DUALITY PRINCIPLES IN FRAME THEORY

PETER G. CASAZZA, GITTA KUTYNIOK, AND MARK C. LAMMERS

Abstract. Duality principles in Gabor theory such as the Ron-Shen duality principle and the Wexler-Raz orthogonality relations play a fundamental role for analyzing Gabor systems. In this paper we present a general approach to derive duality principles in abstract frame theory. For each sequence in a separable Hilbert space we define a corresponding sequence dependent only on two orthonormal bases. Then we characterize exactly properties of the first sequence in terms of the associated one, which yields duality relations for the abstract frame setting. In the last part we apply our results to Gabor systems.

1. Introduction

Frames were first introduced by Duffin and Schaeffer [8] in the context of nonharmonic Fourier series. Outside of signal processing, frames did not seem to generate much interest until the ground breaking work of Daubechies, Grossmann, and Meyer [6] in 1986. Since then the theory of frames began to be more widely studied. Traditionally, frames have been used in signal processing, image processing, data compression, and sampling theory. Recently, the theory is beginning to grow even more rapidly, since several new applications have been developed. For example, frames are now used to migrate the effect of losses in packet-based communication systems and hence to improve the robustness of data transmission [10, 4], and to design high-rate constellations with full diversity in multiple-antenna code design [12].

One of the most important concrete realizations of frames are Gabor frames. Gabor systems were first introduced in 1946 by Gabor [9]. They are generated by modulations and translations of one single function. That is, we choose a fixed function $g \in L^2(\mathbb{R})$ and two parameters $a, b > 0$, and define the associated Gabor system $\mathcal{G}(g, a, b)$ by

$$\mathcal{G}(g, a, b) = \{E_{mb}T_{na}g : m, n \in \mathbb{Z}\},$$

where $T_{na}f(x) = f(x - na)$ and $E_{mb}f(x) = e^{2\pi imbx}f(x)$. One of the most fascinating results is the exact characterization of Gabor frames, known as the Ron-Shen duality principle [15, Theorem 2.2 (e)]. It states that for each $g \in L^2(\mathbb{R})$ and $a, b > 0$ with $ab \leq 1$, $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$ if and only if $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is a Riesz basic sequence. Another very important result is the Wexler-Raz biorthogonality relations [16] (see also [11, Theorem 7.3.1]), which characterize exactly all alternate dual frames of a Gabor frame. Let us mention that a

Date: April 30, 2003.

1991 *Mathematics Subject Classification.* Primary 42C15; Secondary 42A38.

Key words and phrases. duality principle, frame, Gabor system, abstract frame theory, Riesz basis, Ron-Shen duality principle, Schauder basis, Wexler-Raz biorthogonality relations.

detailed mathematical analysis was later given by Janssen [14] and Daubechies, Landau, and Landau [7].

The question arises whether these results, which can be regarded as duality principles, can be generalized to abstract frame theory. If we would have a general duality theory there, we could hope to get an abundance of new duality principles for Gabor systems by using the machinery of frame theory. But we also could apply these results to other frames, e.g. wavelet frames, and obtain for example a kind of Ron-Shen duality principle or Wexler-Raz biorthogonality relations for these types of systems.

In this paper we show that in fact for each sequence in a separable Hilbert space we can construct a corresponding sequence with a kind of duality relation between them. This construction is used to prove duality principles in abstract frame theory, which can be regarded as general versions of several well-known duality principles for Gabor systems.

Let us now give an outline of the paper. We start with a brief review of the definitions and basic properties of frames and bases in Section 2. Given a sequence $(f_i)_{i\in\mathbb{N}}$ in a separable Hilbert space $\mathcal{H}$, we then define the so-called Riesz-dual sequence (R-dual sequence) $(w_j^f)_{j\in\mathbb{N}}$ and show which duality relation exists between them (see Subsection 3.1). This construction is dependent on two fixed orthonormal bases for $\mathcal{H}$. Therefore in Subsection 3.2 we investigate how $(w_j^f)_{j\in\mathbb{N}}$ changes if we choose other orthonormal bases.

The relation between the two sequences turns out to be a powerful tool for deriving duality principles in abstract frame theory. These are obtained by characterizing properties of $(f_i)_{i\in\mathbb{N}}$ in terms of properties of the corresponding sequence $(w_j^f)_{j\in\mathbb{N}}$. In Subsection 4.1 we first study some basic properties as completeness, ω-independence, and minimality, which each Schauder basis satisfies. Then we exactly characterize those sequences $(w_j^f)_{j\in\mathbb{N}}$, which are Schauder basic sequences and show that each Schauder basic sequence is of this form, i.e. is a R-dual sequence of some other sequence. Subsection 4.2 is devoted to the study of frame properties of the sequence $(f_i)_{i\in\mathbb{N}}$. In detail, we examine, when there exists a lower or upper frame bound, when the sequence is a frame sequence, and when it is a tight or exact frame. It is well known that frames, which are equivalent or even unitarily equivalent, often share the same properties. Therefore, in Subsection 4.3, we give equivalent conditions when two frames are (unitarily) equivalent in terms of their R-dual sequences. In addition, we study, when two frames possess the same frame operator. In the last subsection of Section 4 we deal with investigating properties of alternate dual frames and of the canonical dual frame. Since alternate dual frames are crucial for reconstruction algorithms, we give a characterization of them and, in addition, show how the canonical dual frame is distinguished between the set of alternate dual frames.

In the last section we apply our results to Gabor systems. In fact the hypothesis for the construction of a R-dual sequence is satisfied for each Gabor system with some small restrictions on the orthonormal bases (see Subsection 5.1). Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be such that $\mathcal{G}(g, a, b)$ is a tight frame. In Subsection 5.2, we show that in this situation there indeed exist infinitely many pairs of orthonormal bases such that the R-dual sequence equals $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$. Moreover, in the special case $ab = 1$, for each Gabor system $\mathcal{G}(g, a, b)$, we give a concrete pair of orthonormal bases so that the system $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is the R-dual sequence. This seems to indicate the existence of such orthonormal bases for all regular Gabor systems.

A sequel paper will examine the explicit construction of orthonormal bases for $L^2(\mathbb{R})$ such that $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is the R-dual sequence of $\mathcal{G}(g, a, b)$ with respect to these bases for each function $g \in L^2(\mathbb{R})$ and $a, b > 0$. This would give us most of the known duality principles in Gabor theory as corollaries in full generality. Moreover, we would obtain an abundance of new duality principles. Another interesting point for future research is the wavelet setting. We will prove duality principles also in this situation by using our approach in an upcoming paper.

2. Frames and bases review

In this section we will briefly recall the definitions and basic properties of frames and bases. For more information we refer to the excellent survey articles by Casazza [3, 2], the recognized books by Christensen [5], Gröchenig [11], and Young [17] and the very well written research-tutorial by Heil and Walnut [13].

Let $\mathcal{H}$ be a separable Hilbert space. A *Schauder basis* (or simply a *basis*) for $\mathcal{H}$ is a family of functions $(f_i)_{i\in\mathbb{N}}$ such that for all $h \in \mathcal{H}$ there exist unique scalars c_i, $i \in \mathbb{N}$ with

$$h = \sum_{i=1}^{\infty} c_i f_i. \tag{1}$$

In this case, there exist unique elements $\tilde{f}_i \in \mathcal{H}$ such that $c_i = \langle h, \tilde{f}_i \rangle$. Moreover, the sequences $(f_i)_{i\in\mathbb{N}}$ and $(\tilde{f}_i)_{i\in\mathbb{N}}$ are biorthogonal, and $(\tilde{f}_i)_{i\in\mathbb{N}}$ itself forms a basis for $L^2(\mathbb{R})$, the so-called *dual basis* of $(f_i)_{i\in\mathbb{N}}$. A Schauder basis is an *unconditional basis*, if the series in (1) converge unconditionally. Consequently, for a Schauder basis the ordering in (1) can be crucial. If $(f_i)_{i\in\mathbb{N}}$ is a Schauder basis only for its closed linear span, we call it a *Schauder basic sequence*. The following well-known characterization of Schauder bases is sometimes more useful.

Proposition 2.1. *Let $(f_i)_{i\in\mathbb{N}} \subset \mathcal{H}$. Then the following conditions are equivalent.*

(i) *$(f_i)_{i\in\mathbb{N}}$ is a Schauder basic sequence.*

(ii) *There exists some $0 < B < \infty$ such that, for all $N \in \mathbb{N}$ and $(c_i)_{i\in\mathbb{N}} \in l^2(\mathbb{N})$,*

$$\left\| \sum_{i=1}^{N} c_i f_i \right\|^2 \leq B \left\| \sum_{i=1}^{\infty} c_i f_i \right\|^2.$$

A family $(f_i)_{i\in\mathbb{N}}$ is a *frame* for $\mathcal{H}$, if there exist $0 < A \leq B < \infty$ such that for all $h \in \mathcal{H}$,

$$A \|h\|_2^2 \leq \sum_{i=1}^{\infty} |\langle h, f_i \rangle|^2 \leq B \|h\|_2^2. \tag{2}$$

The constants A and B are called a *lower* and *upper frame bound* for the frame. Those sequences which satisfy only the upper inequality in (2) are called *Bessel sequences*. A frame is *tight*, if $A = B$. If $A = B = 1$, it is called a *Parseval frame*. A frame is *exact*, if it ceases to be a frame whenever any single element is deleted from the sequence $(f_i)_{i\in\mathbb{N}}$. We say that two frames $(f_i)_{i\in\mathbb{N}}$, $(g_i)_{i\in\mathbb{N}}$ for $\mathcal{H}$ are *equivalent*, if there exists an invertible operator $U : \mathcal{H} \to \mathcal{H}$ satisfying $Uf_i = g_i$ for all $i \in \mathbb{N}$. If U is a unitary operator, $(f_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$ are called *unitarily equivalent*. Let T_f denote the synthesis operator of $f = (f_i)_{i\in\mathbb{N}}$, i.e. $T_f(c) = \sum_i c_i f_i$

for each sequence of scalars $c = (c_i)_{i\in\mathbb{N}}$. Then the *frame operator* $Sh = T_f T_f^*(h) = \sum_i \langle h, f_i\rangle f_i$ associated with $(f_i)_{i\in\mathbb{N}}$ is a bounded, invertible, and positive operator mapping of $\mathcal{H}$ onto itself. This provides the reconstruction formula

$$h = S^{-1}Sh = \sum_{i=1}^{\infty} \langle h, f_i\rangle \tilde{f}_i = \sum_{i=1}^{\infty} \langle h, \tilde{f}_i\rangle f_i,$$

where $\tilde{f}_i = S^{-1}f_i$. The family $(\tilde{f}_i)_{i\in\mathbb{N}}$ is also a frame for $\mathcal{H}$, called the *canonical dual frame* of $(f_i)_{i\in\mathbb{N}}$. If $(g_i)_{i\in\mathbb{N}}$ is any sequence in $\mathcal{H}$ which satisfies

$$h = \sum_{i=1}^{\infty} \langle h, f_i\rangle g_i = \sum_{i=1}^{\infty} \langle h, g_i\rangle f_i,$$

it is called an *alternate dual frame* of $(f_i)_{i\in\mathbb{N}}$. A sequence is called a *frame sequence*, if it is a frame only for its closed linear span. We have the following well-known characterization of frame sequences.

