

EXCESSES OF GABOR FRAMES

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ABSTRACT. A Gabor system for $L^2(\mathbf{R}^d)$ has the form $\mathcal{G}(g, \Lambda) = \{e^{2\pi i b \cdot x} g(x - a)\}_{(a,b) \in \Lambda}$, where $g \in L^2(\mathbf{R}^d)$ and Λ is a sequence of points in \mathbf{R}^{2d} . We prove that, with only a mild restriction on the generator g and for nearly arbitrary sets of time-frequency shifts Λ , an overcomplete Gabor frame has infinite excess, and in fact there exists an infinite subset that can be removed yet leave a frame. The proof of this result yields an interesting connection between the density of Λ and the excess of the frame.

1. INTRODUCTION

A countable sequence $\mathcal{F} = \{f_i\}_{i \in I}$ of elements of a Hilbert space H is a *frame* for H if there exist constants $A, B > 0$ (called *frame bounds*) such that

$$\forall h \in H, \quad A \|h\|^2 \leq \sum_{i \in I} |\langle h, f_i \rangle|^2 \leq B \|h\|^2.$$

The frame is *tight* if we can take $A = B$. It is a *normalized tight frame* or a *Parseval frame* if we can take $A = B = 1$.

Frames were first introduced by Duffin and Schaeffer [DS52] in the context of nonharmonic Fourier series, and have since seen a wide variety of applications in science, mathematics, and engineering. The *frame operator* $Sh = \sum_{i \in I} \langle h, f_i \rangle f_i$ is a positive, continuous mapping of H onto itself with continuous inverse. The frame \mathcal{F} together with its *standard dual frame* $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i \in I} = \{S^{-1} f_i\}_{i \in I}$ provides the *frame expansions*

$$h = \sum_{i \in I} \langle h, \tilde{f}_i \rangle f_i = \sum_{i \in I} \langle h, f_i \rangle \tilde{f}_i.$$

However, these representations need not be unique, i.e., \mathcal{F} need not be a basis. In fact, \mathcal{F} is a basis if and only if it is a Riesz basis. We say that a frame that is not a basis is *overcomplete* or *redundant*. For each $j \in I$, $\mathcal{F} \setminus \{f_j\} = \{f_i\}_{i \neq j}$ is either incomplete or is itself a frame. The *excess* of \mathcal{F} , denoted $e(\mathcal{F})$, is the supremum of the cardinalities of all subsets $J \subset I$ such that $\{f_i\}_{i \in I \setminus J}$ is complete in H .

The prior paper [BCHL02] studied the excess of frames. Among other results, it was shown that the supremum in the definition of excess is achieved, so in particular if $e(\mathcal{F}) = \infty$ then there is an infinite subset $J \subset I$ such that $\{f_i\}_{i \in I \setminus J}$ is complete. However, it need not be true that $\{f_i\}_{i \in I \setminus J}$ is a frame (even if different choices of J are allowed). Several characterizations

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of when there does exist an infinite subset J such that $\{f_i\}_{i \in I \setminus J}$ is a frame were obtained in [BCHL02], cf. Theorem 2.1 below.

In this paper we are concerned with the special case of *Gabor frames* for the Hilbert space $L^2(\mathbf{R}^d)$. For $x, \omega \in \mathbf{R}^d$, let $T_x f(t) = f(t - x)$ and $M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)$ denote the unitary operators of translation and modulation. Then we define a *time-frequency shift* of a function f on \mathbf{R}^d to be

$$\pi(z)f(t) = M_\omega T_x f(t) = e^{2\pi i \omega \cdot t} f(t - x), \quad z = (x, \omega) \in \mathbf{R}^{2d} = \mathbf{R}^d \times \mathbf{R}^d.$$

Given a fixed *window function* $g \in L^2(\mathbf{R}^d)$ and given a sequence Λ of points in \mathbf{R}^{2d} (repetitions are allowed), the *Gabor system* generated by g and Λ is

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g\}_{\lambda \in \Lambda}.$$

A Gabor system which is a frame is called a Gabor frame. We refer to [Dau92], [Grö01], or [HW89] for background information on Gabor and other frames.

Most results on Gabor systems require some kind of structural assumption on Λ , usually that $\Lambda = \alpha \mathbf{Z}^d \times \beta \mathbf{Z}^d$ (a “rectangular lattice”). Yet Gabor systems with respect to other sequences Λ of time-frequency shifts arise naturally, for example by perturbations of a regular system or directly from the constraints of an application. For example, in [SB01] a Gabor frame with a non-rectangular lattice Λ is applied to wireless coding. It is shown in [LW02] that Gabor systems which are orthonormal bases for $L^2(\mathbf{R}^d)$ can exist even with completely aperiodic Λ .

Very few theoretical results are available for Gabor systems with an arbitrary sequence of time-frequency shifts Λ . Among these, the Ramanathan-Steger density theorem [RS95] stands out as strikingly elegant, both in statement and proof. This result (as extended in [CDH99]) is as follows.

Theorem 1.1 (Density Theorem). *a. If $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbf{R}^d)$, then $1 \leq D^-(\Lambda) \leq D^+(\Lambda) < \infty$.*

b. If $\mathcal{G}(g, \Lambda)$ is a Riesz basis for $L^2(\mathbf{R}^d)$, then $D^-(\Lambda) = D^+(\Lambda) = 1$.

Here $D^\pm(\Lambda)$ are the upper and lower Beurling densities of Λ , which measure in some sense the average number of elements of Λ lying inside sets of unit measure (defined precisely in Section 2.1). To prove Theorem 1.1, Ramanathan and Steger showed that each Gabor frame satisfies a certain Homogeneous Approximation Property (HAP). This HAP seems to be of independent interest, yet, so far as we are aware, no application of it to results other than density conditions has been made.

Our first main result is stated in the next theorem (Theorem 1.2). It states that, with only a mild restriction on g and the assumption that $D^+(\Lambda) > 1$, there is a fundamental connection between a certain quantity directly tied to the excess of a Gabor frame and the density of that frame. Moreover, the HAP plays an important role in the proof. An immediate consequence of this relationship is that not only is the excess of the Gabor frame infinite, but there exists an infinite subset that can be removed yet still leave a frame. The form of the inequality (1) in Theorem 1.2 suggests the potential for additional insights into frame theory in general by examining trace-like features of the projection operator associated to a frame.

The modulation space M^p appearing in the statement of the following theorem is defined precisely in Section 2.4. We remark here only that membership in M^p corresponds to a certain amount of joint localization in both time and frequency, that M^p is dense in L^2 for $p < 2$, and that $M^2 = L^2$. The set $I(r, z)$ appearing in the statement of the theorem is the intersection of Λ with the cube $Q(r, z)$ centered at z with side lengths r .

Theorem 1.2. *Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame for $L^2(\mathbf{R}^d)$. If $g \in \bigcup_{1 \leq p < 2} M^p$, then*

$$\liminf_{r \rightarrow \infty} \inf_{z \in \mathbf{R}^{2d}} \frac{1}{|I(r, z)|} \sum_{\lambda \in I(r, z)} \langle g_\lambda, \tilde{g}_\lambda \rangle \leq \frac{1}{D^+(\Lambda)}. \quad (1)$$

Consequently, if $D^+(\Lambda) > 1$ then there exists an infinite subset J of Λ such that $\mathcal{G}(g, \Lambda \setminus J)$ is a frame for $L^2(\mathbf{R}^d)$.

