I-P$ is the orthogonal projection onto $(\mathbb{H}_L)^{\perp}$. Now, let P_i be the orthogonal rank-1 projection of $\mathbb{H}_M$ onto span φ_i for $0 \le i \le L-1$. Fix a permutation σ of $\{1, 2, \ldots, L\}$ and let $I_j = \{\sigma(0), \sigma(1), \ldots, \sigma(j-1)\}$. Then, again by Proposition 5.6,

$$\{\prod_{m=0}^{j-1} (I - P_{\sigma(m)}) e_i\}_{i=0}^{M-1},$$

is unitarily equivalent to the HTF obtained by turning the row vectors of $\{\varphi_i\}_{i \notin I_j}$ into column vectors. ■

It is possible that the property in Corollary 5.6 characterizes GHFs, but we do not have a proof for this. We can show that the use of permutations is necessary in Proposition 5.6. That is, there are UNTFs which have the step-down property for erasures and for uniformness for one fixed ordering of the rank-1 projections P_i while failing to be equivalent to a GHF. This is the point of the next example.

Example 5.2. In $\mathbb{C}^4$, let

$$\varphi_1 = \sum_{i=1}^{4} e_i, \qquad \varphi_2 = \sum_{i=1}^{4} w_i e_i, \qquad \varphi_3 = e_1 + e_2 - e_3 - e_4,$$

where $w_1 = 1$, $w_2 = -1$, $w_3 = i$ and $w_4 = -i$. Let $\{P_i\}_{i=1}^{3}$ be the rank-1 orthogonal projections of $\mathbb{C}^4$ onto the span of φ_i. Then it follows easily from our results that:

1. $\{(I-P_1)e_i\}_{i=1}^{4}$ is a UNTF which is robust to one erasure.
2. $\{(I-P_2)(I-P_1)e_i\}_{i=1}^{4}$ is a UNTF which is robust to 2 erasures.
3. $\{(I-P_3)(I-P_2)(I-P_1)e_i\}_{i=1}^{4}$ is a UNTF which is robust to 3 erasures, but is not a harmonic frame since $\{(I-P_3)(I-P_1)e_i\}_{i=1}^{4}$ is a UNTF which *is not* robust to 2 erasures.

Example 5.3. If $\varphi_i = (1, w_i)$ in $\mathbb{C}^2$ for $1 \le i \le M$, where $|w_i| = 1$ and the $\{w_i\}_{i=1}^{M}$ are distinct, then $\{\varphi_i\}_{i=1}^{M}$ is a UNTF for $\mathbb{C}^2$ if and only if $\sum_{i=1}^{M} w_i = 0$. In this case, if P_i is an orthogonal rank-1 projection onto the span φ_i, then $\{(I - P_j)e_i\}_{i=1}^{M}$ is a UNTF which is robust to one erasure while $\{(I-P_j)(I-P_k)e_i\}_{i=1}^{M}$ is a UNTF which is robust to 2 erasures for all $1 \le j \ne k \le M$.

Proof. This family is certainly uniform. The assumption $\sum_{i=1}^{M} w_i = 0$ guarantees that the vectors

$$(1, 1, 1, \ldots 1) \text{ and } (w_1, w_2, \ldots, w_M)$$

are orthogonal in $\mathbb{H}_M$ (and conversely). Hence, $\{\varphi_i\}_{i=1}^{M}$ is a UNTF for $\mathbb{C}^2$ by Proposition 5.6. ■

Example 5.4. In general, the frames constructed in Example 5.3 are not equivalent to harmonic frames even after a permutation.

Proof. The reason is that $\{\varphi_j\}_{j=0}^{M-1}$ is a GHF implies that

$$\langle \varphi_j, \varphi_{j+1} \rangle \; = \; \langle \varphi_{j+1}, \varphi_{j+2} \rangle, \quad \text{for all } \; 0 \leq j \leq M-2,$$

(see Proposition 2.4). This would imply in Example 5.3 that

$$\langle \varphi_j, \varphi_{j+1} \rangle \; = \; 1 + w_j \overline{w_{j+1}} \; = \; 1 + w_{j+1} \overline{w_{j+2}} \; = \; \langle \varphi_{j+1}, \varphi_{j+2} \rangle.$$