Proposition 2.2. *Let $(f_i)_{i\in\mathbb{N}} \subset \mathcal{H}$. Then the following conditions are equivalent.*

(i) *$(f_i)_{i\in\mathbb{N}}$ is a frame sequence.*
(ii) *There exist $0 < A \le B < \infty$ such that*

$$A\|c\|_2 \le \left\| \sum_{i=1}^{\infty} c_i f_i \right\| \le B\|c\|_2$$

for all $c = (c_i)_{i\in\mathbb{N}} \in (\ker T_f)^{\perp}$.

Riesz bases are special cases of frames, and can be characterized as those frames which are biorthogonal to their dual frames. An equivalent definition is the following. A family $(f_i)_{i\in\mathbb{N}}$ is a Riesz bases for $\mathcal{H}$, if there exist $0 < A \le B < \infty$ such that for all sequences of scalars $c = (c_i)_{i\in\mathbb{N}}$,

$$A\,\|c\|_2 \le \left\| \sum_{i=1}^{\infty} c_i f_i \right\| \le B\,\|c\|_2.$$

We define the *Riesz basis constants* for $(f_i)_{i\in\mathbb{N}}$ to be the largest number A and the smallest number B such that this inequality holds for all sequences of scalars c. The dual basis of a Riesz basis, the so-called *dual Riesz basis* is itself a Riesz basis. If $(f_i)_{i\in\mathbb{N}}$ is a Riesz basis only for its closed linear span, we call it a *Riesz basic sequence*.

An arbitrary sequence $(f_i)_{i\in\mathbb{N}}$ in $\mathcal{H}$ is *minimal*, if there exists a sequence $(\tilde{f}_i)_{i\in\mathbb{N}}$, which is biorthogonal to $(f_i)_{i\in\mathbb{N}}$, or equivalently, if the distance $d(f_i, \mathrm{span}_{j\ne i} f_j) \ne 0$ for all $i \in \mathbb{N}$. It is *complete*, if the span of $(f_i)_{i\in\mathbb{N}}$ is dense in $\mathcal{H}$. A sequence $(f_i)_{i\in\mathbb{N}}$ is called *ω-independent for $\ell^2(\mathbb{N})$-sequences* if, whenever $c = (c_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{N})$ is a sequence of scalars and $\sum_i c_i f_i = 0$, it follows that $c = 0$. If this holds for any sequences of scalars $c = (c_i)_{i\in\mathbb{N}}$, we say that $(f_i)_{i\in\mathbb{N}}$ is *ω-independent*.

3. The R-dual sequence

3.1. **Definition.** Let $(f_i)_{i\in\mathbb{N}}$ be a sequence in a separable Hilbert space $\mathcal{H}$ and let $(e_j)_{j\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ be orthonormal bases for $\mathcal{H}$. We will construct a sequence $(w_j^f)_{j\in\mathbb{N}}$ in the following way. For each $i \in \mathbb{N}$, we expand f_i with respect to the basis $(e_j)_{j\in\mathbb{N}}$ and define a matrix with these coefficients as entries of the row vectors. Then, for each $j \in \mathbb{N}$, we define w_j^f to be the linear combination of $h_i, i \in \mathbb{N}$ with the column entries of this matrix as coefficients. This process leads to the following definition.

Definition 3.1. Let $(e_j)_{j\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ be orthonormal bases for a separable Hilbert space $\mathcal{H}$. Let $(f_i)_{i\in\mathbb{N}} \subset \mathcal{H}$ be such that $\sum_i |\langle f_i, e_j\rangle|^2 < \infty$ for each $j \in \mathbb{N}$, and let

$$w_j^f = \sum_{i=1}^{\infty} \langle f_i, e_j\rangle h_i \quad (j \in \mathbb{N}).$$

Then $(w_j^f)_{j\in\mathbb{N}}$ is called the *Riesz-dual sequence (R-dual sequence) for* $(f_i)_{i\in\mathbb{N}}(= f)$ *with respect to* $(e_j)_{j\in\mathbb{N}}$ *and* $(h_i)_{i\in\mathbb{N}}$.

Notice that the hypothesis $\sum_i |\langle f_i, e_j\rangle|^2 < \infty$ for each $j \in \mathbb{N}$ is always fulfilled if the sequence $(f_i)_{i\in\mathbb{N}}$ is Bessel.

This simple construction gives us a powerful tool for deriving duality principles in general frame theory (see Section 4). These duality principles are stated in terms of the sequences $(f_i)_{i\in\mathbb{N}}$ and $(w_j^f)_{j\in\mathbb{N}}$. Therefore, knowing the orthonormal bases $(e_j)_{j\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ which belong to the construction and the R-dual sequence $(w_j^f)_{j\in\mathbb{N}}$, we also need an algorithm to invert the process and calculate $(f_i)_{i\in\mathbb{N}}$ from this data.

Lemma 3.2. *Let* $(e_j)_{j\in\mathbb{N}}$ *and* $(h_i)_{i\in\mathbb{N}}$ *be orthonormal bases for* $\mathcal{H}$. *Let* $(f_i)_{i\in\mathbb{N}} \subset \mathcal{H}$ *be such that* $\sum_i |\langle f_i, e_j\rangle|^2 < \infty$ *for each* $j \in \mathbb{N}$. *Then, for all* $i \in \mathbb{N}$,

$$f_i = \sum_{j=1}^{\infty} \langle w_j^f, h_i\rangle e_j.$$

In particular, this shows that $(f_i)_{i\in\mathbb{N}}$ *is the R-dual sequence for* $(w_j^f)_{j\in\mathbb{N}}$ *with respect to* $(h_i)_{i\in\mathbb{N}}$ *and* $(e_j)_{j\in\mathbb{N}}$.

Proof. The definition of $(w_j^f)_{j\in\mathbb{N}}$ implies that

$$\langle f_i, e_j\rangle = \langle w_j^f, h_i\rangle \quad \text{for all } i \in \mathbb{N},\ j \in \mathbb{N}.$$

Thus we have

$$f_i = \sum_{j=1}^{\infty} \langle f_i, e_j\rangle e_j = \sum_{j=1}^{\infty} \langle w_j^f, h_i\rangle e_j.$$

□

This is a kind of duality relation between the sequence $(f_i)_{i\in\mathbb{N}}$ and its R-dual sequence $(w_j^f)_{j\in\mathbb{N}}$, which implies that we can interchange their roles in results in Section 4.

Remark 3.3. Let $(f_i)_{i\in\mathbb{N}} \subset \mathcal{H}$ be such that $\sum_i |\langle f_i, e_j\rangle|^2 < \infty$ for each $j \in \mathbb{N}$. Suppose that $(f_i)_{i\in\mathbb{N}}$ satisfies property A if and only if $(w_j^f)_{j\in\mathbb{N}}$ satisfies property B, where A and B are operator theoretic properties (e.g. being a frame, a Riesz basic sequence, etc.). By Lemma 3.2, this implies that $(f_i)_{i\in\mathbb{N}}$ satisfies property B if and only if $(w_j^f)_{j\in\mathbb{N}}$ satisfies property A. Therefore each result in Section 4 gives rise to a similar result with $(f_i)_{i\in\mathbb{N}}$ and $(w_j^f)_{j\in\mathbb{N}}$ being interchanged.

3.2. **Dependency on the orthonormal bases.** We will examine to which extent the sequence $(w_j^f)_{j\in\mathbb{N}}$ depends on the chosen orthonormal bases $(e_j)_{j\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$. The next result seems to indicate that producing $(w_j^f)_{j\in\mathbb{N}}$ is nearly almost a unique process. That is, if we change one of the orthonormal bases $(e_j)_{j\in\mathbb{N}}$ or $(h_i)_{i\in\mathbb{N}}$, then we may not be able to change the other basis to get the same R-dual sequence.

Proposition 3.4. *Let $(e_j)_{j\in\mathbb{N}}$, $(e_j')_{j\in\mathbb{N}}$, $(h_i)_{i\in\mathbb{N}}$, and $(h_i')_{i\in\mathbb{N}}$ be orthonormal bases for a separable Hilbert space $\mathcal{H}$. Let $(f_i)_{i\in\mathbb{N}} \subset \mathcal{H}$ be such that $\sum_i |\langle f_i, e_j\rangle|^2 < \infty$ as well as $\sum_i |\langle f_i, e_j'\rangle|^2 < \infty$ for all $j \in \mathbb{N}$. Denote the R-dual sequence of $(f_i)_{i\in\mathbb{N}}$ with respect to $(e_j)_{j\in\mathbb{N}}$, $(h_i)_{i\in\mathbb{N}}$ and $(e_j')_{j\in\mathbb{N}}$, $(h_i')_{i\in\mathbb{N}}$ by $(w_j^f)_{j\in\mathbb{N}}$ and $(w_j'^f)_{j\in\mathbb{N}}$, respectively. Then the following conditions are equivalent.*

(i) $w_j'^f = w_j^f$ *for all* $j \in \mathbb{N}$.
(ii) *If B is the matrix of $(e_j')_{j\in\mathbb{N}}$ with respect to $(e_j)_{j\in\mathbb{N}}$, C is the matrix of $(h_i')_{i\in\mathbb{N}}$ with respect to $(h_i)_{i\in\mathbb{N}}$ and $A = (\langle f_i, e_j\rangle)_{i\in\mathbb{N},j\in\mathbb{N}}$, then $AB^* = \overline{C}A$.*

Proof. Let $B = (b_{ij})_{i,j\in\mathbb{N}}$ and $C = (c_{ij})_{i,j\in\mathbb{N}}$. By definition of $(w_j^f)_{j\in\mathbb{N}}$ and $(w_j'^f)_{j\in\mathbb{N}}$ we have

$$\langle w_k^f, h_i\rangle = \langle f_i, e_k\rangle \quad \text{and} \quad \langle w_k'^f, h_i'\rangle = \langle f_i, e_k'\rangle.$$

Hence $(w_j^f)_{j\in\mathbb{N}} = (w_j'^f)_{j\in\mathbb{N}}$ if and only if, for every $i \in \mathbb{N}, k \in \mathbb{N}$ we have

$$\sum_{j=1}^{\infty} \langle f_i, e_j\rangle \overline{b_{kj}} = \langle f_i, e_k'\rangle = \langle w_k'^f, h_i'\rangle = \langle w_k^f, h_i'\rangle = \sum_{j=1}^{\infty} \overline{c_{ij}} \langle w_k^f, h_j\rangle = \sum_{j=1}^{\infty} \overline{c_{ij}} \langle f_j, e_k\rangle.$$

Since, for all $i \in \mathbb{N}, k \in \mathbb{N}$,

$$\sum_{j=1}^{\infty} \langle f_i, e_j\rangle \overline{b_{kj}} = (AB^*)_{ik} \quad \text{and} \quad \sum_{j=1}^{\infty} \overline{c_{ij}} \langle f_j, e_k\rangle = (\overline{C}A)_{ik},$$

the result follows. □

This indicates that if $(f_i)_{i\in\mathbb{N}}$ is a frame there only exist different pairs of orthonormal bases $((e_j)_{j\in\mathbb{N}}, (h_i)_{i\in\mathbb{N}})$ and $((e_j')_{j\in\mathbb{N}}, (h_i')_{i\in\mathbb{N}})$, which lead to the same R-dual sequence, under very rare circumstances.