If $\mathcal{G}(g, \Lambda)$ is an overcomplete frame and Λ is a *lattice* in \mathbf{R}^{2d} , then necessarily $D^+(\Lambda) > 1$ (a lattice is the image of \mathbf{Z}^{2d} under an invertible linear transformation). However, this is not the case when Λ is not a lattice. For example, we can start with a Gabor Riesz basis $\mathcal{G}(g, \Lambda)$, which by the Density Theorem must satisfy $D^-(\Lambda) = D^+(\Lambda) = 1$, and add finitely many points to Λ (or even infinitely many if judiciously chosen) to obtain an overcomplete Gabor frame with the same density. This marginal case is not addressed by Theorem 1.2.

Theorem 1.2 can be extended to the case of frames of the form $\mathcal{G}(g_1, \Lambda_1) \cup \dots \cup \mathcal{G}(g_r, \Lambda_r)$. Our second main result states that in the rectangular lattice setting, i.e., the case where Λ_k has the form $\alpha_k \mathbf{Z}^d \times \beta_k \mathbf{Z}^d$, the assumption in Theorem 1.2 that g lies in some modulation space M^p can be removed. Additionally, for this result we only need to require that the system be a frame for its closed span, not for the entire space. This result was obtained in [BCHL02] for the special case that either $(\alpha_1, \dots, \alpha_r)$ or $(\beta_1, \dots, \beta_r)$ are rationally related, including in particular the case $r = 1$. We present in this paper a new approach that applies even to the irrationally related case.

Theorem 1.3. *Let $g_1, \dots, g_r \in L^2(\mathbf{R}^d)$, and let $\Lambda_k = \alpha_k \mathbf{Z}^d \times \beta_k \mathbf{Z}^d$ for $k = 1, \dots, r$. If $\mathcal{F} = \mathcal{G}(g_1, \Lambda_1) \cup \dots \cup \mathcal{G}(g_r, \Lambda_r)$ is an overcomplete frame for its closed span H in $L^2(\mathbf{R}^d)$, then this frame has infinite excess and there exists an infinite subset of \mathcal{F} that can be removed yet leave a frame for H . In fact, this subset can be taken to have the form $\{T_{\alpha_k n_j} g_k\}_{j=1}^\infty$, i.e., translates of one of the generators g_k .*

Theorem 1.3 can be extended to more general lattices by applying a metaplectic transformation, cf. [Grö01, Sec. 9.4] or [GHHK02] for background information on this type of extension. In particular, by applying a metaplectic transformation, Theorem 1.3 can be extended to the case where each Λ_i is a symplectic lattice in \mathbf{R}^{2d} with respect to the same symplectic matrix, i.e., each Λ_i has the form $\Lambda_i = A(\alpha_i \mathbf{Z}^d \times \beta_i \mathbf{Z}^d)$ where A is a fixed $2d \times 2d$ symplectic matrix. When $d = 1$, every lattice in \mathbf{R}^2 is a symplectic lattice, but this is not the case when $d > 1$. Specializing to the case $d = 1$ and a single generator therefore yields the following corollary.

Corollary 1.4. *Let $g \in L^2(\mathbf{R})$, and let Λ be a lattice in \mathbf{R}^2 . If $\mathcal{F} = \mathcal{G}(g, \Lambda)$ is an overcomplete frame for its closed span H in $L^2(\mathbf{R})$, then this frame has infinite excess and there exists an infinite subset of \mathcal{F} that can be removed yet leave a frame for H .*

The techniques used to prove Theorem 1.3 can also be applied to the case of wavelets. Given $a > 1$ and $b > 0$, define the wavelet system generated by g , a , b to be

$$\mathcal{W}(g, a, b) = \{a^{nd/2}g(a^n x - mb)\}_{m \in \mathbf{Z}^d, n \in \mathbf{Z}}.$$

Then the following result can be proved similarly to Theorem 1.3.

Theorem 1.5. *Let $g_1, \dots, g_r \in L^2(\mathbf{R}^d)$, let $a_1, \dots, a_r > 1$, and let $b_1, \dots, b_r > 0$ be given. If $\mathcal{F} = \mathcal{W}(g_1, a_1, b_1) \cup \dots \cup \mathcal{W}(g_r, a_r, b_r)$ is an overcomplete frame for its closed span H in $L^2(\mathbf{R}^d)$, then this frame has infinite excess and there exists an infinite subset of \mathcal{F} that can be removed yet leave a frame for H . In fact, this subset can be taken to consist of dilates of one of the generators g_k .*

2. PRELIMINARIES

2.1. General Notation. Let Λ be a sequence of points in \mathbf{R}^{2d} . Then the *lower and upper Beurling densities* of Λ are, respectively,

$$D^-(\Lambda) = \liminf_{r \rightarrow \infty} \inf_{z \in \mathbf{R}^{2d}} \frac{|\Lambda \cap Q(r, z)|}{r^{2d}} \quad \text{and} \quad D^+(\Lambda) = \limsup_{r \rightarrow \infty} \sup_{z \in \mathbf{R}^{2d}} \frac{|\Lambda \cap Q(r, z)|}{r^{2d}},$$

where $|E|$ denotes the cardinality of a set E , and where $Q(r, z)$ is the cube centered at $z = (z_1, \dots, z_{2d}) \in \mathbf{R}^{2d}$ with side lengths r , i.e.,

$$Q(r, z) = \prod_{i=1}^{2d} [z_i - \frac{r}{2}, z_i + \frac{r}{2}].$$

2.2. Excess. The following result from [BCHL02] will play an important role.

Theorem 2.1. *Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a frame for a Hilbert space H , with standard dual frame $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i \in I}$. Then the following statements are equivalent.*

- a. *There exists an infinite $J_1 \subset I$ such that $\{f_i\}_{i \in J_1}$ is a frame for H .*
- b. *There exists $L > 0$ and an infinite $J_2 \subset I$ such that for each $j \in J_2$, $\{f_i\}_{i \neq j}$ is a frame for H with lower frame bound L .*
- c. *There exists an infinite $J_3 \subset I$ such that $\sup_{i \in J_3} \langle f_i, \tilde{f}_i \rangle < 1$.*

Remark 2.2. a. It is easy to see that $0 \leq \langle f_i, \tilde{f}_i \rangle \leq 1$ for any frame, because $\langle f_i, \tilde{f}_i \rangle = \|S^{-1/2} f_i\|^2$ and $\{S^{-1/2} f_i\}_{i \in I}$ is a Parseval frame for H .

b. Although the sets J_1, J_2, J_3 in Theorem 2.1 need not coincide in general, we do have $J_1 \subset J_2$.

c. Theorem 2.1 can be refined to include sharp frame bound estimates, cf. [BCHL02] for details.

d. If $T(f) = \{\langle f, f_i \rangle\}_{i \in I}$ is the *analysis operator* for the frame \mathcal{F} , then $P = T(T^*T)^{-1}T^*$ is the orthogonal projection of $\ell^2(I)$ onto $\text{range}(T)$. The diagonal elements of the matrix representation for P in the standard basis for $\ell^2(I)$ are $\langle f_i, \tilde{f}_i \rangle$.

The following result provides a useful sufficient condition for ensuring that statement c in Theorem 2.1 will hold.

Lemma 2.3. *Let $a = (a_i)_{i \in I}$ be a countable sequence of real numbers with $0 \leq a_i \leq 1$ for each i . If there exist finite subsets I_n of I such that $\lim |I_n| = \infty$ and*

$$\liminf_{n \rightarrow \infty} \frac{1}{|I_n|} \sum_{i \in I_n} a_i < 1, \quad (2)$$

then there is an infinite subset $J \subset I$ such that $\sup_{j \in J} a_j < 1$.