Hence, $w_j \overline{w_{j+1}} = w_{j+1} \overline{w_{j+2}}$. Now, let $w_1 = 1$, $w_2 = -1/2 + \sqrt{3/4} i$ and $w_2 = -1/2 - \sqrt{3/4} i$. Then $|w_j| = 1$ and $\sum_{j=1}^{3} w_j = 0$. Hence, by Example 5.3, $\{(1, w_j)\}_{j=1}^{3}$ is a UNTF for $\mathbb{C}^2$. However, for any permutation σ of $\{1, 2, 3\}$, we do not have $w_{\sigma(j)} \overline{w_{\sigma(j+1)}} = w_{\sigma(j+1)} \overline{w_{\sigma(j+2)}}$, for all $0 \leq j \leq 2$ (with $w_4 = w_1$). ■

It is not hard to see that in the case of $\mathbb{R}^2$, the condition in Corollary 5.6 is sufficient.

APPENDIX: PROOFS

A.1. PROOF OF PROPOSITION 4.1

First note that if $\{c_i\}_{i=1}^{N}$ are distinct Mth roots of c with $|c| = 1$, then

$$\sum_{k=0}^{M-1} c_i^k \; = \; 0, \qquad \sum_{k=0}^{M-1} |c_i^k|^2 \; = \; M, \qquad \sum_{\substack{k=0 \\ i \neq l}}^{M-1} (c_i c_l)^k \; = \; 0, \tag{A.1}$$

and thus

$$\sum_{k=0}^{M-1} (c_i \overline{c_l})^k \; = \; M \delta_{i-l}. \tag{A.2}$$

We now check the necessity of our condition for a frame to be a general harmonic frame. If $\{\varphi_k\}_{k=1}^{M}$ is a general harmonic frame for $\mathbb{H}_N$ then by the definition:

$$\varphi_k \; = \; (c_1^k b_1, c_2^k b_2, \cdots, c_N^k b_N) \; = \; \sum_{i=1}^{N} c_i^k b_i e_i,$$

where $\{e_i\}_{i=1}^N$ is the natural unit vector orthonormal basis of $\mathbb{H}_N$, $|b_i| = 1/\sqrt{M}$, and $\{c_i\}_{i=1}^N$ are distinct Mth roots of c with $|c| = 1$. Now let

$$\varphi_0 = \sum_{i=1}^n b_i e_i.$$

Define a unitary operator U on $\mathbb{H}_N$ by $Ue_i = c_i e_i$. Then, for all $0 \le k \le M-1$ we have $U^k e_i = c_i^k e_i$, and thus, $U^k \varphi_0 = \varphi_k$ for all $0 \le k \le M-1$. So our frame has the form given in the proposition.

The sufficiency of the condition is checked similarly. If we assume that $\{\varphi_k\}$ is of the form described in the proposition, then:

$$\|\varphi_k\| = \|U^k \varphi_0\| \;=\; \|\varphi_0\| \;=\; \sqrt{\frac{N}{M}},$$

since U is a unitary operator. So $\{\varphi_k\}$ is a uniform frame. To see that $\{\varphi_k\}$ is a normalized tight frame, we let $f = \sum_{i=1}^N a_i e_i \in \mathbb{H}_N$ and compute:

$$\begin{aligned}
\sum_{k=0}^{M-1} |\langle f, U^k \varphi_0\rangle|^2 \;&=\; \sum_{k=0}^{M-1} |\sum_{i=1}^N a_i c_i^k b_i|^2 \;= \\
&=\; \sum_{k=0}^{M-1}\sum_{i=1}^N |a_i c_i^k b_i|^2 + \sum_{m\neq \ell} a_m \overline{b_\ell} \sum_{k=0}^{M-1} (c_m \overline{c_\ell})^k \\
&=\; M \sum_{i=1}^N |a_i|^2 |b_i|^2 + 0 \;=\; M \sum_{i=1}^N |a_i|^2 |\frac{1}{\sqrt{M}}|^2 \;=\; \|f\|^2,
\end{aligned}$$

where in the next to last equality we used the fact that the $\{c_i\}$ is a family of distinct Mth roots of c (see (A.2)). Hence, $\{\varphi_k\}$ is a UNTF. Thus, using (15) and writing $f = e_j$, we see that

$$\|e_j\|^2 \;=\; 1 \;=\; \sum_{k=1}^M |\langle e_j, U^k \varphi_0\rangle|^2 \;=\; \sum_{k=1}^M |\langle e_j, \sum_{i=1}^N c_i^k b_i e_i\rangle|^2 \;=\; \sum_{k=1}^M |c_j^k b_j|^2 \;=\; M|b_j|^2.$$