Lemma 3.5. *Let $(e_j)_{j\in\mathbb{N}}$, $(e_j')_{j\in\mathbb{N}}$, $(h_i)_{i\in\mathbb{N}}$, $(h_i')_{i\in\mathbb{N}}$, $(f_i)_{i\in\mathbb{N}}$, $(w_j^f)_{j\in\mathbb{N}}$, $(w_j'^f)_{j\in\mathbb{N}}$, A, B, and C be defined as in Proposition 3.4. If $(f_i)_{i\in\mathbb{N}}$ is a frame for $L^2(\mathbb{R})$ with frame operator S, then the following conditions are equivalent.*

(i) $w_j'^f = w_j^f$ *for all* $j \in \mathbb{N}$.
(ii) $A^*C^tAS^{-1}B^* = I$.

Proof. If $(f_i)_{i\in\mathbb{N}}$ is a frame, then there exists a matrix V so that $VA = I$ and $V^*V = S^{-1}$. So $AB^* = \overline{C}A$ implies $B^* = VAB^* = V\overline{C}A$. Hence we have $B = A^*C^tV^*$. But B has to be unitary, which yields

$$I = BB^* = A^*C^tV^*VAB^*.$$

On the other hand there exists a unitary matrix E such that $A = ES^{1/2}$. Thus $V = S^{-1/2}E^*$ and $V^*V = ES^{-1/2}S^{-1/2}E^* = ES^{-1}E^*$. Now,

$$I = A^*C^tES^{-1}E^*AB^* = A^*C^tES^{-1/2}B^* = A^*C^tAS^{-1}B^*.$$

□

But in the situation of Gabor frames we have examples, where the conditions in (ii) are fulfilled (compare Proposition 5.4).

It is a very strong requirement to demand the sequences $(w_j^{'f})_{j\in\mathbb{N}}$ and $(w_j^f)_{j\in\mathbb{N}}$ to coincide. If we want those sequences to satisfy similar properties, it is often enough to require them to be (unitarily) equivalent. The following result characterizes those orthonormal bases which yield (unitarily) equivalent R-dual sequences. It is remarkable that these properties do not dependent on the choice of the orthonormal bases $(h_i)_{i\in\mathbb{N}}$ and $(h_i')_{i\in\mathbb{N}}$.

Proposition 3.6. *Let $(e_j)_{j\in\mathbb{N}}$, $(e_j')_{j\in\mathbb{N}}$, $(h_i)_{i\in\mathbb{N}}$, $(h_i')_{i\in\mathbb{N}}$, $(f_i)_{i\in\mathbb{N}}$, $(w_j^f)_{j\in\mathbb{N}}$, and $(w_j^{'f})_{j\in\mathbb{N}}$ be defined as in Proposition 3.4. If $(w_j^f)_{j\in\mathbb{N}}$, and $(w_j^{'f})_{j\in\mathbb{N}}$ are frames, then the following statements hold.*

- (i) *The following conditions are equivalent.*
  - (a) *$(w_j^{'f})_{j\in\mathbb{N}}$ is equivalent to $(w_j^f)_{j\in\mathbb{N}}$.*
  - (b) *$ker(A) = ker(AB^*)$.*
- (ii) *The following conditions are equivalent.*
  - (a) *$(w_j^{'f})_{j\in\mathbb{N}}$ is unitarily equivalent to $(w_j^f)_{j\in\mathbb{N}}$.*
  - (b) *$A^*A = (AB^*)^*(AB^*)$.*

  If $(f_i)_{i\in\mathbb{N}}$ is a frame with frame operator S, the above is equivalent to $S = BSB^*$.

Proof. Let $a = (a_j)_{j\in\mathbb{N}}$ be any sequence of scalars. First we observe that

$$\sum_{j=1}^{\infty} a_j w_j^{'f} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_j\langle f_i, e_j'\rangle h_i' = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_j\langle f_i, e_k\rangle\overline{b_{jk}}h_i' = \sum_{i=1}^{\infty}(AB^*a)_i h_i'.$$

This implies

$$\sum_{j=1}^{\infty} a_j w_j^{'f} = 0 \quad\Longleftrightarrow\quad AB^*a = 0.$$

Secondly, we have

$$\sum_{j=1}^{\infty} a_j w_j^{f} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_j\langle f_i, e_j\rangle h_i = \sum_{i=1}^{\infty}(Aa)_i h_i.$$

Hence

$$\sum_{j=1}^{\infty} a_j w_j^{f} = 0 \quad\Longleftrightarrow\quad Aa = 0.$$

Now $(w_j^{'f})_{j\in\mathbb{N}}$ is equivalent to $(w_j^f)_{j\in\mathbb{N}}$ if and only if for all sequences of scalars $(a_j)_{j\in\mathbb{N}}$, we have $\sum_j a_j w_j^{'f} = 0$ if and only if $\sum_j a_j w_j^f = 0$. This proves (i).

To show (ii), it suffices to prove that $\langle w_j'^f, w_k'^f \rangle = \langle w_j^f, w_k^f \rangle$ for all $j,k \in \mathbb{N}$ if and only if (b) holds. First, we compute

$$\begin{aligned} \langle w_j'^f, w_k'^f \rangle &= \sum_{i=1}^{\infty} \langle f_i, e_j' \rangle \langle e_k', f_i \rangle \\ &= \sum_{i=1}^{\infty} \left[\sum_{m=1}^{\infty} \langle f_i, e_m \rangle \overline{b_{jm}} \right] \left[\sum_{n=1}^{\infty} b_{kn} \langle e_n, f_i \rangle \right] \\ &= (BA^*AB^*)_{kj}. \end{aligned}$$

Since also

$$\langle w_j^f, w_k^f \rangle = \sum_{i=1}^{\infty} \langle f_i, e_j \rangle \langle e_k, f_i \rangle = (A^*A)_{kj},$$

we obtain that $\langle w_j'^f, w_k'^f \rangle = \langle w_j^f, w_k^f \rangle$ holds for all $j,k \in \mathbb{N}$ if and only if we have $A^*A = (AB^*)^*(AB^*)$.

Finally, if $(f_i)_{i\in\mathbb{N}}$ is a frame, we have $A^*A = S$. Thus

$$S = A^*A = BA^*AB^* = BSB^*.$$

□

4. Duality principles

Throughout this section let $\mathcal{H}$ be a separable Hilbert space. We fix two orthonormal bases $(e_j)_{j\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ for $\mathcal{H}$. Let $(f_i)_{i\in\mathbb{N}} \subset \mathcal{H}$ be such that $\sum_i |\langle f_i, e_j \rangle|^2 < \infty$ for all $j \in \mathbb{N}$. Recall that this inequality is automatically fulfilled whenever $(f_i)_{i\in\mathbb{N}}$ is a Bessel sequence. Let $(w_j^f)_{j\in\mathbb{N}}$ denotes its R-dual sequence with respect to $(e_j)_{j\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$. We will show that we can characterize all properties of $(f_i)_{i\in\mathbb{N}}$ in terms of properties of $(w_j^f)_{j\in\mathbb{N}}$.

First we establish a basic connection between a sequence and its R-dual sequence which will be used frequently in what follows.

Proposition 4.1. *For all $(a_j)_{j\in\mathbb{N}} \in l^2(\mathbb{N})$ and $(b_i)_{i\in\mathbb{N}} \in l^2(\mathbb{N})$,*

$$\left\| \sum_{j=1}^{\infty} a_j w_j^f \right\|^2 = \sum_{i=1}^{\infty} |\langle \phi, f_i \rangle|^2 \quad \textit{and} \quad \left\| \sum_{i=1}^{\infty} b_i f_i \right\|^2 = \sum_{j=1}^{\infty} |\langle g, w_j^f \rangle|^2,$$

where $\phi = \sum_j a_j e_j$ and $g = \sum_i b_i h_i$.

Proof. We compute

$$\left\| \sum_{j=1}^{\infty} a_j w_j^f \right\|^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_j \overline{a_k} \langle f_i, e_j \rangle \overline{\langle f_i, e_k \rangle} = \sum_{i=1}^{\infty} |\langle \phi, f_i \rangle|^2.$$

The second claim follows from Remark 3.3. □

4.1. **Schauder bases.** The basic properties, which a Schauder basis satisfy, are completeness, ω-independence, and minimality. Therefore before characterizing those sequences $(f_i)_{i\in\mathbb{N}}$, which are Schauder basic sequences, we start by giving equivalent conditions for the sequence $(f_i)_{i\in\mathbb{N}}$ to be complete.

Proposition 4.2. *The following conditions are equivalent.*

(i) $(f_i)_{i\in\mathbb{N}}$ *is complete.*
(ii) $(w_j^f)_{j\in\mathbb{N}}$ *is ω-independent for $\ell^2(\mathbb{N})$-sequences.*

In particular, if $(w_j^f)_{j\in\mathbb{N}}$ is ω-independent, then $(f_i)_{i\in\mathbb{N}}$ is complete.