Proof. Let $r = \liminf_{n \rightarrow \infty} \frac{1}{|I_n|} \sum_{i \in I_n} a_i$, and choose s, ε so that $r < s - \varepsilon < s < 1$. Define $F_n = \{i \in I_n : a_i \leq s\}$. By (2), there exist $n_k \rightarrow \infty$ such that

$$\frac{1}{|I_{n_k}|} \sum_{i \in I_{n_k}} a_i \leq s - \varepsilon. \quad (3)$$

At most $|F_{n_k}|$ terms in the summation on the left side of (3) are smaller than s , so we have

$$s - \varepsilon \geq \frac{1}{|I_{n_k}|} \sum_{i \in I_{n_k}} a_i \geq \frac{s(|I_{n_k}| - |F_{n_k}|)}{|I_{n_k}|} = s - \frac{|F_{n_k}|}{|I_{n_k}|}.$$

Hence $|F_{n_k}|/|I_{n_k}| \geq \varepsilon$ for each k . Since $\lim |I_n| = \infty$, it follows that $|\bigcup F_n| = \infty$. \square

2.3. Deletions from Frames. In this section we will prove some new results which extend Theorem 2.1 further.

Theorem 2.4. *Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a frame for a Hilbert space H , with frame bounds A, B . Let $J \subset I$ be given, and define truncated analysis operators $T_J : H \rightarrow \ell^2(J)$ and $T_{I \setminus J} : H \rightarrow \ell^2(I \setminus J)$ by*

$$T_J(f) = (\langle f, f_i \rangle)_{i \in J} \quad \text{and} \quad T_{I \setminus J}(f) = (\langle f, f_i \rangle)_{i \in I \setminus J}.$$

Then the following statements hold.

a. *If there exists a bounded operator $L : \ell^2(J) \rightarrow \ell^2(I \setminus J)$ such that*

$$\gamma = \|T_J^* - T_{I \setminus J}^* L\|^2 < \frac{A}{2}, \quad (4)$$

then $\mathcal{F}' = \{f_i\}_{i \in I \setminus J}$ is a frame for H , with frame bounds $A' = \frac{A-2\gamma}{1+2\|L\|^2}$, $B' = B$.

b. *If $\mathcal{F}' = \{f_i\}_{i \in I \setminus J}$ is a frame for H , then there exists a bounded operator $L : \ell^2(J) \rightarrow \ell^2(I \setminus J)$ such that (4) holds with $\gamma = 0$.*

Proof. a. Assume L satisfies (4). Then

$$\begin{aligned} \|T_J f\|^2 &\leq (\|T_J f - L^* T_{I \setminus J} f\| + \|L^* T_{I \setminus J} f\|)^2 \\ &\leq 2\|T_J f - L^* T_{I \setminus J} f\|^2 + 2\|L^* T_{I \setminus J} f\|^2 \\ &\leq 2\gamma \|f\|^2 + 2\|L\|^2 \|T_{I \setminus J} f\|^2. \end{aligned}$$

Therefore,

$$A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 = \|T_J f\|^2 + \|T_{I \setminus J} f\|^2 \leq 2\gamma \|f\|^2 + (1 + 2\|L\|^2) \|T_{I \setminus J} f\|^2.$$

Consequently,

$$A' \|f\|^2 = \frac{A - 2\gamma}{1 + 2\|L\|^2} \|f\|^2 \leq \|T_{I \setminus J} f\|^2 = \sum_{i \in I \setminus J} |\langle f, f_i \rangle|^2,$$

which establishes that \mathcal{F}' has a lower frame bound of A' . The upper frame bound is trivial since \mathcal{F}' is a subset of \mathcal{F} .

b. Assume $\mathcal{F}' = \{f_i\}_{i \in I \setminus J}$ is a frame for H , and let $\tilde{\mathcal{F}}'$ be the standard dual frame of \mathcal{F}' (in general, this will not be the same as the standard dual frame of \mathcal{F}). Let $\tilde{T}_{I \setminus J}$ be the analysis operator for $\tilde{\mathcal{F}}'$. Then $T_{I \setminus J}^* \tilde{T}_{I \setminus J} = \mathbf{1}$, the identity operator on H . Define $L = \tilde{T}_{I \setminus J} T_J^*$. Then $T_J^* - T_{I \setminus J}^* L = 0$, so (4) is satisfied with $\gamma = 0$. \square

Specializing to the case of removing a single element yields the following corollary, which will play an important role in the proof of Theorem 1.3 that is presented in Section 4 below.

Corollary 2.5. *Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a frame for a Hilbert space H , with frame bounds A, B . Let $j \in I$ be given. If there exists a sequence $a = (a_i)_{i \neq j} \in \ell^2(I \setminus \{j\})$ such that*

$$\gamma = \left\| f_j - \sum_{i \neq j} a_i f_i \right\|^2 < \frac{A}{2},$$

then $\mathcal{F}' = \{f_i\}_{i \neq j}$ is a frame for H with frame bounds $A' = \frac{A - 2\gamma}{1 + 2\|a\|_{\ell^2}^2}$, $B' = B$.

Proof. Set $J = \{j\}$, and define $L: \mathbf{C} \rightarrow \ell^2(I \setminus \{j\})$ by $L(c) = ca$. Then $\|L\| = \|a\|_{\ell^2}$, and

$$(T_J^* - T_{I \setminus J}^* L)(c) = cf_j - \sum_{i \neq j} ca_i f_i, \quad c \in \mathbf{C}.$$

Hence

$$\gamma = \|T_J^* - T_{I \setminus J}^* L\|^2 = \left\| f_j - \sum_{i \neq j} a_i f_i \right\|^2,$$

so the result follows from Theorem 2.4. \square

2.4. Modulation Spaces. The modulation spaces quantify the time-frequency content of a function or distribution. They appear naturally in time-frequency analysis, and we refer to [Grö01] for detailed discussion and applications.

For our purposes, the following special case of unweighted modulation spaces will be sufficient. Let $G(x) = 2^{d/4}e^{-x \cdot x}$ be the Gaussian function, normalized so that $\|G\|_2 = 1$. Then for $1 \leq p \leq \infty$, the modulation space M^p consists of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{M^p} = \left(\int_{\mathbf{R}^{2d}} |\langle f, \pi(z)G \rangle|^p dz \right)^{1/p} < \infty, \quad (5)$$

with the usual adjustment if $p = \infty$.

Remark 2.6. a. M^p is a Banach space for each $1 \leq p \leq \infty$. Any nonzero function $g \in M^1$ (including all Schwartz-class functions in particular) can be substituted for the Gaussian G in (5) to produce an equivalent norm for M^p , cf. Lemma A.2 below.

b. $M^2 = L^2$, and $\mathcal{S} \subsetneq M^p \subsetneq M^q \subsetneq \mathcal{S}'$ for $1 \leq p < q \leq \infty$, where \mathcal{S} is the Schwartz class.

c. $(M^p)' = M^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

d. M^p is isometric under time-frequency shifts, i.e.,

$$\forall z \in \mathbf{R}^{2d}, \quad \|\pi(z)f\|_{M^p} = \|f\|_{M^p}.$$

e. If Q is a cube in \mathbf{R}^d , then $h = \mathbf{1}_Q \in M^p$ for $1 < p \leq \infty$.

The case $q = 1$ of the following proposition is a standard result for the modulation spaces. We will require the following extension to other values of q . A proof of this proposition was provided to us by K. Gröchenig and E. Cordero, and is reported in the Appendix.