So $|b_j| = 1/\sqrt{M}$, for all $j, 0 \le j \le N-1$. This further means that

$$\varphi_0 \;=\; (b_1, b_2, \ldots, b_N)$$

with $|b_j| = 1/\sqrt{M}$ and since $\varphi_k = U^k \varphi_0$,

$$\varphi_k \;=\; (c_1^k b_1, c_2^k b_2, \ldots, c_N^k b_N).$$

A.2. PROOF OF THEOREM 4.1

We prove the theorem in steps.

Step I: We first prove that there is a constant $|c| = 1$ so that $U^M = cI$. This will show that the operators $\{U^i\}_{i=0}^{M-1}$, in fact, form a . Since the frame is an NTF, for every $f \in \mathbb{H}_N$ we have,

$$f = \sum_{k=0}^{M-1} \langle f, U^k \varphi_0 \rangle U^k \varphi_0.$$

Hence,

$$\begin{aligned} Uf &= \sum_{k=0}^{M-1} \langle Uf, U^k \varphi_0 \rangle U^k \varphi_0 = \\ &= \sum_{k=0}^{M-1} \langle f, U^{k-1} \varphi_0 \rangle U^k \varphi_0, \qquad \text{since } U^* = U^{-1}, \\ &= U \left(\sum_{k=0}^{M-1} \langle f, U^{k-1} \varphi_0 \rangle U^{k-1} \varphi_0 \right) = U \left(\sum_{k=-1}^{M-2} \langle f, U^k \varphi_0 \rangle U^k \varphi_0 \right). \end{aligned} \tag{A.3}$$

Since U is one-to-one,

$$f = \sum_{k=-1}^{M-2} \langle f, U^k \varphi_0 \rangle U^k \varphi_0 = \sum_{k=0}^{M-1} \langle f, U^k \varphi_0 \rangle U^k \varphi_0. \tag{A.4}$$

From (A.3) and (A.4), it follows that,

$$\langle f, U^{M-1} \varphi_0 \rangle U^{M-1} \varphi_0 = \langle f, U^{-1} \varphi_0 \rangle U^{-1} \varphi_0.$$

Applying U we have

$$\langle f, U^{M-1} \varphi_0 \rangle U^{M} \varphi_0 = \langle f, U^{-1} \varphi_0 \rangle U^{0} \varphi_0 = \langle f, U^{-1} \varphi_0 \rangle \varphi_0. \tag{A.5}$$

Hence, there is a $c \in \mathbb{C}$ so that $U^M \varphi_0 = c\varphi_0$. Replacing f by $U^{-1}f$ in (A.5) gives:

$$\begin{aligned} \langle U^{-1} f, U^{M-1} \varphi_0 \rangle U^M \varphi_0 &= \langle f, U^M \varphi_0 \rangle U^M \varphi_0 \\ &= \langle f, c\varphi_0 \rangle c\varphi_0 = |c|^2 \langle f, \varphi_0 \rangle \varphi_0 \\ &= \langle U^{-1} f, U^{-1} \varphi_0 \rangle \varphi_0 = \langle f, \varphi_0 \rangle \varphi_0. \end{aligned}$$

So $|c|^2 = 1$. Also, for all $0 \le k \le M-1$,

$$U^M U^k \varphi_0 = U^k U^M \varphi_0 = U^k c \varphi_0 = c U^k \varphi_0.$$

Since $\{U^k \varphi_0\}$ spans $\mathbb{H}_N$, that is, for any $f \in \mathbb{H}_N$ $f = \sum_k \langle f, U^k \varphi_0 \rangle U^k \varphi_0$, it follows that $U^M = cI$. This completes Step I.

Step II: We want to prove:

1. U is diagonalizable with respect to an orthonormal basis $\{e_i\}_{i=1}^N$ with diagonal elements $\{c_i\}_{i=1}^N$ with c_i an Mth root of c.