Proof. Let $(a_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{N})$ and define ϕ by $\phi = \sum_j a_j e_j$. Using Proposition 4.1, we obtain

$$\left\| \sum_{j=1}^{\infty} a_j w_j^f \right\|^2 = \sum_{i=1}^{\infty} |\langle \phi, f_i\rangle|^2.$$

Therefore $(w_j^f)_{j\in\mathbb{N}}$ is ω-independent for $\ell^2(\mathbb{N})$-sequences if and only if

$$\sum_{i=1}^{\infty} |\langle \phi, f_i\rangle|^2 \neq 0 \quad \text{for all } \phi = \sum_{j=1}^{\infty} a_j e_j,\ a = (a_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{N}),\ a \neq 0.$$

This in turn is equivalent to $(f_i)_{i\in\mathbb{N}}$ being complete. □

The set of sequences $(w_j^f)_{j\in\mathbb{N}}$ in $\mathcal{H}$, which are ω-independent for $\ell^2(\mathbb{N})$-sequences, is a proper subset of the set of ω-independent sequences. Therefore we also ask for a characterization of ω-independent sequences $(w_j^f)_{j\in\mathbb{N}}$ in terms of properties of $(f_i)_{i\in\mathbb{N}}$. This is provided by the following proposition.

Proposition 4.3. *The following conditions are equivalent.*

(i) $(w_j^f)_{j\in\mathbb{N}}$ *is ω-independent.*
(ii) *If $a = (a_j)_{j\in\mathbb{N}}$ is a sequence of scalars and $\phi_n = \sum_{j=1}^n a_j e_j \in \mathcal{H}$ for every $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} |\langle \phi_n, f_i\rangle|^2 = 0,$$

then $a = 0$.

Proof. By Proposition 4.1, we have

$$\sum_{i=1}^{\infty} |\langle \phi_n, f_i\rangle|^2 = \left\| \sum_{j=1}^{n} a_j w_j^f \right\|^2.$$

Now the claim follows immediately. □

The next result investigates when the R-dual sequence is minimal.

Proposition 4.4. *The following conditions are equivalent.*

(i) $(w_j^f)_{j\in\mathbb{N}}$ *is minimal.*

(ii) *There exist constants* $0 < c_j \leq 1$, $j \in \mathbb{N}$ *such that, for every sequence of scalars* $(a_j)_{j\in\mathbb{N}}$ *and* $k \in \mathbb{N}$, *we have*

$$\left\| \sum_{j=1}^{\infty} a_j w_j^f \right\| \geq |c_k a_k|.$$

(iii) *There exist constants* $c_j > 0$, $j \in \mathbb{N}$ *such that, for every* $\phi = \sum_j a_j e_j \in \mathcal{H}$, *where* $(a_j)_{j\in\mathbb{N}}$ *is a sequence of scalars, we have*

$$\sum_{i=1}^{\infty} |\langle \phi, f_i \rangle|^2 \geq \|(c_j a_j)_{j\in\mathbb{N}}\|_2^2.$$

Proof. First suppose that $(w_j^f)_{j\in\mathbb{N}}$ is minimal. Then there exists a sequence $(g_j)_{j\in\mathbb{N}}$, which is biorthogonal to $(w_j^f)_{j\in\mathbb{N}}$, i.e., for all $j, k \in \mathbb{N}$, we have

$$\langle g_k, w_j^f \rangle = \delta_{kj}.$$

For each $j \in \mathbb{N}$, define c_j by $c_j = \frac{1}{\|g_j\|} > 0$. Then, for each sequence of scalars $(a_j)_{j\in\mathbb{N}}$ and each $k \in \mathbb{N}$, we obtain

$$\left\| \sum_{j=1}^{\infty} a_j w_j^f \right\| \geq \frac{1}{\|g_k\|} \left| \left\langle g_k, \sum_{j=1}^{\infty} a_j w_j^f \right\rangle \right| = |c_k a_k|.$$

This shows (i) $\Rightarrow$ (ii).

Now assume that (ii) holds. Let $(a_j)_{j\in\mathbb{N}}$ be a sequence of scalars with $\phi = \sum_j a_j e_j \in \mathcal{H}$. Suppose that $\sum_j a_j w_j^f = 0$. By (ii), we have $0 = \|\sum_j a_j w_j^f\| \geq |c_k a_k|$. This implies $a_k = 0$ for all $k \in \mathbb{N}$, and hence (i).

To prove (ii) $\Rightarrow$ (iii), let c_j', $j \in \mathbb{N}$ be constants which satisfy condition (ii). Define $c_j = \frac{c_j'}{2^j}$, $j \in \mathbb{N}$. Using Proposition 4.1, we have

$$\sum_{j=1}^{\infty} |c_j a_j|^2 = \sum_{j=1}^{\infty} |\frac{1}{2^j} c_j' a_j|^2 \leq \left(\sum_{j=1}^{\infty} \frac{1}{2^j} \right) \sup_{k\in\mathbb{N}} |c_k' a_k|^2 \leq \left\| \sum_{j=1}^{\infty} a_j w_j^f \right\|^2 = \sum_{i=1}^{\infty} |\langle \phi, f_i \rangle|^2.$$

The converse follows immediately from Proposition 4.1. □

Now we turn our attention towards Schauder bases, or even more generally Schauder basic sequences. We will characterize exactly those sequences $(w_j^f)_{j\in\mathbb{N}}$, which are Schauder basic sequences, in terms of properties of $(f_i)_{i\in\mathbb{N}}$.

Theorem 4.5. *Let the frame operator for* $(f_i)_{i\in\mathbb{N}}$ *be denoted by* S. *For each* $N \in \mathbb{N}$, *let* P_N *denote the orthogonal projection of* $\mathcal{H}$ *onto* $\text{span}_{1\leq j\leq N}\{e_j\}$ *and let* S_N *denote the frame operator for* $(P_N f_i)_{i\in\mathbb{N}}$. *Then the following conditions are equivalent.*

(i) *The non-zero elements of the sequence* $(w_j^f)_{j\in\mathbb{N}}$ *form a Schauder basic sequence.*

(ii) *There exists some* $0 < B < \infty$ *such that, for all* $N \in \mathbb{N}$ *and* $\phi \in \mathcal{H}$,

$$\sum_{i=1}^{\infty} |\langle \phi, P_N f_i \rangle|^2 \leq B \sum_{i=1}^{\infty} |\langle \phi, f_i \rangle|^2.$$

(iii) *There exists some* $0 < B < \infty$ *such that, for all* $N \in \mathbb{N}$,

$$S_N \leq BS.$$

Moreover, in this case $(f_i)_{i\in\mathbb{N}}$ *is complete in* $F = span\{e_j : \sum_i |\langle f_i, e_j\rangle|^2 \neq 0\}$.

Proof. Let $E = \{j \in \mathbb{N} : \sum_i |\langle f_i, e_j\rangle|^2 \neq 0\}$. Deleting all elements e_j with $j \in E$, ensures that $w_j^f \neq 0$ for all $j \in \mathbb{N}$. Hence by considering only those elements e_j with $j \in E$ without restriction we can assume that $\sum_i |\langle f_i, e_j\rangle|^2 \neq 0$ for all $j \in \mathbb{N}$.

Fix $N \in \mathbb{N}$ and $M \geq N$. By Proposition 2.1 and 4.1, (i) is equivalent to

$$\left\|\sum_{j=1}^{N} a_j w_j^f\right\|^2 \leq B \left\|\sum_{j=1}^{M} a_j w_j^f\right\|^2 = B \sum_{i=1}^{\infty} |\langle \phi, f_i\rangle|^2 \quad \text{for all } (a_j)_{j\in\mathbb{N}} \in l^2(\mathbb{N}),$$

where $\phi = \sum_{j=1}^{M} a_j e_j$. Now

$$\left\|\sum_{j=1}^{N} a_j w_j^f\right\|^2 = \sum_{i=1}^{\infty} |\langle \phi, P_N f_i\rangle|^2$$

implies that (i) and (ii) are equivalent.

The equivalence of (ii) and (iii) follows immediately from $\sum_i |\langle \phi, P_N f_i\rangle|^2 = \langle S_N\phi, \phi\rangle$ and $\sum_i |\langle \phi, f_i\rangle|^2 = \langle S\phi, \phi\rangle$, whenever the sum is finite.

Finally, we prove the moreover part. Towards a contradiction, assume that $(f_i)_{i\in\mathbb{N}}$ is not complete in F. Hence there exists some $\phi \in F$ such that $\langle \phi, f_i\rangle = 0$ for all $i \in \mathbb{N}$. Now we expand ϕ with respect to the orthonormal basis $(e_j)_{j\in\mathbb{N}}$, i.e. $\phi = \sum_j a_j e_j$. Let $k \in \mathbb{N}$ denote the smallest positive integer such that $a_k \neq 0$. Then (ii) implies

$$0 \neq \sum_{i=1}^{\infty} |\langle \phi, P_k f_i\rangle|^2 \leq B \sum_{i=1}^{\infty} |\langle \phi, f_i\rangle|^2.$$

But since ϕ is contained in the orthogonal complement of $\text{span}_{i\in\mathbb{N}}\{f_i\}$, we obtain

$$\sum_{i=1}^{\infty} |\langle \phi, f_i\rangle|^2 = 0,$$

a contradiction. □

We will show that in fact any Schauder basic sequence in $\mathcal{H}$ is of the form $(w_j^f)_{j\in\mathbb{N}}$, i.e. is a R-dual sequence of some other sequence. Hence we derive all Riesz basic sequences, all Schauder bases and all Riesz bases by using our construction.

Proposition 4.6. *If* $(g_j)_{j\in\mathbb{N}}$ *is a Schauder basic sequence in* $\mathcal{H}$*, then there exists a sequence* $(f_i)_{i\in\mathbb{N}}$ *in* $\mathcal{H}$ *and orthonormal bases* $(e_j)_{j\in\mathbb{N}}$ *and* $(h_i)_{i\in\mathbb{N}}$ *for* $\mathcal{H}$ *with* $w_j^f = g_j$ *for every* $j \in \mathbb{N}$. *Moreover, if* $(g_j)_{j\in\mathbb{N}}$ *is a Schauder basis for* $\mathcal{H}$*, then* $(f_i)_{i\in\mathbb{N}}$ *is* ω*-independent for* $\ell^2(\mathbb{N})$*-sequences.*

Proof. Since $(g_j)_{j\in\mathbb{N}}$ is a Schauder basic sequence in $\mathcal{H}$, we have that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1 = \text{span}_{j\in\mathbb{N}}\{g_j\}$. Since $(g_j)_{j\in\mathbb{N}}$ is a basic sequence, there exists a sequence $(g_j')_{j\in\mathbb{N}}$ in

$\mathcal{H}_1$, which is biorthogonal to $(g_j)_{j\in\mathbb{N}}$, i.e. we have $\langle g_j^{'}, g_k\rangle = \delta_{jk}$ for all $j,k \in \mathbb{N}$. Let $D = \{\phi \in \mathcal{H}_1 : \sum_j |\langle \phi, g_j\rangle|^2 < \infty\}$. Since

$$\sum_{k=1}^{\infty} |\langle g_j', g_k\rangle|^2 = |\langle g_j', g_j\rangle|^2 = 1,$$

we have that $g_j^{'} \in D$ for every $j \in \mathbb{N}$. Hence,

$$E = \{\sum_{j=1}^{n} a_j g_j^{'} : (a_j)_{j\in\mathbb{N}} \text{ a sequence of scalars and } n \in \mathbb{N}\} \subset D.$$