Proposition 2.7. *Let $1 \leq p, q, r \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, and let Λ be a sequence of points in \mathbf{R}^{2d} satisfying $D^+(\Lambda) < \infty$. Then there exists a constant $C = C(p, q, \Lambda) > 0$ such that*

$$\forall g \in M^p, \quad \forall f \in M^q, \quad \left(\sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^r \right)^{1/r} \leq C \|g\|_{M^p} \|f\|_{M^q}.$$

2.5. A Recurrence Lemma. The following recurrence lemma will be used later in the proof of Theorem 1.3.

Lemma 2.8. *Let $\alpha_1, \dots, \alpha_r > 0$ be given, and fix $\delta > 0$. Then there exist infinitely many points $(n_j^1, \dots, n_j^r) \in \mathbf{Z}^d \times \dots \times \mathbf{Z}^d$ such that for each $j = 1, 2, \dots$ we have*

$$|\alpha_1 n_j^1 - \alpha_k n_j^k| < \delta, \quad k = 2, \dots, r.$$

Proof. It suffices to prove the case $d = 1$. Let $\mathbf{T}^r = [0, \alpha_1) \times \dots \times [0, \alpha_r)$ be the r -torus, and define a translation $T: \mathbf{T}^r \rightarrow \mathbf{T}^r$ by $T(x_1, \dots, x_r) = (x_1 + 1 \bmod \alpha_1, \dots, x_r + 1 \bmod \alpha_r)$. Let U be the open ball of radius $\delta/2$ centered at 0 in \mathbf{T}^r . Then, since T is a measure-preserving mapping, we have by the Poincaré Recurrence Theorem ([Sin89, p. 11] or [Wal82, Thm. 1.4]) that almost every point of U returns to U infinitely often under iteration by T . Let $a \in U$

be any such point. Then there exist infinitely many positive integers $N_1 < N_2 < \dots$ such that $T^{N_j}(a) \in U$, i.e., for each j we have

$$|(N_j + a) \bmod \alpha_k| < \frac{\delta}{2}, \quad k = 1, \dots, r.$$

Hence, there exist integers n_j^k such that

$$|(N_j + a) - n_j^k \alpha_k| < \frac{\delta}{2}, \quad k = 1, \dots, r,$$

and consequently,

$$|n_j^1 \alpha_1 - n_j^k \alpha_k| < \delta, \quad k = 2, \dots, r.$$

By taking δ small enough, we are assured that the integers n_j^1 are distinct, which completes the proof. \square

3. PROOF OF THEOREM 1.2

We will prove Theorem 1.2 in this section. We break the proof down into several smaller steps. We assume throughout this section that $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbf{R}^d)$ with frame bounds A, B , that g lies in M^p for some $1 \leq p < 2$, and that $1 < D^+(\Lambda) < \infty$.

The frame operator for $\mathcal{G}(g, \Lambda)$ is $Sf = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g$. The standard dual frame of $\mathcal{G}(g, \Lambda)$ is $\tilde{\mathcal{G}} = \{\tilde{g}_\lambda\}_{\lambda \in \Lambda}$ where $\tilde{g}_\lambda = S^{-1}(\pi(\lambda)g)$. If Λ is a lattice in \mathbf{R}^{2d} , then it can be shown that this dual frame is itself a Gabor frame of the form $\mathcal{G}(\gamma, \Lambda)$, but this need not be the case when Λ is not a lattice. For simplicity of notation, we will write

$$g_\lambda = \pi(\lambda)g,$$

but we emphasize that \tilde{g}_λ need not be of the form $\pi(\lambda)\gamma$.

3.1. Goal. Our goal is to show that equation (1) holds, and that this implies that an infinite subset J of Λ can be found such that $\mathcal{G}(g, \Lambda \setminus J)$ is still a frame. To see how this second statement is a consequence of the first, recall from Theorem 2.1 that to show that there is an infinite subset of Λ which may be removed yet leave a frame, we need to show that there exists some (possibly different) infinite subset $J \subset \Lambda$ such that

$$\sup_{\lambda \in J} \langle g_\lambda, \tilde{g}_\lambda \rangle < 1. \quad (6)$$

Further, by Lemma 2.3, to do this it suffices to show that

$$\liminf_{r \rightarrow \infty} \inf_{z \in \mathbf{R}^{2d}} \frac{1}{|I(r, z)|} \sum_{\lambda \in I(r, z)} \langle g_\lambda, \tilde{g}_\lambda \rangle < 1,$$

where $I(r, z) = \Lambda \cap Q(r, z)$. We will show that the quantity on the left side of the preceding equation is actually bounded by $1/D^+(\Lambda)$. Hence, when $D^+(\Lambda) > 1$, there will be an infinite J for which (6) is satisfied.

For simplicity of notation, we will often use the abbreviation $I = I(r, z)$. Define the truncated frame operators

$$S_I f = \sum_{\lambda \in I} \langle f, g_\lambda \rangle g_\lambda \quad \text{and} \quad S_{\Lambda \setminus I} f = \sum_{\lambda \in \Lambda \setminus I} \langle f, g_\lambda \rangle g_\lambda.$$

We note the following basic facts.

- Lemma 3.1.** a. $\|g_\lambda\|_2^2 \leq B$ for each $\lambda \in \Lambda$.
 b. $\|\tilde{g}_\lambda\|_2^2 \leq \frac{1}{A}$ for each $\lambda \in \Lambda$.
 c. $\|S_I\|, \|S_{\Lambda \setminus I}\| \leq \|S\| \leq B$ (operator norm).
 d. $\|S^{-1}\| \leq \frac{1}{A}$.
 e. The trace of $S_I S^{-1}$ is $\text{tr}(S_I S^{-1}) = \sum_{\lambda \in I} \langle g_\lambda, \tilde{g}_\lambda \rangle$.

Using this notation, our goal is to show that

$$\liminf_{r \rightarrow \infty} \inf_{z \in \mathbf{R}^{2d}} \frac{1}{|I|} \text{tr}(S_I S^{-1}) \leq \frac{1}{D^+(\Lambda)}, \quad I = I(r, z). \quad (7)$$

3.2. **Fix ε .** Let $\varepsilon > 0$ be fixed for the remainder of this proof. Since

$$D^+(\Lambda) = \limsup_{r \rightarrow \infty} \sup_{z \in \mathbf{R}^{2d}} \frac{|I(r, z)|}{r^{2d}},$$

there exists a strictly increasing sequence $r_k \rightarrow \infty$ and points $z_k \in \mathbf{R}^{2d}$ such that

$$\forall k, \quad |I| = |I(r_k, z_k)| \geq r_k^{2d} (D^+(\Lambda) - \varepsilon). \quad (8)$$

3.3. **One Gabor Orthonormal Basis and the HAP.** Let $Q = Q(1, 0) = [-\frac{1}{2}, \frac{1}{2}]^d$, and set $h = \mathbf{1}_Q$, the characteristic function of Q . Then

$$\mathcal{G}(h, \mathbf{Z}^{2d}) = \{\pi(\delta)h : \delta \in \mathbf{Z}^{2d}\} = \{h_\delta : \delta \in \mathbf{Z}^{2d}\}$$

is a Gabor orthonormal basis for $L^2(\mathbf{R}^d)$. We will use this system as a reference for comparison to $\mathcal{G}(g, \Lambda)$. We could use another Gabor orthonormal or Riesz basis, but by the Balian-Low theorem the generator of any such basis is limited in its joint time-frequency concentration. In particular, no generator of a Gabor Riesz basis can lie in M^1 , cf. [FG97], [GHHK02]. The particular Gabor system $\mathcal{G}(h, \mathbf{Z}^{2d})$ has the advantage that $h \in M^q$ for each $q > 1$.