2. $\varphi_0 = \sum_{i=1}^N b_i e_i$ and $|b_i| = \sqrt{\frac{1}{M}}$, for all $1 \le i \le N$.

These two steps give us frame elements as in Definition 2.1.

Since U is unitary (and hence normal), and our space is finite dimensional, it follows that U is diagonalizable with respect to an orthonormal basis $\{e_i\}_{i=1}^N$ with diagonal elements $\{c_i\}_{i=1}^N$. Writing φ_0 as in (2), we have that

$$U^k\varphi_0 \;=\; \sum_{i=1}^N c_i^k b_i e_i.$$

Since $U^M = cI$, it follows that each c_i is an Mth root of c. Also, since the frame is an NTF, for all $1 \le j \le N$ we have

$$1 \;=\; \sum_{k=1}^M |\langle e_j, U^k\varphi_0\rangle|^2 \;=\; \sum_{k=1}^M |\langle e_j, \sum_{i=1}^N c_i^k b_i e_i\rangle|^2 \;=\; \sum_{k=1}^N |c_j^k b_j|^2 \;=\; M|b_j|^2.$$

So $|b_j| = 1/\sqrt{M}$. This completes Step II.

Step III: We finally prove that the c_j are distinct Mth roots of c. This will complete the proof.

Fix $1 \le m \ne \ell \le M$. Since the columns of a normalized tight frame are orthogonal vectors, that is, $F^*F = I$, we have for columns m, ℓ,

$$0 \;=\; \sum_{k=0}^{M-1} b_m c_m^k \overline{b_\ell c_\ell^k} \;=\; b_m \overline{b_\ell} \sum_{k=0}^{M-1} (c_m \overline{c_\ell})^k.$$

Hence,

$$\sum_{k=0}^{M-1} (c_m \overline{c_\ell})^k \;=\; 0.$$

Thus, c_m and c_ℓ are distinct Mth roots of c. This completes the proof of Step III and the theorem.

A.3. PROOF OF PROPOSITION 5.5

For all $1 \le j \le L$, let

$$g_j \;=\; \sum_{i=1}^M \langle g_j, e_i\rangle e_i,$$

be an orthonormal basis for $\mathbb{H}_L$. We will construct the projections $\{P_j\}_{j=1}^L$ by finite induction. We start with P_1. Since $\{(I-P)e_i\}_{i=1}^M$ is robust to 1 erasure, by Theorem 5.1, for every $1 \le i \le M$ there exists a $1 \le j \le L$ so that $\langle g_j, e_i\rangle \ne 0$. Choose scalars $a_1, a_2 \ldots, a_L$ so that for any $1 \le j \le L$, if $\langle g_j, e_i\rangle \ne 0$ for some $1 \le i \le M$, then

$$|a_j\langle g_j, e_i\rangle| \;\ge\; 2\sum_{k=1}^{j-1} |a_k\langle g_k, e_i\rangle|.$$

It follows easily that

$$\varphi_1 \;=\; \frac{\sum_{j=1}^L a_j g_j}{\|\sum_{j=1}^L a_j g_j\|},$$

then $\langle \varphi_1, e_i\rangle \neq 0$, for all $1 \le i \le M$. Now, let P_1 be the orthogonal projection onto the span $\{\varphi_1\}$, then $\{(I-P_1)e_i\}_{i=1}^M$ is robust to 1 erasure by Corollary 5.1.

Now assume we have found orthonormal vectors $\{\varphi_j\}_{j=1}^n$, $n \le k-1$ so that the rank-1 projections P_j onto span $\{\varphi_j\}$ satisfy the proposition and span $\{\varphi_j\}_{j=1}^n =$ span $\{g_j\}_{j=1}^n$. We will construct φ_{n+1} and P_{n+1}. Let $\{g_j\}_{j=1}^L$ be an orthonormal basis for the orthogonal complement of span $\{\varphi_j\}_{j=1}^n$ in $\mathbb{H}_L$. Fix $1 \le i_1 < i_2 < \ldots < i_{n+1} \le M$. Since