Since $(g_j^{'})_{j\in\mathbb{N}}$ is a basis for $\mathcal{H}_1$, E is dense in $\mathcal{H}_1$. That is, D is dense in $\mathcal{H}_1$. Hence D contains an orthonormal basis $(h_j)_{j\in\mathbb{N}}$ for $\mathcal{H}_1$. Now we can extend $(h_j)_{j\in\mathbb{N}}$ to $(h_i)_{i\in\mathbb{N}}$ an orthonormal basis for $\mathcal{H}$ by adding to $(h_j)_{j\in\mathbb{N}}$ an orthonormal basis for $\mathcal{H}_2$. Then $(h_i)_{i\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$ and $\sum_j |\langle h_i, g_j\rangle|^2 < \infty$ for all $i \in \mathbb{N}$. It follows that

$$f_i := w_i^g = \sum_{j=1}^{\infty} \langle g_j, h_i\rangle e_j,$$

exists for $(e_j)_{j\in\mathbb{N}}$ an orthonormal basis for $\mathcal{H}$. Moreover, by Proposition 4.2, $(f_i)_{i\in\mathbb{N}}$ is ω-independent for $\ell^2(\mathbb{N})$-sequences, if $(g_j)_{j\in\mathbb{N}}$ spans $\mathcal{H}$. By construction and Lemma 3.2, $w_j^f = g_j$ for every $j \in \mathbb{N}$. □

4.2. **Frame properties.** We first characterize all Bessel sequences and sequences with lower frame bounds. Then we deal with frame sequences and frames, and at last we obtain equivalent conditions for a frame to be tight or exact.

4.2.1. *Existence of frame bounds.* We can classify exactly those R-dual sequences, which belong to a Bessel sequence or a sequence with lower frame bound.

Proposition 4.7. *The following conditions are equivalent.*

(i) $(f_i)_{i\in\mathbb{N}}$ *is a Bessel sequence with Bessel bound* B.

(ii) $(w_j^f)_{j\in\mathbb{N}}$ *satisfies*

$$\left\| \sum_{j=1}^{\infty} a_j w_j^f \right\| \le \sqrt{B}\|a\|_2$$

for all $a = (a_j)_{j\in\mathbb{N}} \in l^2(\mathbb{N})$.

Proof. Let $(a_j)_{j\in\mathbb{N}} \in l^2(\mathbb{N})$ and set $\phi = \sum_j a_j e_j$. By Proposition 4.1, we obtain

$$\left\| \sum_{j=1}^{\infty} a_j w_j^f \right\|^2 = \sum_{i=1}^{\infty} |\langle \phi, f_i\rangle|^2.$$

Since $\|\phi\|^2 = \|a\|_2^2$, the claim follows immediately. □

Proposition 4.8. *The following conditions are equivalent.*

(i) $(f_i)_{i\in\mathbb{N}}$ *has lower frame bound* A.

(ii) $(w_j^f)_{j\in\mathbb{N}}$ *satisfies*

$$\left\|\sum_{j=1}^{\infty} a_j w_j^f\right\| \geq \sqrt{A}\|a\|_2$$

for all $a = (a_j)_{j\in\mathbb{N}} \in l^2(\mathbb{N})$.

Proof. We can use the same arguments as in the proof of Proposition 4.7. □

Provided that the sequence $(f_i)_{i\in\mathbb{N}}$ is Bessel, we show that the R-dual sequence is unitarily equivalent to $(S^{1/2}e_j)_{j\in\mathbb{N}}$, where S is the frame operator for $(f_i)_{i\in\mathbb{N}}$.

Proposition 4.9. *If* $(f_i)_{i\in\mathbb{N}} \subset \mathcal{H}$ *is Bessel then*

$$\langle w_j^f, w_k^f\rangle = \langle S^{1/2}e_k, S^{1/2}e_j\rangle,$$

where S *is the frame operator for* $(f_i)_{i\in\mathbb{N}}$.

Proof. We compute

$$\langle w_j^f, w_k^f\rangle = \sum_{i=1}^{\infty}\langle f_i, e_j\rangle\overline{\langle f_i, e_k\rangle} = \left\langle \sum_{i=1}^{\infty}\langle e_k, f_i\rangle f_i, e_j\right\rangle = \langle Se_k, e_j\rangle = \langle S^{1/2}e_k, S^{1/2}e_j\rangle.$$

□

Corollary 4.10. *Let* $(f_i)_{i\in\mathbb{N}} \subset \mathcal{H}$ *be a Bessel sequence. Then, for every family of scalars* $(a_j)_{j\in\mathbb{N}}$, *we have*

$$\left\|\sum_{j=1}^{\infty} a_j w_j^f\right\| = \left\|\sum_{j=1}^{\infty} \overline{a_j} S^{1/2} e_j\right\|.$$

Proof. We have

$$\left\|\sum_{j=1}^{\infty} a_j w_j^f\right\|^2 = \sum_{j,k=1}^{\infty} a_j\overline{a_k}\langle w_j^f, w_k^f\rangle = \sum_{j,k=1}^{\infty} a_j\overline{a_k}\langle S^{1/2}e_k, S^{1/2}e_j\rangle = \left\|\sum_{j=1}^{\infty}\overline{a_j}S^{1/2}e_j\right\|^2.$$

□

4.2.2. *Frame sequences.* There exists an interesting relation between the synthesis operator of f and the span of $(w_j^f)_{j\in\mathbb{N}}$, which will turn out to be very useful in the sequel.

Lemma 4.11. *We have*

$$(\operatorname{span}_{j\in\mathbb{N}}\{w_j^f\})^\perp = \ker T_f$$

in the sense that $g \in (\operatorname{span}_{j\in\mathbb{N}}\{w_j^f\})^\perp$ *if and only if* $(\langle h_i, g\rangle)_{i\in\mathbb{N}} \in \ker T_f$.

Proof. Let $g \in \mathcal{H}$. Then $g \in (\operatorname{span}_{j\in\mathbb{N}}\{w_j^f\})^\perp$ if and only if $\langle g, w_j^f\rangle = 0$ for all $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, we have

$$\langle g, w_j^f\rangle = \left\langle g, \sum_{i=1}^{\infty}\langle f_i, e_j\rangle h_i\right\rangle = \sum_{i=1}^{\infty}\langle g, h_i\rangle\overline{\langle f_i, e_j\rangle} = \left\langle e_j, \sum_{i=1}^{\infty}\langle h_i, g\rangle f_i\right\rangle.$$

Since $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis, $\langle e_j, \sum_i \langle h_i, g\rangle f_i\rangle = 0$ for all $j \in \mathbb{N}$ if and only if $\sum_i \langle h_i, g\rangle f_i = 0$. Hence, by definition of T_f, $g \in (\mathrm{span}_{j\in\mathbb{N}}\{w_j^f\})^\perp$ is equivalent to $(\langle h_i, g\rangle)_{i\in\mathbb{N}} \in \ker T_f$. □

The next result shows a kind of equilibrium between a sequence and its R-dual sequence. It can be viewed as a general version of [15, Theorem 2.2 (c)].

Proposition 4.12. *The following conditions are equivalent.*

(i) $(f_i)_{i\in\mathbb{N}}$ *is a frame sequence.*
(ii) $(w_j^f)_{j\in\mathbb{N}}$ *is a frame sequence.*

Proof. Let $(b_i)_{i\in\mathbb{N}} \in l^2(\mathbb{N})$. By Proposition 4.1, we have

$$\sum_{j=1}^{\infty} |\langle g, w_j^f\rangle|^2 = \left\| \sum_{i=1}^{\infty} b_i f_i \right\|^2,$$

where $g = \sum_i b_i h_i$. Using Lemma 4.11 and Proposition 2.2 finishes the proof. □

The following result gives equivalent conditions for the sequence $(f_i)_{i\in\mathbb{N}}$ to be a frame for $\mathcal{H}$. This can be regarded as the Ron-Shen duality principle in abstract frame theory.

Theorem 4.13. *Suppose that there exist* $0 < A \le B < \infty$ *such that* $A \le \sum_i |\langle f_i, e_j\rangle|^2 \le B$ *for all* $j \in \mathbb{N}$. *For each subset* $V \subseteq \mathbb{N}$, *let* P_V *denote the orthogonal projection of* $\mathcal{H}$ *onto* $\mathrm{span}_{j\in V}\{e_j\}$. *Then the following conditions are equivalent.*

(i) $(f_i)_{i\in\mathbb{N}}$ *is a frame for* $\mathcal{H}$ *with frame bounds* A *and* B.
(ii) $(w_j^f)_{j\in\mathbb{N}}$ *is a Riesz basic sequence with Riesz basis constants* $\sqrt{A}$ *and* $\sqrt{B}$.
(iii) *There exists some* $0 < B < \infty$ *such that, for all orthogonal projections* P *on* $\mathcal{H}$ *and for all* $\phi \in \mathcal{H}$, *we have*

$$\sum_{i=1}^{\infty} |\langle \phi, Pf_i\rangle|^2 \le B \sum_{i=1}^{\infty} |\langle \phi, f_i\rangle|^2.$$

(iv) *There exists some* $0 < B < \infty$ *such that, for all subsets* $V \subseteq \mathbb{N}$ *and for all* $\phi \in \mathcal{H}$, *we have*

$$\sum_{i=1}^{\infty} |\langle \phi, P_V f_i\rangle|^2 \le B \sum_{i=1}^{\infty} |\langle \phi, f_i\rangle|^2.$$

Proof. The equivalence of (i) and (ii) follows immediately from Proposition 4.7 and 4.8.

Now suppose that $(f_i)_{i\in\mathbb{N}}$ is a frame for $\mathcal{H}$. For any orthogonal projection P on $\mathcal{H}$ and for any $\phi \in P\mathcal{H}$, we have

$$\sum_{i=1}^{\infty} |\langle \phi, Pf_i\rangle|^2 = \sum_{i=1}^{\infty} |\langle P\phi, f_i\rangle|^2 = \sum_{i=1}^{\infty} |\langle \phi, f_i\rangle|^2.$$

This shows (i) ⇒ (iii).

Obviously, (iii) implies (iv).