We apply the Homogeneous Approximation Property for the Gabor orthonormal basis $\mathcal{G}(h, \mathbf{Z}^{2d})$ to the function g . In particular, by [CDH99, Cor. 3.5], there exists an $R > 0$ such that

$$\forall z \in \mathbf{R}^{2d}, \quad \forall r \geq 0, \quad \forall \mu \in Q(r, z), \quad \|(\mathbf{1} - P_V)g_\mu\|_2 < \varepsilon, \quad (9)$$

where P_V is the orthogonal projection of $L^2(\mathbf{R}^d)$ onto

$$V = V(r + R, z) = \text{span}\{h_\delta : \delta \in \mathbf{Z}^{2d} \cap Q(r + R, z)\}.$$

We will concentrate for a while on a specific $r = r_k > R$ and $z = z_k \in \mathbf{R}^{2d}$. We will suppress some indices and write

$$\begin{aligned} I &= I(r_k, z_k) = \Lambda \cap Q(r_k, z_k), \\ V &= V(r_k + R, z_k) = \text{span}\{h_\delta : \delta \in \mathbf{Z}^{2d} \cap Q(r_k + R, z_k)\}, \\ W &= V(r_k - R, z_k) = \text{span}\{h_\delta : \delta \in \mathbf{Z}^{2d} \cap Q(r_k - R, z_k)\}, \\ U &= V(R, \lambda) = \text{span}\{h_\delta : \delta \in \mathbf{Z}^{2d} \cap Q(R, \lambda)\}, \end{aligned}$$

and let P_V, P_W, P_U denote the orthogonal projection of $L^2(\mathbf{R}^d)$ onto V, W , and U , respectively. In this notation, note that the HAP (9) implies in particular that

$$\forall \lambda \in I, \quad \|(\mathbf{1} - P_V)g_\lambda\|_2 < \varepsilon, \quad (10)$$

and that

$$\forall \lambda \in \Lambda, \quad \|(\mathbf{1} - P_U)g_\lambda\|_2 < \varepsilon. \quad (11)$$

3.4. First Estimate. Recall that our goal is to estimate $\frac{1}{|I|} \text{tr}(S_I S^{-1})$. Write

$$\text{tr}(S_I S^{-1}) = \text{tr}((\mathbf{1} - P_V)S_I S^{-1}) + \text{tr}(P_V S_I S^{-1}). \quad (12)$$

For the first term on the right of (12), observe that

$$(\mathbf{1} - P_V)S_I S^{-1}f = (\mathbf{1} - P_V) \left(\sum_{\lambda \in I} \langle S^{-1}f, g_\lambda \rangle g_\lambda \right) = \sum_{\lambda \in I} \langle f, \tilde{g}_\lambda \rangle (\mathbf{1} - P_V)g_\lambda.$$

Computing the trace, applying the HAP in the form of (10), and using the boundedness of the norms of the dual frame elements, we have

$$\text{tr}((\mathbf{1} - P_V)S_I S^{-1}) = \sum_{\lambda \in I} \langle (\mathbf{1} - P_V)g_\lambda, \tilde{g}_\lambda \rangle \leq \sum_{\lambda \in I} \|(\mathbf{1} - P_V)g_\lambda\|_2 \|\tilde{g}_\lambda\|_2 \leq \frac{\varepsilon |I|}{A^{1/2}}. \quad (13)$$

3.5. Second Estimate. Now we will work on the second term on the right of (12). We will expand that term into three parts and then simplify by using the relations

$$\begin{aligned} P_V(\mathbf{1} - P_W) &= P_{V \cap W^\perp}, \\ P_V P_W &= P_W \quad (\text{since } W \subset V), \\ S &= S_I + S_{\Lambda \setminus I}. \end{aligned}$$

The three terms in the expansion are obtained as follows:

$$\begin{aligned} \text{tr}(P_V S_I S^{-1}) &= \text{tr}(P_V(S - S_{\Lambda \setminus I})S^{-1}) \\ &= \text{tr}(P_V) - \text{tr}(P_V(\mathbf{1} - P_W)S_{\Lambda \setminus I}S^{-1}) - \text{tr}(P_V P_W S_{\Lambda \setminus I}S^{-1}) \\ &= \text{tr}(P_V) - \text{tr}(P_{V \cap W^\perp} S_{\Lambda \setminus I}S^{-1}) - \text{tr}(P_W S_{\Lambda \setminus I}S^{-1}). \end{aligned} \quad (14)$$

3.5.1. *First term.* To estimate the first term in (14), note that the dimension of V is known because $\mathcal{G}(h, \mathbf{Z}^{2d})$ is an orthonormal basis. Consequently,

$$\mathrm{tr}(P_V) = \dim(V) = |\mathbf{Z}^{2d} \cap Q(r_k + R, z_k)| \leq (r_k + R + 1)^{2d}. \quad (15)$$

3.5.2. *Second term.* For the second term in (14), note that since $\mathcal{G}(h, \mathbf{Z}^{2d})$ is an orthonormal basis, we have that

$$V \cap W^\perp = \mathrm{span}\{h_\delta : \delta \in \mathbf{Z}^{2d} \cap [Q(r_k + R, z_k) \setminus Q(r_k - R, z_k)]\}.$$

The set $Q(r_k + R, z_k) \setminus Q(r_k - R, z_k)$ is a “square annulus,” so

$$\begin{aligned} \dim(V \cap W^\perp) &= |\mathbf{Z}^{2d} \cap [Q(r_k + R, z_k) \setminus Q(r_k - R, z_k)]| \\ &\leq (r_k + R + 1)^{2d} - (r_k - R - 1)^{2d}. \end{aligned}$$

Now we apply the fact that

$$X = X^* \geq 0 \implies |\mathrm{tr}(XY)| \leq \mathrm{tr}(X) \|Y\|$$

to compute that

$$\begin{aligned} |\mathrm{tr}(P_{V \cap W^\perp} S_{\Lambda \setminus I} S^{-1})| &\leq \mathrm{tr}(P_{V \cap W^\perp}) \|S_{\Lambda \setminus I} S^{-1}\| \\ &\leq \dim(V \cap W^\perp) \|S_{\Lambda \setminus I}\| \|S^{-1}\| \\ &\leq \frac{B}{A} ((r_k + R + 1)^{2d} - (r_k - R - 1)^{2d}). \end{aligned}$$

3.5.3. *Third term.* Now we come to the third term in (14). Observe that

$$P_W S_{\Lambda \setminus I} S^{-1} f = P_W \sum_{\lambda \in \Lambda \setminus I} \langle S^{-1} f, g_\lambda \rangle g_\lambda = \sum_{\lambda \in \Lambda \setminus I} \langle f, \tilde{g}_\lambda \rangle P_W g_\lambda.$$

Let

$$D = \mathbf{Z}^{2d} \cap Q(r_k - R, z_k).$$

Then since $\mathcal{G}(h, \mathbf{Z}^{2d})$ is an orthonormal basis and since $W = \{h_\delta : \delta \in D\}$, we have $P_W h_\delta = h_\delta$ when $\delta \in D$ and $P_W h_\delta = 0$ otherwise. Hence

$$\begin{aligned} |\mathrm{tr}(P_W S_{\Lambda \setminus I} S^{-1})|^2 &= \left| \sum_{\delta \in \mathbf{Z}^{2d}} \sum_{\lambda \in \Lambda \setminus I} \langle h_\delta, \tilde{g}_\lambda \rangle \langle P_W g_\lambda, h_\delta \rangle \right|^2 \\ &= \left| \sum_{\delta \in \mathbf{Z}^{2d}} \sum_{\lambda \in \Lambda \setminus I} \langle h_\delta, \tilde{g}_\lambda \rangle \langle g_\lambda, P_W h_\delta \rangle \right|^2 \end{aligned}$$

$$\begin{aligned}
&= \left| \sum_{\delta \in D} \sum_{\lambda \in \Lambda \setminus I} \langle h_\delta, \tilde{g}_\lambda \rangle \langle g_\lambda, h_\delta \rangle \right|^2 \\
&\leq \left(\sum_{\delta \in D} \sum_{\lambda \in \Lambda \setminus I} |\langle h_\delta, \tilde{g}_\lambda \rangle|^2 \right) \left(\sum_{\delta \in D} \sum_{\lambda \in \Lambda \setminus I} |\langle g_\lambda, h_\delta \rangle|^2 \right). \tag{16}
\end{aligned}$$

