

# FRAMES OF TRANSLATES

PETER G. CASAZZA, OLE CHRISTENSEN, AND NIGEL J. KALTON

ABSTRACT. Frames consisting of translates of a single function play an important role in sampling theory as well as in wavelet theory and Gabor analysis. We give a necessary and sufficient condition for a subfamily of regularly spaced translates of a function  $\phi \in L^2(\mathbb{R})$ ,  $(\tau_{nb}\phi)_{n \in \Lambda}$ ,  $\Lambda \subset \mathbb{Z}$ , to form a frame (resp. Riesz basis) for its closed linear span. One consequence is that if  $\Lambda \subset \mathbb{N}$ , then this family is a frame if and only if it is a Riesz basis. For the case of arbitrary translates of a function  $\phi \in L^1(\mathbb{R})$  we show that for sparse sets, having an upper frame bound is equivalent to the family being a frame sequence. Also we give some relatively mild density conditions which will yield frame sequences. Finally, we use the fractional Hausdorff dimension to identify classes of exact frame sequences.

## 1. PRELIMINARIES

Let  $H$  be a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  linear in the first entry. Let  $\Lambda$  be any countable index set. Recall that a sequence  $(f_n)_{n \in \Lambda} \subseteq H$  is a *frame* for  $H$  if

$$(1.1) \quad \exists A, B > 0 : A\|f\|^2 \leq \sum_{n \in \Lambda} |\langle f, f_n \rangle|^2 \leq B\|f\|^2, \forall f \in H.$$

$A, B$  are called the *lower and upper frame bounds*. They are not unique: the biggest lower bound and the smallest upper bound are called the *optimal frame bounds*. It can be shown that every element  $f \in H$  has a representation as an infinite linear combination of the frame elements: using the *frame operator*

$$S : H \rightarrow H, \quad Sf = \sum \langle f, f_n \rangle f_n.$$

Then

$$f = SS^{-1}f = \sum \langle f, S^{-1}f_n \rangle f_n, \quad \forall f \in H.$$

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Clearly a frame  $(f_n)$  is total, i.e.,  $\overline{\text{span}}(f_n) = H$ . In case  $(f_n)$  is a frame for  $\overline{\text{span}}(f_n)$  (which in general might be a proper subspace of  $H$ ) we say that  $(f_n)$  is a *frame sequence*. We say that  $(f_n)$  is an *exact frame sequence* if  $(f_n)$  is a Riesz basis for  $\overline{\text{span}}(f_n)$ ; i.e., if

$$(1.2) \quad \exists A, B > 0 : A \sum |c_n|^2 \leq \left\| \sum_{n \in \Lambda} c_n f_n \right\|^2 \leq B \sum |c_n|^2,$$

for all  $(c_n) \in c_{00}(\Lambda)$ , the space of finitely nonzero sequences.

Let us remark here that (1.1) implies that  $(f_n)_{n \in \Lambda}$  is a frame for  $H$  with frame bounds  $A$  and  $B$ , if and only if there is bounded operator  $V : H \rightarrow \ell^2(\Lambda)$  defined by  $Vf = \{\langle f, f_n \rangle\}_{n \in \Lambda}$  with  $A\|f\|^2 \leq \|Vf\|^2 \leq B\|f\|^2$  for  $f \in H$ . Thus  $V$  must be an isomorphism onto its range  $V(H)$ . Let  $(e_n)_{n \in \Lambda}$  denote the canonical basis vectors in  $\ell^2(\Lambda)$ . We then see, taking adjoints, that the above condition is also equivalent to the existence of a bounded surjective operator  $T : \ell^2(\Lambda) \rightarrow H$  so that  $Te_n = f_n$  and  $A\|a\|^2 \leq \|Ta\|^2 \leq B\|a\|^2$  for  $a \in (\ker T)^\perp$  or, equivalently  $Ad(a, \ker T)^2 \leq \|Ta\|^2 \leq B\|f\|^2$ . The frame operator is  $S = TV = TT^*$ .

Thus in particular  $(f_n)_{n \in \Lambda}$  is a frame sequence with constants  $A, B$  if and only if we have the linear map  $T : c_{00}(\Lambda) \rightarrow H$  defined by  $Te_n = f_n$  extends to a bounded operator on  $\ell^2(\Lambda)$  and satisfies

$$(1.3) \quad A\|a\|^2 \leq \|Ta\|^2 \leq B\|a\|^2$$

for  $a \in (\ker T)^\perp$ . Comparing with (1.2) we see that  $(f_n)$  is an exact frame sequence if and only if  $T$  is one-one.

The most important examples of frame sequences comes from sampling theory [5]. For  $x \in \mathbb{R}$  we define *translation by  $x$*  by

$$\tau_x : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), (\tau_x f)(y) = f(y - x), y \in \mathbb{R}.$$

It is proved in [4] that a collection  $(\tau_{x_n} \phi)$ , where  $\phi \in L^2(\mathbb{R})$ ,  $(x_n) \subseteq \mathbb{R}$  can never be a frame for  $L^2(\mathbb{R})$ . However, frame sequences of this form exists and play an important role in sampling theory as well as wavelet theory, cite5,6,8.