It remains to prove (iv) $\Rightarrow$ (ii). For this, suppose that there exists some $0 < B < \infty$ such that, for all subsets $V \subseteq \mathbb{N}$, we have

$$\sum_{i=1}^{\infty} |\langle f, P_V f_i \rangle|^2 \le B \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2.$$

For each finite subset V of $\mathbb{N}$, we can permute the orthonormal basis $(e_j)_{j\in\mathbb{N}}$, so that P_V equals P_N, $N = |V|$, where P_N denotes the orthogonal projection of $\mathcal{H}$ onto $\text{span}_{1\le j\le N}\{e_j\}$. By taking all subsets $V \subseteq \mathbb{N}$, we obtain

$$\sum_{i=1}^{\infty} |\langle f, P_N f_i \rangle|^2 \le B \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2$$

for all $N \in \mathbb{N}$ and $f \in \mathcal{H}$. Now Theorem 4.5 implies that $(w_j^f)_{j\in\mathbb{N}}$ is a Schauder basic sequence for any permutation of the basis $(e_j)_{j\in\mathbb{N}}$. Hence $(w_j^f)_{j\in\mathbb{N}}$ is an unconditional basis. Moreover, the hypothesis $A \le \sum_i |\langle f_i, e_j \rangle|^2 \le B$ for all $j \in \mathbb{N}$ implies that $(\|w_j^f\|)_{j\in\mathbb{N}}$ is bounded. Thus $(w_j^f)_{j\in\mathbb{N}}$ is a Riesz basic sequence. □

Let $(f_i)_{i\in\mathbb{N}}$ be a frame for $\mathcal{H}$. By the previous theorem, the R-dual sequence $(w_j^f)_{j\in\mathbb{N}}$ is a Riesz basic sequence. We have an explicit form for the dual Riesz basis of $(w_j^f)_{j\in\mathbb{N}}$.

Proposition 4.14. *Let $(f_i)_{i\in\mathbb{N}}$ be a frame for $\mathcal{H}$ with frame operator denoted by S. Define $w_j^{f\,*}$ for each $j \in \mathbb{N}$ by*

$$w_j^{f\,*} = \sum_{i=1}^{\infty} \langle S^{-1} f_i, e_j \rangle h_i.$$

Then we have

$$\langle w_j^{f\,*}, w_i^f \rangle = \delta_{ij} \quad \text{for all } i, j \in \mathbb{N},$$

i.e. $(w_j^{f\,*})_{j\in\mathbb{N}}$ is the dual Riesz basis of $(w_j^f)_{j\in\mathbb{N}}$.

Proof. For all $i, j \in \mathbb{N}$, we have

$$\langle w_j^{f\,*}, w_k^f \rangle = \sum_{i=1}^{\infty} \langle S^{-1} f_i, e_j \rangle \overline{\langle f_i, e_k \rangle} = \left\langle e_k, \sum_{i=1}^{\infty} \langle e_j, S^{-1} f_i \rangle f_i \right\rangle = \langle e_j, e_k \rangle = \delta_{jk}.$$

□

4.2.3. *Tight and exact frames.* First we give a characterization of λ-tight frames in terms of their R-dual sequences. This is a kind of general version of an important duality principle from Gabor theory [11, Corollary 7.3.2], which follows from the Wexler-Raz biorthogonality relations.

Proposition 4.15. *The following conditions are equivalent.*

(i) *$(f_i)_{i\in\mathbb{N}}$ is a λ-tight frame.*
(ii) *$(\frac{1}{\sqrt{\lambda}} w_j^f)_{j\in\mathbb{N}}$ is an orthonormal system.*

Proof. Using Theorem 4.13, condition (i) holds if and only if, for all $\phi = \sum_j a_j e_j$, where $a = (a_j)_{j\in\mathbb{N}}$ is a sequence of scalars, we have

$$\lambda\|a\|_2^2 = \lambda\|\phi\|^2 = \sum_{i=1}^{\infty} |\langle \phi, f_i\rangle|^2 = \left\|\sum_{j=1}^{\infty} a_j w_j^f\right\|^2 .$$

But $\lambda\|a\|_2^2 = \|\sum_j a_j w_j^f\|^2$ is equivalent to $(\frac{1}{\sqrt{\lambda}} w_j^f)_{j\in\mathbb{N}}$ being an orthonormal system. □

This shows in particular that the sequence $(f_i)_{i\in\mathbb{N}}$ is a Parseval frame if and only if its R-dual sequence is an orthonormal system.

By using well-known equivalent conditions for a frame to be exact, we can characterize these frames in terms of their R-dual sequence by using previous results.

Proposition 4.16. *The following conditions are equivalent.*

(i) $(f_i)_{i\in\mathbb{N}}$ *is an exact frame.*
(ii) $(f_i)_{i\in\mathbb{N}}$ *is a Riesz basis.*
(iii) $(f_i)_{i\in\mathbb{N}}$ *is a frame that is* ω*-independent.*
(iv) $(w_j^f)_{j\in\mathbb{N}}$ *is a Riesz basic sequence such that whenever* $b = (b_i)_{i\in\mathbb{N}}$ *is a sequence of scalars,* $g_n = \sum_{i=1}^n b_i h_i$ *for every* $n \in \mathbb{N}$, *and*

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} |\langle g_n, w_j^f\rangle|^2 = 0,$$

then $b = 0$.
(v) $(w_j^f)_{j\in\mathbb{N}}$ *is a Riesz basis.*

Proof. The equivalence of (i), (ii), and (iii) is well known. Moreover, (iii) ⇔ (iv) follows immediately from Theorem 4.13, Proposition 4.3, and Remark 3.3. To prove (ii) ⇔ (v) first assume that $(f_i)_{i\in\mathbb{N}}$ is a Riesz basis. Then Theorem 4.13, Lemma 4.11, and the fact that $\ker T_f = \{0\}$ imply that $(w_j^f)_{j\in\mathbb{N}}$ is also a Riesz basis. The converse can be shown the same way because of Remark 3.3. □

4.3. **Relationships between frames.** Given two frames for $\mathcal{H}$, if they are equivalent or even unitarily equivalent, they often share the same properties. Therefore in the following we characterize those pairs of frames, which are (unitarily) equivalent.

If $(f_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$ are tight frames for $\mathcal{H}$, the following proposition is a kind of generalization of the Gabor frame result by Balan and Landau [1], which classifies the equivalent Gabor frames.

Proposition 4.17. *Let* $(f_i)_{i\in\mathbb{N}}$ *and* $(g_i)_{i\in\mathbb{N}}$ *be frames for* $\mathcal{H}$. *Then the following conditions are equivalent.*

(i) $(f_i)_{i\in\mathbb{N}}$ *is equivalent to* $(g_i)_{i\in\mathbb{N}}$.
(ii) *The spans of* $(w_j^f)_{j\in\mathbb{N}}$ *and* $(w_j^g)_{j\in\mathbb{N}}$ *are equal.*

Proof. Recall that $(f_i)_{i\in\mathbb{N}}$ is equivalent to $(g_i)_{i\in\mathbb{N}}$ if and only if $\ker T_f = \ker T_g$. Now the claim follows from Lemma 4.11. □

The following proposition deals with unitarily equivalent frames.

Proposition 4.18. *Let $(f_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$ be frames for $\mathcal{H}$. Then the following conditions are equivalent.*

(i) *$(f_i)_{i\in\mathbb{N}}$ is unitarily equivalent to $(g_i)_{i\in\mathbb{N}}$.*
(ii) *The frame operators S_{w^f} and S_{w^g} for $w^f = (w_j^f)_{j\in\mathbb{N}}$ and $w^g = (w_j^g)_{j\in\mathbb{N}}$ are equal.*

Proof. $(f_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$ are unitarily equivalent if and only if, for all $(b_i)_{i\in\mathbb{N}} \in (\ker T_f)^\perp$,

$$\left\| \sum_{i=1}^\infty b_i f_i \right\|^2 = \left\| \sum_{i=1}^\infty b_i g_i \right\|^2.$$

By Proposition 4.1, this in turn is equivalent to

$$\sum_{j=1}^\infty |\langle \phi, w_j^f \rangle|^2 = \sum_{j=1}^\infty |\langle \phi, w_j^g \rangle|^2$$

for all $\phi = \sum_i b_i h_i$. The claim now follows immediately from $\sum_j |\langle \phi, w_j^f \rangle|^2 = \langle S_{w^f}\phi, \phi\rangle$ and $\sum_j |\langle \phi, w_j^g \rangle|^2 = \langle S_{w^g}\phi, \phi\rangle$. □

We may also ask under which conditions both frames possess the same frame operator. This is an immediate corollary from the previous proposition by using the duality relation between a sequence and its R-dual sequence.

Proposition 4.19. *Let $(f_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$ be frames for $\mathcal{H}$. Then the following conditions are equivalent.*

(i) *$(f_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$ have the same frame operator.*
(ii) *$(w_j^f)_{j\in\mathbb{N}}$ is unitarily equivalent to $(w_j^g)_{j\in\mathbb{N}}$.*

Proof. The proof follows immediately from Proposition 4.18 by applying Remark 3.3. □

4.4. **The dual frame.** We will study properties of alternate dual frames and canonical dual frames.

First we characterize all alternate dual frames of a frame $(f_i)_{i\in\mathbb{N}}$ in terms of the R-dual sequence. This can be regarded as Wexler-Raz biorthogonality relations in abstract frame theory.

Theorem 4.20. *Let $(f_i)_{i\in\mathbb{N}}$ and $(\psi_i)_{i\in\mathbb{N}}$ be frames for $\mathcal{H}$. Define the frame operator for $(f_i)_{i\in\mathbb{N}}$ relative to $(\psi_i)_{i\in\mathbb{N}}$ by*

$$S_{f,\psi}\phi = \sum_{i=1}^\infty \langle \phi, f_i \rangle \psi_i.$$

Then

$$S_{f,\psi}\phi = \sum_{j=1}^\infty \left\langle \phi, \sum_{k=1}^\infty \langle w_k^f, w_j^\psi \rangle e_k \right\rangle e_j.$$

Moreover, the following conditions are equivalent.

(i) *$(\psi_i)_{i\in\mathbb{N}}$ is an alternate dual frame of $(f_i)_{i\in\mathbb{N}}$.*
(ii) *$S_{f,\psi} = S_{\psi,f} = I$.*
(iii) *$\langle w_k^f, w_j^\psi \rangle = \delta_{jk}$ for all $j, k \in \mathbb{N}$.*

Proof. The equivalence of (i) and (ii) follows from the definition.