We can bound the first term on the right of (16) by using the fact that $\{\tilde{g}_\lambda\}_{\lambda \in \Lambda}$ is a frame for $L^2(\mathbf{R}^d)$ with frame bounds $\frac{1}{B}$, $\frac{1}{A}$:

$$\sum_{\delta \in D} \sum_{\lambda \in \Lambda \setminus I} |\langle h_\delta, \tilde{g}_\lambda \rangle|^2 \leq \sum_{\delta \in D} \frac{1}{A} \|h_\delta\|_2^2 = \frac{|D|}{A} \leq \frac{(r_k - R + 1)^{2d}}{A}. \tag{17}$$

For the second term on the right of (16), recall that $g \in M^p$ where $1 \leq p < 2$. Fix $p < s < 2$. Then

$$\sum_{\delta \in D} \sum_{\lambda \in \Lambda \setminus I} |\langle g_\lambda, h_\delta \rangle|^2 \leq \left(\sup_{\lambda \in \Lambda \setminus I, \delta \in D} |\langle g_\lambda, h_\delta \rangle|^{2-s} \right) \sum_{\delta \in D} \sum_{\lambda \in \Lambda \setminus I} |\langle g_\lambda, h_\delta \rangle|^s. \tag{18}$$

We will estimate each of these two pieces separately.

To bound the first term on the right of (18), consider a typical $\lambda \in \Lambda \setminus I$ and $\delta \in D$. We have that

$$|\langle g_\lambda, h_\delta \rangle| = |\langle g_\lambda, P_W h_\delta \rangle| \leq \|P_W g_\lambda\|_2 \|h_\delta\|_2 = \|P_W g_\lambda\|_2.$$

Letting $U = V(R, \lambda)$, we have by the HAP in the form of (11) that

$$\|(\mathbf{1} - P_U)g_\lambda\|_2 < \varepsilon.$$

However, $W = V(r_k - R, z_k)$ and $\lambda \notin I = I(r_k, z_k)$, so it follows that $U \subset W^\perp$ and $W \subset U^\perp$. Therefore $P_W \leq P_{U^\perp} = \mathbf{1} - P_U$, so

$$\forall \delta \in D, \quad \forall \lambda \in \Lambda \setminus I, \quad |\langle g_\lambda, h_\delta \rangle| \leq \|P_W g_\lambda\|_2 \leq \|(\mathbf{1} - P_U)g_\lambda\|_2 < \varepsilon. \tag{19}$$

To estimate the second term on the right of (18), let q be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{s}$. Note that $q > 1$, so $h_\delta \in M^q$. Since M^q is invariant under time-frequency shifts, we have $\|h_\delta\|_{M^q} = \|h\|_{M^q}$ for every δ . Since $g \in M^p$, we therefore have from Proposition 2.7 that

$$\begin{aligned}
\sum_{\delta \in D} \sum_{\lambda \in \Lambda \setminus I} |\langle g_\lambda, h_\delta \rangle|^s &\leq \sum_{\delta \in D} C^s \|g\|_{M^p}^s \|h_\delta\|_{M^s}^s \\
&= C^s \|g\|_{M^p}^s \|h\|_{M^s}^s |D| \\
&\leq C_1 (r_k - R + 1)^{2d}. \tag{20}
\end{aligned}$$

Combining the estimates (16)–(20) yields

$$|\mathrm{tr}(P_W S_{\Lambda \setminus I} S^{-1})|^2 \leq \frac{(r_k - R + 1)^{2d}}{A} \varepsilon^{2-s} C_1 (r_k - R + 1)^{2d},$$

or

$$|\mathrm{tr}(P_W S_{\Lambda \setminus I} S^{-1})| \leq C_2 \varepsilon^{\frac{2-s}{2}} (r_k - R + 1)^{2d}.$$

3.6. Combine the Terms. Combining all the previous estimates yields for each $I = I(r_k, z_k)$ that

$$\begin{aligned} \frac{1}{|I|} \operatorname{tr}(S_I S^{-1}) &\leq \frac{1}{|I|} \left(\left| \operatorname{tr}((\mathbf{1} - P_V) S_I S^{-1}) \right| + \left| \operatorname{tr}(P_V) \right| + \right. \\ &\quad \left. \left| \operatorname{tr}(P_{V \cap W^\perp} S_{\Lambda \setminus I} S^{-1}) \right| + \left| \operatorname{tr}(P_W S_{\Lambda \setminus I} S^{-1}) \right| \right) \\ &\leq \frac{\varepsilon}{A^{1/2}} + \frac{(r_k + R + 1)^{2d}}{r_k^{2d} (D^+(\Lambda) - \varepsilon)} + \frac{B((r_k + R + 1)^{2d} - (r_k - R - 1)^{2d})}{A r_k^{2d} (D^+(\Lambda) - \varepsilon)} + \\ &\quad \frac{C_2 \varepsilon^{\frac{2-s}{2}} (r_k - R + 1)^{2d}}{r_k^{2d} (D^+(\Lambda) - \varepsilon)}, \end{aligned}$$

where we have bounded $1/|I|$ by using (8). Since z_k is one point in \mathbf{R}^{2d} and since $r_k \rightarrow \infty$, we therefore have

$$\liminf_{r \rightarrow \infty} \inf_{z \in \mathbf{R}^{2d}} \frac{1}{|I|} \operatorname{tr}(S_I S^{-1}) \leq \frac{\varepsilon}{A^{1/2}} + \frac{1}{D^+(\Lambda) - \varepsilon} + 0 + \frac{C_2 \varepsilon^{\frac{2-p}{2}}}{D^+(\Lambda) - \varepsilon},$$

where $I = I(r, z)$ in the equation above. Since ε was arbitrary, we conclude that

$$\liminf_{r \rightarrow \infty} \inf_{z \in \mathbf{R}^{2d}} \frac{1}{|I|} \operatorname{tr}(S_I S^{-1}) \leq \frac{1}{D^+(\Lambda)},$$

which was our goal. This completes the proof of Theorem 1.2.

4. PROOF OF THEOREM 1.3

We will prove Theorem 1.3 in this section. We are given $g_k \in L^2(\mathbf{R}^d)$ and $\Lambda_k = \alpha_k \mathbf{Z}^d \times \beta_k \mathbf{Z}^d$ for $k = 1, \dots, r$, and we assume that $\mathcal{F} = \bigcup_{k=1}^r \mathcal{G}(g_k, \Lambda_k)$ is an overcomplete frame for its closed span H in $L^2(\mathbf{R})$. We must show that some infinite subset of some $\mathcal{G}(g_j, \Lambda_j)$ can be removed from \mathcal{F} so that the remaining set is still a frame for H , and further this set can be chosen to consist of translates of a single generator g_i .