$$\{(I-P_n)(I-P_{n-1})\cdots(I-P_1)e_i\}_{i=1}^M$$

is robust to n erasures, if we row reduce $\{\varphi_j\}_{j=1}^n$ using the columns $\{i_1, i_2, \ldots i_n\}$ (and switching rows if necessary) we obtain vectors $\{h_j\}_{j=1}^n$ with span $\{h_j\}_{j=1}^n =$ span $\{\varphi_j\}_{j=1}^n$ and for every $1 \le j \le n$ we have,

$$\langle h_j, e_{i_l}\rangle \;=\; \delta_{j\ell}.$$

Now we verify the following claim:
Claim: For all but finitely many $0 < x < 1$, if

$$h_{n+1} \;=\; \sum_{j=n+1}^{L} x^j g_j,$$

then the row rank of

$$(\langle h_j, e_{i_\ell}\rangle)_{j,\ell=1}^{n+1},$$

is $n+1$.
Proof of Claim: Fix $0 < x < 1$. We row reduce this matrix by taking, for all $1 \le \ell \le M$,

$$-\langle \sum_{j=n+1}^{L} x^j g_j, e_{i_\ell}\rangle h_\ell \;+\; h_{n+1}.$$

We then arrive at a matrix with 1s in the jth row and the i_jth column for $1 \le j \le n$, zeroes otherwise, zeroes in the $(n+1)$st row for columns i_j for all $1 \le j \le n$, and in the $(n+1, n+1)$st position we have:

$$\langle h_{n+1}, e_{i_{n+1}}\rangle \;-\; \sum_{\ell=1}^{n}\langle \sum_{j=n+1}^{L} x^j g_j, e_{i_\ell}\rangle\langle h_\ell, e_{i_\ell}\rangle.$$

If this number is nonzero, then our matrix is of rank $n+1$. Now we check what it takes for this to be nonzero.

$$\langle h_{n+1}, e_{i_{n+1}}\rangle \;-\; \sum_{\ell=1}^{n}\langle \sum_{j=n+1}^{L} x^j g_j, e_{i_\ell}\rangle\langle h_\ell, e_{i_\ell}\rangle$$

$$= \sum_{j=n+1}^{L} x^j \langle g_j, e_{i_{n+1}}\rangle - \sum_{j=n+1}^{L} x^j \sum_{j=1}^{n}\langle g_j, e_{i_\ell}\rangle\langle h_\ell, e_{i_\ell}\rangle$$

$$= \sum_{j=n+1}^{L} x^j \left[\langle g_j, e_{i_{n+1}} \rangle - \sum_{\ell=1}^{n} \langle g_j, e_{i_\ell} \rangle \langle h_\ell, e_{i_\ell} \rangle \right].$$

By the hypotheses on the proposition,

$$(\langle g_j, e_{i_\ell} \rangle)_{j=1,\ell=1}^{L \quad n+1},$$

has rank $n+1$. It follows that there is a $n+1 \leq j \leq L$ so that

$$\langle g_j, e_{i_{n+1}} \rangle - \sum_{\ell=1}^{n} \langle g_j, e_{i_\ell} \rangle \langle h_\ell, e_{i_\ell} \rangle \neq 0.$$

Hence, the number of x's with

$$\langle h_{n+1}, e_{i_{n+1}} \rangle - \sum_{\ell=1}^{n} \langle \sum_{j=n+1}^{L} x^j g_j, e_{i_\ell} \rangle \langle h_\ell, e_{i_\ell} \rangle = 0,$$

is finite. This establishes the claim.

Applying the claim to all choices of $1 \leq i_1 < i_2 < \cdots i_{n+1} \leq M$, we see that for all but finitely many $0 < x < 1$, for all $1 \leq i_1 < i_2 < \cdots < i_{n+1} \leq L$, the rank of

$$(\langle h_j, e_{i_\ell} \rangle)_{j=1,\ell=1}^{M \quad n+1}$$

is $n+1$. Let

$$\varphi_{n+1} = \frac{h_{n+1}}{\|h_{n+1}\|},$$

and let P_{n+1} be the orthogonal projection of $\mathbb{H}_M$ onto span $\{\varphi_{n+1}\}$. Now,

$$\{(I - P_{n+1})(I - P_n) \cdots (I - P_1) e_i\}_{i=1}^{M}$$

is robust to $(n+1)$ erasures by Theorem 5.1. This completes the proof of the proposition.

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