To prove the concrete form of $S_{f,\psi}$ and (ii) $\Leftrightarrow$ (iii), for each $\phi \in \mathcal{H}$, we compute

$$\begin{aligned}
S_{f,\psi}\phi &= \sum_{i=1}^{\infty} \langle \phi, f_i \rangle \psi_i \\
&= \sum_{i=1}^{\infty} \left\langle \phi, \sum_{k=1}^{\infty} \langle w_k^f, h_i \rangle e_k \right\rangle \sum_{j=1}^{\infty} \langle w_j^\psi, h_i \rangle e_j \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle \phi, e_k \rangle \langle h_i, w_k^f \rangle \langle w_j^\psi, h_i \rangle e_j \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle \phi, e_k \rangle \left\langle w_j^\psi, \sum_{i=1}^{\infty} \langle w_k^f, h_i \rangle h_i \right\rangle e_j \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle \phi, e_k \rangle \langle w_j^\psi, w_k^f \rangle e_j \\
&= \sum_{j=1}^{\infty} \left\langle \phi, \sum_{k=1}^{\infty} \langle w_k^f, w_j^\psi \rangle e_k \right\rangle e_j.
\end{aligned}$$

This proves the first part of the claim.

Now $S_{f,\psi} = I$ if and only if

$$\left\langle \phi, \sum_{k=1}^{\infty} \langle w_k^f, w_j^\psi \rangle e_k \right\rangle = \langle \phi, e_j \rangle \quad \text{for all } j \in \mathbb{N},\ \phi \in \mathcal{H}.$$

This in turn is equivalent to

$$\sum_{k=1}^{\infty} \langle w_k^f, w_j^\psi \rangle e_k = e_j \quad \text{for all } j \in \mathbb{N}.$$

Combining this with the fact that $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis finishes the proof. □

Now we investigate the properties and the construction of alternate dual frames in more detail. They can be viewed as a kind of generalization of results from [16, 7, 14], which are combined in [11, Lemma 7.6.1] and [11, Proposition 7.6.2].

Theorem 4.21. *Let $(f_i)_{i\in\mathbb{N}}$ be a frame for $\mathcal{H}$ with frame operator denoted by S. Then $w_j^{S^{-1}f} \in \operatorname{span}_{l\in\mathbb{N}}\{w_l^f\}$ for all $j \in \mathbb{N}$. Moreover, the following conditions are equivalent.*

(i) *$(\psi_i)_{i\in\mathbb{N}}$ is an alternate dual frame of $(f_i)_{i\in\mathbb{N}}$.*

(ii) *There exists a Bessel sequence $(k_j)_{j\in\mathbb{N}}$ in $(\operatorname{span}_{l\in\mathbb{N}}\{w_l^f\})^\perp$ so that*

$$w_j^\psi = w_j^{S^{-1}f} + k_j \quad \textit{for all } j \in \mathbb{N}.$$

Proof. Since $(S^{-1}f_i)_{i\in\mathbb{N}}$ is equivalent to $(f_i)_{i\in\mathbb{N}}$, Theorem 4.17 implies that $\operatorname{span}_{l\in\mathbb{N}}\{w_j^{S^{-1}f}\} = \operatorname{span}_{l\in\mathbb{N}}\{w_l^f\}$. This proves the first claim.

Now suppose that $(\psi_i)_{i\in\mathbb{N}}$ is an alternate dual frame of $(f_i)_{i\in\mathbb{N}}$. Then

$$e_j = \sum_{i=1}^{\infty} \langle e_j, \psi_i \rangle f_i = \sum_{l=1}^{\infty}\sum_{i=1}^{\infty} \langle e_j, \psi_i \rangle \langle f_i, e_l \rangle e_l.$$

Hence,

$$\sum_{i=1}^{\infty} \langle e_j, \psi_i \rangle \langle f_i, e_j \rangle = 1 \tag{3}$$

and for all $l \neq j$

$$\sum_{i=1}^{\infty} \langle e_j, \psi_i \rangle \langle f_i, e_l \rangle = 0. \tag{4}$$

Hence, letting

$$\psi_i = S^{-1}f_i + (\psi_i - S^{-1}f_i),$$

we have

$$\begin{aligned} &\sum_{i=1}^{\infty} \langle e_j, S^{-1}f_i + (\psi_i - S^{-1}f_i) \rangle \langle f_i, e_l \rangle \\ &= \left\langle \sum_{i=1}^{\infty} \langle e_j, S^{-1}f_i \rangle f_i, e_l \right\rangle + \left\langle \sum_{i=1}^{\infty} \langle e_j, \psi_i - S^{-1}f_i \rangle f_i, e_l \right\rangle \\ &= \langle e_j, e_l \rangle + \left\langle \sum_{i=1}^{\infty} \langle e_j, \psi_i - S^{-1}f_i \rangle f_i, e_l \right\rangle, \end{aligned}$$

since $(f_i)_{i\in\mathbb{N}}$ is a frame. Applying (3) and (4) yields

$$\sum_{i=1}^{\infty} \langle e_j, \psi_i - S^{-1}f_i \rangle f_i = 0 \quad \text{for all } j \in \mathbb{N}.$$

Thus $k_l = \sum_i \langle \psi_i - S^{-1}f_i, e_l \rangle h_i$ is a Bessel sequence in $(\mathrm{span}_{j\in\mathbb{N}}\{w_j^f\})^\perp$ by Lemma 4.11 and $w_j^\psi = w_j^{S^{-1}f} + k_j$. This shows (i) $\Rightarrow$ (ii).

Now suppose (ii) holds. It is easy to check that

$$w_j^\psi = w_j^{S^{-1}f} + k_j \quad \text{for all } j \in \mathbb{N}$$

implies

$$\psi_i = \sum_{j=1}^{\infty} \left[\langle S^{-1}f_i, e_j \rangle + \langle k_j, h_i \rangle \right] e_j \quad \text{for all } i \in \mathbb{N}.$$

So, for each $l \in J$,

$$\begin{aligned}\sum_{i=1}^{\infty}\langle e_l,\psi_i\rangle f_i &= \sum_{i=1}^{\infty}\left\langle e_l,\sum_{j=1}^{\infty}\left[\langle S^{-1}f_i,e_j\rangle+\langle k_j,h_i\rangle\right]e_j\right\rangle f_i\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\langle e_j,S^{-1}f_i\rangle+\langle h_i,k_j\rangle\right]\langle e_l,e_j\rangle f_i\\ &= \sum_{i=1}^{\infty}\left[\langle e_l,S^{-1}f_i\rangle+\langle h_i,k_l\rangle\right]f_i\\ &= \sum_{i=1}^{\infty}\langle e_l,S^{-1}f_i\rangle f_i+\sum_{i=1}^{\infty}\langle h_i,k_l\rangle f_i.\end{aligned}$$

Since $(f_i)_{i\in\mathbb{N}}$ is a frame, $\sum_i\langle e_l,S^{-1}f_i\rangle f_i = e_l$. Moreover, $(k_l)_{l\in\mathbb{N}} \in (\mathrm{span}_{j\in\mathbb{N}}\{w_j^f\})^\perp$. Hence Lemma 4.11 implies that $\sum_i\langle h_i,k_l\rangle f_i = 0$. This proves that $(\psi_i)_{i\in\mathbb{N}}$ is an alternate dual frame of $(f_i)_{i\in\mathbb{N}}$. □

Among the alternate dual frames the canonical dual frame is distinguished by the following properties.

Proposition 4.22. *Let $(f_i)_{i\in\mathbb{N}}$ be a frame for $\mathcal{H}$ with frame operator denoted by S and let $(\psi_i)_{i\in\mathbb{N}}$ be an alternate dual frame of $(f_i)_{i\in\mathbb{N}}$. Then the following conditions are equivalent.*

(i) *$\psi_i = S^{-1}f_i$ for all $i \in \mathbb{N}$.*

(ii) *If $(\tilde{\psi}_i)_{i\in\mathbb{N}}$ is an alternate dual frame of $(f_i)_{i\in\mathbb{N}}$, we have*

$$\|w_j^\psi\| < \|w_j^{\tilde{\psi}}\|$$

for all $j \in J$ for which $\tilde{\psi}_j \neq \psi_j$.

(iii) *If $(\tilde{\psi}_i)_{i\in\mathbb{N}}$ is an alternate dual frame of $(f_i)_{i\in\mathbb{N}}$, we have*

$$\left\|\frac{w_j^\psi}{\|w_j^\psi\|}-\frac{w_j^f}{\|w_j^f\|}\right\| < \left\|\frac{w_j^{\tilde{\psi}}}{\|w_j^{\tilde{\psi}}\|}-\frac{w_j^f}{\|w_j^f\|}\right\|$$

for all $j \in \mathbb{N}$ for which $\tilde{\psi}_j \neq \psi_j$.

Proof. By Theorem 4.21, an alternate dual is of the form $w_j^\psi = w_j^{S^{-1}f} + k_j$, $j \in \mathbb{N}$, where $w_j^{S^{-1}f} \in \mathrm{span}_{l\in\mathbb{N}}\{w_l^f\}$ for all $j \in \mathbb{N}$ and $(k_j)_{j\in\mathbb{N}}$ is a Bessel sequence in $(\mathrm{span}_{l\in\mathbb{N}}\{w_l^f\})^\perp$. Hence $\|w_j^\psi\|^2 = \|w_j^{S^{-1}f}\|^2 + \|k_j\|^2 \geq \|w_j^{S^{-1}f}\|^2$ with equality if and only if $(\psi_i)_{i\in\mathbb{N}} = (S^{-1}f_i)_{i\in\mathbb{N}}$.

To prove (ii) ⇔ (iii), we compute

$$\left\|\frac{w_j^\psi}{\|w_j^\psi\|}-\frac{w_j^f}{\|w_j^f\|}\right\|^2 = 2-\frac{\langle w_j^f,w_j^\psi\rangle+\langle w_j^\psi,w_j^f\rangle}{\|w_j^f\|\|w_j^\psi\|}.$$

Now Theorem 4.20 implies $\langle w_j^\psi, w_j^f\rangle = 1$ for all $j \in \mathbb{N}$. Therefore (ii) and (iii) are equivalent. □

5. Application to Gabor systems

In this section we will apply the general results obtained in the preceding sections to the special case of Gabor systems. Moreover, in special cases we prove the existence of pairs of orthonormal bases, which gives rise to particular R-dual sequences. This raises the hope to obtain the general duality theory for Gabor frames from our theory. But for this we need to resolve Problem 5.6.

5.1. Necessary conditions.

In order to apply the results from Section 4 to Gabor systems, we first need to check whether the R-dual sequence exists for each Gabor system.

We start with a technical lemma which will be needed in the following.