Let A, B denote the frame bounds for \mathcal{F} . For simplicity of notation, we will write the elements of \mathcal{F} as

$$g_{m,n}^k(x) = \pi(\beta_k m, \alpha_k n) g_k(x) = M_{\beta_k m} T_{\alpha_k n} g_k(x) = e^{2\pi i \beta_k m \cdot x} g_k(x - \alpha_k n),$$

where $m, n \in \mathbf{Z}^d$ and $k = 1, \dots, r$.

Since \mathcal{F} is overcomplete, there is some element which may be removed yet still leave a frame for H . Without loss of generality we may assume it is an element of $\mathcal{G}(g_1, \Lambda_1)$, say $h = g_{k,\ell}^1$. Since $h_{m,n} = e^{-2\pi i \beta_1 k \cdot \alpha_1 n} g_{m+k, n+\ell}^1$, the elements of $\mathcal{G}(h, \Lambda_1)$ are exactly the elements of $\mathcal{G}(g_1, \Lambda_1)$ except in a different order and multiplied by constants of magnitude 1 (here is one point where we make use of the assumption that the Λ_k are lattices). Without loss of generality we may therefore assume that the element removed is $g_1 = g_{0,0}^1$.

Define an index set

$$\Gamma_0 = (\{1, \dots, r\} \times \mathbf{Z}^d \times \mathbf{Z}^d) \setminus \{(1, 0, 0)\}.$$

Then

$$\mathcal{F}'_0 = \mathcal{F} \setminus \{g_{0,0}^1\} = \{g_{m,n}^k\}_{(k,m,n) \in \Gamma_0}$$

is a frame for H . We will show that there exist infinitely many indices $n_j \in \mathbf{Z}^d$ such that if we set

$$\Gamma_{n_j} = (\{1, \dots, r\} \times \mathbf{Z}^d \times \mathbf{Z}^d) \setminus \{(1, 0, n_j)\},$$

then

$$\mathcal{F}'_{n_j} = \mathcal{F} \setminus \{g_{0,n_j}^1\} = \{g_{m,n}^k\}_{(k,m,n) \in \Gamma_{n_j}}$$

is also frame for h , and furthermore all of these frames \mathcal{F}'_{n_j} have the same lower frame bound $L > 0$. It then follows from Theorem 2.1 that infinitely many elements may be removed from \mathcal{F} yet leave a frame, and furthermore, this set to be removed is a subset of $\{g_{0,n_j}^1\}_{j=1}^\infty$, which is a set of translates of g_1 .

Let S_0 be the frame operator for the frame \mathcal{F}'_0 . Define

$$a_{m,n}^k = \langle S_0^{-1}g_1, g_{m,n}^k \rangle,$$

and set $a = (a_{m,n}^k)_{(k,m,n) \in \Gamma_0}$. Note that $a \in \ell^2(\Gamma_0)$ since the scalars $a_{m,n}^k$ are the frame coefficients of $S_0^{-1}g$ with respect to the frame \mathcal{F}'_0 .

Define

$$h_1 = \sum_{(m,n) \neq (0,0)} a_{m,n}^k g_{m,n}^k$$

and

$$h_k = \sum_{m,n \in \mathbf{Z}^d} a_{m,n}^k g_{m,n}^k, \quad k = 2, \dots, r.$$

Then

$$\sum_{k=1}^r h_k = \sum_{(k,m,n) \in \Gamma_0} a_{m,n}^k g_{m,n}^k = \sum_{(k,m,n) \in \Gamma_0} \langle S_0^{-1}g_1, g_{m,n}^k \rangle g_{m,n}^k = S_0(S_0^{-1}g_1) = g_1.$$

Fix $\varepsilon < \sqrt{A/2}$. Since translation is strongly continuous in $L^2(\mathbf{R}^d)$, there exists a $\delta > 0$ such that

$$|t| < \delta \implies \|h_k - T_t h_k\|_2 < \frac{\varepsilon}{r-1}, \quad k = 2, \dots, r. \quad (21)$$

By Lemma 2.8, there exist points $(n_j^1, \dots, n_j^r) \in \mathbf{Z}^d \times \dots \times \mathbf{Z}^d$ for $j = 1, 2, \dots$ such that

$$|\alpha_1 n_j^1 - \alpha_k n_j^k| < \delta, \quad k = 2, \dots, r. \quad (22)$$

We will show that for each j ,

$$\gamma_j = \left\| g_{0,n_j^1}^1 - \sum_{(k,m,n) \in \Gamma_{n_j^1}} a_{m,n-n_j^k}^k g_{m,n}^k \right\|_2^2 \leq \varepsilon^2 < \frac{A}{2}. \quad (23)$$

Consequently, by Corollary 2.5 we will have that $\mathcal{F}'_{n_j^1}$ is a frame for H with lower frame bound $A'_j = \frac{A-2\gamma_j}{1+2\|a\|_2^2}$. Since $A'_j \geq \frac{A-2\varepsilon}{1+2\|a\|_2^2} > 0$, all the frames $\mathcal{F}'_{n_j^1}$ will share the same single positive lower frame bound, and the proof will be complete.

To prove (23), we first use (21) and (22) to we compute that

$$\left\| \sum_{k=2}^r (T_{\alpha_k n_j^k} h_k - T_{\alpha_1 n_j^1} h_k) \right\|_2 \leq \sum_{k=2}^r \|h_k - T_{\alpha_1 n_j^1 - \alpha_k n_j^k} h_k\|_2 < \varepsilon.$$

Therefore, since $g_{0, n_j^1}^1 = T_{\alpha_1 n_j^1} g_1$,

$$\left\| g_{0, n_j^1}^1 - \sum_{k=1}^r T_{\alpha_k n_j^k} h_k \right\|_2 \leq \left\| T_{\alpha_1 n_j^1} g_1 - \sum_{k=1}^r T_{\alpha_1 n_j^1} h_k \right\|_2 + \left\| \sum_{k=2}^r (T_{\alpha_1 n_j^1} h_k - T_{\alpha_k n_j^k} h_k) \right\|_2 < 0 + \varepsilon = \varepsilon.$$

Finally,

$$\sum_{k=1}^r T_{\alpha_k n_j^k} h_k = \sum_{(k, m, n) \in \Gamma_0} a_{m, n}^k g_{m, n + n_j^k}^k = \sum_{(k, m, n) \in \Gamma_{n_j^1}} a_{m, n - n_j^k}^k g_{m, n}^k,$$

so the proof is complete.

APPENDIX A. PROOF OF PROPOSITION 2.7

We will prove Proposition 2.7 in this section. We thank Karlheinz Gröchenig and Elena Cordero for providing this proof, which is related to techniques developed in [CG02].

We obtain this proof by applying interpolation. We refer to [BL76] for background on interpolation in general, and to [Fei80], [Fei83], [FG89] for results on the interpolation properties of the modulation and Wiener amalgam spaces.

Definition A.1. The *Short-Time Fourier Transform* of a tempered distribution f with respect to a window function g is

$$V_g f(z) = \langle f, \pi(z)g \rangle, \quad z \in \mathbf{R}^{2d},$$

whenever this is defined.

In particular, letting $G(x) = 2^{d/4} e^{-x \cdot x}$ denote the Gaussian window (normalized so that $\|G\|_2 = 1$), we can restate the definition of the modulation space M^p given in Section 2.4 as follows:

$$M^p = \{f \in \mathcal{S}'(\mathbf{R}^d) : \|f\|_{M^p} = \|V_G f\|_{L^p} < \infty\}.$$

The following lemma deals with the effect of replacing the Gaussian G by another function g . An extension of this lemma shows that $\|V_g f\|_{L^p}$ is actually an equivalent norm for M^p for each nonzero function $g \in M^1$.