Lemma 5.1. *Let $a, b > 0$ and define $c = \min\{a, \frac{1}{b}\}$. Fix some $l \in \mathbb{Z}$. Then, for all $g \in L^2(\mathbb{R})$ and for all $f \in L^2([lc,(l+1)c])$, we have*

$$\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2 = b\int_{lc}^{(l+1)c} |f(x)|^2 \sum_{n\in\mathbb{Z}} |g(x-nc)|^2 dx.$$

Proof. First note that $\sum_{n\in\mathbb{Z}} |g(x-nc)|^2$ converges almost everywhere, since $g \in L^2(\mathbb{R})$ and $\int_{lc}^{(l+1)c} \sum_{n\in\mathbb{Z}} |g(x-nc)|^2 dx = \|g\|_2^2 < \infty$. Then we have

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2 &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\langle E_{mb}, f\cdot T_{na}\overline{g}\rangle|^2 \\ &= \sum_{n\in\mathbb{Z}} b\|f\cdot T_{na}\overline{g}\|^2 \\ &= b\int_{lc}^{(l+1)c} |f(x)|^2 \sum_{n\in\mathbb{Z}} |g(x-nc)|^2 dx\end{aligned}$$

□

The next result will determine a large class of orthonormal bases for which the hypothesis of Definition 3.1 is satisfied.

Proposition 5.2. *Let $a, b > 0$. Then, for all $g \in L^2(\mathbb{R})$ and for all bounded compactly supported $f \in L^2(\mathbb{R})$, we have*

$$\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2 < \infty.$$

Proof. Let $c = \min\{a, \frac{1}{b}\}$. We can write f as

$$\sum_{k=-N}^{N} f\chi_{[kc,(k+1)c]}.$$

Using Lemma 5.1 and with $\operatorname{esssup}_{x\in\mathbb{R}}|f(x)| \le B$, we obtain

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\rangle|^2 &= \sqrt{\sum_{m,n\in\mathbb{Z}} \left|\left\langle E_{mb}T_{na}g, \sum_{k=-N}^{N} f\chi_{[kc,(k+1)c]}\right\rangle\right|^2} \\ &\le \sum_{k=-N}^{N} \sqrt{\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}g, f\chi_{[kc,(k+1)c]}\rangle|^2} \\ &\le \sqrt{b}\sqrt{\sum_{k=-N}^{N} \int_{kc}^{(k+1)c} |f(x)|^2 \sum_{n\in\mathbb{Z}} |g(x-nc)|^2 dx} \\ &\le B\sqrt{b}2N\|g\|^2.\end{aligned}$$

□

Now it follows easily that w_j^f exists for all $j \in \mathbb{N}$ and for every Gabor family $\mathcal{G}(g,a,b) = (E_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$.

Corollary 5.3. *Let $a, b > 0$, let $g \in L^2(\mathbb{R})$ and let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R})$ with e_j bounded and compactly supported for each $j \in \mathbb{N}$. Then, for each orthonormal basis $(h_i)_{i\in\mathbb{N}}$ for $L^2(\mathbb{R})$, $w_j^{\mathcal{G}(g,a,b)}$ exists for all $j \in \mathbb{N}$.*

Proof. This follows immediately from Proposition 5.2. □

5.2. **A special form of $w^{\mathcal{G}(g,a,b)}$.** Recall that the Ron-Shen duality principle [15, Theorem 2.2 (e)] states that, for $g \in L^2(\mathbb{R})$ and $a, b > 0$, the Gabor system $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbb{R})$ if and only if $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is a Riesz basic sequence. Hence we are interested in whether the system $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is indeed a R-dual sequence of $\mathcal{G}(g,a,b)$. For this, we need to find suitable orthonormal bases. Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be such that $\mathcal{G}(g,a,b)$ is a λ-tight frame for $L^2(\mathbb{R})$. For this class of sequences we can indeed find such orthonormal bases. Moreover, the proof of the following proposition gives an explicit construction.

Proposition 5.4. *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be such that $\mathcal{G}(g,a,b)$ is a λ-tight frame for $L^2(\mathbb{R})$. Then, for each bounded and compactly supported orthonormal basis $(e_j)_{j\in\mathbb{N}}$ for $L^2(\mathbb{R})$, there exists a unitary operator $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ so that $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is the R-dual sequence of $\mathcal{G}(g,a,b)$ with respect to $(e_j)_{j\in\mathbb{N}}$ and $(U(e_j))_{j\in\mathbb{N}}$.*

Proof. Without loss of generality we can assume that the index set J is of the form $\mathbb{Z}^2$ and $\lambda = 1$. By Corollary 5.3, the R-dual sequence $(w_{jk}^{\mathcal{G}(g,a,b)})_{j,k\in\mathbb{Z}}$ defined by

$$w_{jk}^{\mathcal{G}(g,a,b)} = \sum_{m,n\in\mathbb{Z}} \langle E_{mb}T_{na}g, e_{jk}\rangle e_{mn}$$

does exist. By Proposition 4.15, the system $(w_{jk}^{\mathcal{G}(g,a,b)})_{j,k\in\mathbb{Z}}$ is an orthonormal basis for its closed linear span. Now let $(\varphi_{jk})_{j,k\in\mathbb{Z}}$ and $(\psi_{jk})_{j,k\in\mathbb{Z}}$ be fixed orthonormal bases for $\operatorname{span}\{w_{jk}^{\mathcal{G}(g,a,b)} : j,k \in \mathbb{Z}\}^\perp$ and $\operatorname{span}\{E_{\frac{j}{a}}T_{\frac{k}{b}}g : j,k \in \mathbb{Z}\}^\perp$, respectively. For this to work, we need to know that these two spaces have the same dimension. To see this, recall that for a Gabor frame $\mathcal{G}(g,a,b)$ with analysis operator T_g, if $ab = 1$ then $\ker T_g = \{0\}$ and $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ spans

$L^2(\mathbb{R})$. Also, if $ab < 1$, then $\dim(\ker T_g) = \infty$ and $\dim(\text{span}\{E_{\frac{j}{a}} T_{\frac{k}{b}} g : j,k \in \mathbb{Z}\}^{\perp}) = \infty$. Combined with Lemma 4.11, this implies that our spaces have the same dimension. Now define the operator $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$U(w_{jk}^{\mathcal{G}(g,a,b)}) = \frac{1}{\|g\|_2} E_{\frac{j}{a}} T_{\frac{k}{b}} g \quad \text{and} \quad U(\varphi_{jk}) = \psi_{jk} \quad \text{for all } j,k \in \mathbb{Z}.$$

By [11, Corollary 7.3.2], the system $(E_{\frac{j}{a}} T_{\frac{k}{b}} g)_{j,k\in\mathbb{Z}}$ is orthogonal with $\|E_{\frac{j}{a}} T_{\frac{k}{b}} g\|_2 = \|g\|_2$ and hence U is a unitary operator which satisfies

$$\frac{1}{\|g\|_2} E_{\frac{j}{a}} T_{\frac{k}{b}} g = U(w_{jk}^{\mathcal{G}(g,a,b)}) = \sum_{m,n\in\mathbb{Z}} \langle E_{mb} T_{na} g, e_{jk}\rangle U(e_{mn}).$$

□

In the special case $ab = 1$ we can say even more.

Remark 5.5. Let $g \in L^2(\mathbb{R})$ and $a, b > 0$, $ab = 1$. Further we define $(e_{mn})_{m,n\in\mathbb{N}}$ and $(h_{ij})_{i,j\in\mathbb{N}}$ by

$$e_{mn} = \frac{1}{\sqrt{a}} E_{\frac{m}{a}} T_{na} \chi_{[0,a]} \quad \text{and} \quad h_{ij} = \frac{1}{\sqrt{b}} E_{mb} T_{\frac{n}{b}} \chi_{[0,\frac{1}{b}]}.$$

Then $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is the R-dual sequence of $\mathcal{G}(g,a,b)$ with respect to the orthonormal bases $(e_{mn})_{m,n\in\mathbb{N}}$ and $(h_{ij})_{i,j\in\mathbb{N}}$.

This can be proven by a straightforward but long calculation.

This seems to indicate that the following problem can be solved.

Problem 5.6. *Given a Gabor frame (sequence) $\mathcal{G}(g,a,b)$, find orthonormal bases $(e_j)_{j\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ for $L^2(\mathbb{R})$ so that $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a}) = (w_{jk}^{\mathcal{G}(g,a,b)})_{j,k\in\mathbb{Z}}$.*

If this is true, we obtain most known duality principles in Gabor theory e.g. the Ron-Shen duality principle as corollaries from our results.

References

[1] R. Balan and Z. Landau, *A note on equivalence of Weyl-Heisenberg frames*, preprint.

[2] P.G. Casazza, *Modern tools for Weyl-Heisenberg (Gabor) Frame Theory*, Adv. in Imag. and Electron. Physics, **115** (2001), 1–127.

[3] P.G. Casazza, *The art of frame theory*, Taiwanese J. of Math., **4** (2000), 129–201.

[4] P.G. Casazza and J. Kovačević, *Uniform Tight Frames with Erasures*, preprint.

[5] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[6] I. Daubechies, A. Grossmann, and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., **27** (1986), 1271–1283.

[7] I. Daubechies, H.J. Landau, and Z. Landau, *Gabor time-frequency lattices and the Wexler-Raz identity*, J. Fourier Anal. Appl., **1** (1995), 437–478.

[8] R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., **72** (1952), 341–366.

[9] D. Gabor, *Theory of communications*, J. Inst. Elec. Engrg., **93** (1946), 429–457.

[10] V.K. Goyal, J. Kovačević, and J.A. Kelner, *Quantized frame expansions with erasures*, Appl. Comput. Harmon. Anal., **10** (2001), 203–233.

[11] K. Gröchenig, *Foundations of time-frequency analysis*, Birkhäuser, Boston, 2001.

[12] B. Hassibi, B. Hochwald, A. Shokrollahi, and W. Sweldens, *Representation theory for high-rate multiple-antenna code design*, IEEE Trans. Inform. Th., **47** (2001), 2355–2367.

[13] C. Heil and D. Walnut *Continuous and discrete Wavelet transforms*, SIAM Review, **31** (1989), 628–666.
[14] A.J.E.M. Janssen, *Duality and biorthogonality for Weyl-Heisenberg frames*, J. Fourier Anal. Appl., **1** (1995), 403–436.
[15] A. Ron and Z. Shen, *Weyl-Heisenberg frames and Riesz bases in* $L_2(\mathbb{R}^d)$, Duke Math. J., **89** (1997), 237–282.
[16] J. Wexler and S. Raz, *Discrete Gabor expansions*, Signal Proc., **21** (1990), 207–221.
[17] R. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, (1980).

(P. G. Casazza) Department of Mathematics, University of Missouri, Columbia, Missouri 65211, USA
E-mail address: pete@math.missouri.edu

(G. Kutyniok) Department of Mathematics and Computer Science, University of Paderborn, 33095 Paderborn, Germany
E-mail address: gittak@uni-paderborn.de

(M. C. Lammers) Department of Mathematics, Western Washington University, Bellingham, Washington 98225, USA
E-mail address: lammerm@cc.wwu.edu