Lemma A.2. *If $1 \leq p \leq \infty$, then*

$$\forall g \in M^1, \quad \forall f \in M^p, \quad \|V_g f\|_{L^p} \leq \|g\|_{M^1} \|f\|_{M^p}.$$

Proof. From [Grö01, eq. (11.31)], there exists a constant C such that

$$\|V_g f\|_{L^p} \leq C \|V_g G\|_{L^1} \|V_G f\|_{L^p} = C \|g\|_{M^1} \|f\|_{M^p}.$$

Further, the value of C is determined by the weight used to define the modulation space. Since we are dealing only with the unweighted case (or, equivalently, the weight is identically 1), it can be shown that the constant is $C = 1$. \square

Let $\mathcal{C} = \mathcal{C}(\mathbf{R}^d)$ denote the space of all bounded, continuous functions on \mathbf{R}^d under the L^∞ norm.

Let $Q = [-\frac{1}{2}, \frac{1}{2}]^d$. Define the Wiener amalgam space $W(\mathcal{C}, \ell^p)$ to be the space of all continuous functions f on \mathbf{R}^d for which the norm

$$\|f\|_{W(\mathcal{C}, \ell^p)} = \left(\sum_{k \in \mathbf{Z}^d} \|f \cdot \mathbf{1}_{Q+k}\|_\infty^p \right)^{1/p}$$

is finite, with the usual adjustment if $p = \infty$. Note that $W(\mathcal{C}, \ell^\infty) = L^\infty \cap \mathcal{C} = \mathcal{C}$, while $W(\mathcal{C}, \ell^p)$ is a proper subset of $L^p \cap \mathcal{C}$ when $p < \infty$. We refer to [Grö01], [HW89] for background information on Wiener amalgam spaces,

The following proposition is the key step in the proof of Proposition 2.7.

Proposition A.3. *Let $1 \leq p, q, r \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $g \in M^p$ and $f \in M^q$, then $V_g f \in W(\mathcal{C}, \ell^r)$. Further, there exists a constant $C = C(p, q)$ such that*

$$\forall g \in M^p, \quad \forall f \in M^q, \quad \|V_g f\|_{W(\mathcal{C}, \ell^r)} \leq C \|g\|_{M^p} \|f\|_{M^q}.$$

Proof. Case 1: $p = 1, 1 \leq q \leq \infty, r = q$.

It is easy to see in this case that $V_g f$ is continuous.

If $q = 1$, then we have $f, g \in M^1$, so $V_g f \in W(\mathcal{C}, \ell^1)$ by [Grö01, Prop. 12.1.11]. Further, by [Grö01], there exists a constant $C > 0$ such that

$$\|V_g f\|_{W(\mathcal{C}, \ell^1)} \leq C \|g\|_{M^1} \|f\|_{M^1}.$$

In fact, the proof of this result shows that it suffices to take $C = \|\mathbf{1}_{[-2,1]^d}\|_{L^1} = 3^d$.

If $q = \infty$, then, by Lemma A.2, we have

$$\|V_g f\|_{W(\mathcal{C}, \ell^\infty)} = \|V_g f\|_{L^\infty} \leq \|g\|_{M^1} \|f\|_{M^\infty}.$$

The result for $1 < q < \infty$ now follows from interpolation. In fact, if we set $\theta = 1/q$ then

$$\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{\infty},$$

and we have

$$[W(\mathcal{C}, \ell^1), W(\mathcal{C}, \ell^\infty)]_\theta = W(\mathcal{C}, \ell^q) \quad \text{and} \quad [M^1, M^\infty]_\theta = M^q.$$

With g fixed, applying interpolation to the mapping $f \mapsto V_g f$ therefore yields

$$\|V_g f\|_{W(\mathcal{C}, \ell^q)} \leq (3^d \|g\|_{M^1})^\theta (\|g\|_{M^1})^{1-\theta} \|f\|_{M^q} = 3^{d/q} \|g\|_{M^1} \|f\|_{M^q}.$$

Case 2: $\frac{1}{p} + \frac{1}{q} = 1, r = \infty$.

In this case we have $p = q'$, the dual index to q . Then since M^q and $M^{q'}$ are dual spaces,

$$|V_g f(z)| = |\langle f, \pi(z)g \rangle| \leq \|f\|_{M^q} \|\pi(z)g\|_{M^{q'}} = \|f\|_{M^q} \|g\|_{M^{q'}}.$$

Hence $V_g f \in L^\infty$, and it is again easy to see that $V_g f$ is continuous, so $V_g f \in L^\infty \cap \mathcal{C} = W(\mathcal{C}, \ell^\infty)$. Further,

$$\|V_g f\|_{W(\mathcal{C}, \ell^\infty)} = \|V_g f\|_{L^\infty} \leq \|g\|_{M^{q'}} \|f\|_{M^q}.$$

Case 3: $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.

By Case 1, we have

$$\|V_g f\|_{W(\mathcal{C}, \ell^q)} \leq (3^{d/q} \|f\|_{M^q}) \|g\|_{M^1},$$

and by Case 2, we have

$$\|V_g f\|_{W(\mathcal{C}, \ell^\infty)} \leq \|f\|_{M^q} \|g\|_{M^{q'}}.$$

Set $\theta = q/r$. Then $0 \leq \theta \leq 1$ and

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{\infty} \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{q'}.$$

Further,

$$[W(\mathcal{C}, \ell^q), W(\mathcal{C}, \ell^\infty)]_\theta = W(\mathcal{C}, \ell^r) \quad \text{and} \quad [M^1, M^{q'}]_\theta = M^p.$$

With f fixed, applying interpolation to the mapping $g \mapsto V_g f$ therefore yields

$$\|V_g f\|_{W(\mathcal{C}, \ell^r)} \leq (3^{d/q} \|f\|_{M^q})^\theta (\|f\|_{M^q})^{1-\theta} \|g\|_{M^p} = 3^{d/r} \|f\|_{M^q} \|g\|_{M^p}. \quad \square$$

The significance of membership of $V_g f$ in the Wiener amalgam space is made clear by the next result, which follows from [Grö01, Prop. 11.1.4] and states that the ℓ^r sequence space norm of a set of separated samples of a function in $W(\mathcal{C}, \ell^r)$ is bounded by the amalgam space norm of that function. A sequence Λ is *separated* if there exists some $\delta > 0$ such that any two different points of Λ are at least a distance δ apart.

Proposition A.4. *Let $1 \leq r \leq \infty$ be given. If Λ is separated, then there exists a constant $C = C(r, \Lambda) > 0$ such that*

$$\forall F \in W(\mathcal{C}, \ell^r), \quad \left(\sum_{\lambda \in \Lambda} |F(\lambda)|^r \right)^{1/r} \leq C \|F\|_{W(\mathcal{C}, \ell^r)}.$$

A proof of Proposition 2.7 now follows from combining Lemma A.3 with Proposition A.4.

Proof of Proposition 2.7. Assume $D^+(\Lambda) < \infty$. Then by [CDH99, Lemma 2.3], Λ can be written as a finite union of disjoint sequences $\Lambda_1, \dots, \Lambda_k$ each of which is separated. If $g \in M^p$ and $f \in M^q$, then $V_g f \in W(\mathcal{C}, \ell^r)$ by Lemma A.3. Hence, by Proposition A.4, we have for each $j = 1, \dots, k$ that

$$\sum_{\lambda \in \Lambda_j} |\langle f, \pi(\lambda)g \rangle|^r = \sum_{\lambda \in \Lambda_j} |V_g f(\lambda)|^r \leq C_1 \|V_g f\|_{W(\mathcal{C}, \ell^r)}^r \leq C_2 \|g\|_{M^p}^r \|f\|_{M^q}^r. \quad \square$$

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