We start by considering sequences of the form  $(\tau_{nb} \phi)_{n \in \Lambda}$ ,  $\phi \in L^2(\mathbb{R})$ ,  $\Lambda \subset \mathbb{Z}$ , In section 2 we give necessary and sufficient conditions for such sets to form frame (resp. exact frame) sequences. This extends work of Benedetto and Walnut [2] and Benedetto and Li [1] as well as removing an unnecessary hypothesis in their

results. Kim and Lim [12] also showed that this was an unnecessary hypothesis in [2]. As one of several applications of this result, we show that if  $\Lambda \subset \mathbb{N}$ , then  $(\tau_{nb}\phi)_{n \in \Lambda}$  is a frame sequence if and only if it is an exact frame sequence. In section 3 we give conditions for an arbitrary sequence of translates to have finite upper frame bound. In section 4 we consider the case where  $\Phi_b$  (see section 2 for the definition) is continuous and use the cardinality of the zero set of  $\Phi_b$  and the density of our set to produce frame sequences. Finally, in section 5 we relate the fractional Hausdorff dimension to exact frame sequences of translates.

## 2. FRAMES OF TRANSLATES

Our first theorem is a generalization of a result of Benedetto and Li [1]. The proof we give is considerably simpler.

We first introduce some notation. For a function  $\phi \in L^1(\mathbb{R})$  we denote by  $\hat{\phi}$  the *Fourier transform of  $\phi$*

$$\hat{\phi}(\xi) = \int \phi(x) e^{-2\pi i \xi x} dx.$$

As usual the definition of the Fourier transform extends to an isometry  $\phi \rightarrow \hat{\phi}$  on  $L^2(\mathbb{R})$ .

Now suppose  $\phi \in L^2(\mathbb{R})$  and that  $b > 0$ . Let us identify the circle  $\mathbb{T}$  with the interval  $[0, 1)$  via the standard map  $\xi \rightarrow e^{2\pi i \xi}$ . We define the function  $\Phi_b : \mathbb{T} \rightarrow \mathbb{R}$  by

$$\Phi_b(\xi) = \sum_{n \in \mathbb{Z}} \left| \hat{\phi}\left(\frac{\xi + n}{b}\right) \right|^2.$$

Note that  $\Phi_b \in L^1(\mathbb{T})$ .

For any  $n \in \mathbb{Z}$  we note that

$$\langle \tau_{nb}\phi, \phi \rangle = \langle e^{-2\pi i n \xi b} \hat{\phi}, \hat{\phi} \rangle = \frac{1}{b} \int_0^1 \Phi_b(\xi) e^{-2\pi i n \xi} d\xi = \frac{1}{b} \hat{\Phi}_b(n).$$

If  $\Lambda \subset \mathbb{Z}$  we let  $H_\Lambda$  be the closed subspace of  $L^2(\mathbb{T})$  generated by the characters  $e^{2\pi i n \xi}$  for  $n \in \Lambda$ . We let  $E_\Lambda$  be the closed subspace of  $H_\Lambda$  of all  $f$  such that  $\Phi_b(\xi)f(\xi) = 0$  a.e. If  $f \in H_\Lambda$  we denote by  $d(f, E_\Lambda)$  the distance of  $f$  to the subspace  $E_\Lambda$ .

**Theorem 2.1.** *Suppose  $\phi \in L^2(\mathbb{R})$  and  $b > 0$ . If  $\Lambda \subset \mathbb{Z}$  then  $(\tau_{nb}\phi)_{n \in \Lambda}$  is a frame sequence with frame bounds  $A$  and  $B$  if and only if for every  $f \in H_\Lambda$  we have*

$$Ad(f, E_\Lambda)^2 \leq \frac{1}{b} \int_0^1 |f(\xi)|^2 \Phi_b(\xi) d\xi \leq B \|f\|^2,$$

or equivalently, for all  $f \in H_\Lambda \cap E_\Lambda^\perp$ ,

$$A \|f\|^2 \leq \frac{1}{b} \int_0^1 |f(\xi)|^2 \Phi_b(\xi) d\xi \leq B \|f\|^2.$$

Furthermore, if this condition is satisfied,  $(\tau_{nb}\phi)_{n \in \Lambda}$  is an exact frame sequence with the same bounds if and only if  $E_\Lambda = \{0\}$ .

*Proof.* By our remarks and the definition  $(\tau_{nb}\phi)_{n \in \Lambda}$  is a frame sequence with frame bounds  $A, B$  if and only if the linear map  $T : c_{00}(\Lambda) \rightarrow L^2(\mathbb{R})$  defined by  $Te_n = \tau_{nb}\phi$  extends to a bounded linear operator  $T : \ell^2(\Lambda) \rightarrow L^2(\mathbb{R})$  such that  $Te_n = \tau_{nb}\phi$  and

$$Ad(u, \ker T)^2 \leq \|Tu\|^2 \leq B \|u\|^2$$

for  $u \in \ell^2(\Lambda)$ .

Let  $U : H_\Lambda \rightarrow \ell^2(\Lambda)$  be the natural isometry  $Uf = \{\hat{f}(n)\}_{n \in \Lambda}$ . Then for any trigonometric polynomial  $f \in H_\Lambda$

$$\begin{aligned} \|TUF\|^2 &= \left\| \sum_{n \in \Lambda} \hat{f}(n) \tau_{nb}\phi \right\|^2 \\ &= \int_{-\infty}^{\infty} \left| \sum_{n \in \Lambda} \hat{f}(n) e^{-2\pi i n b \xi} \hat{\phi}(\xi) \right|^2 d\xi \\ &= \frac{1}{b} \sum_{n \in \mathbb{Z}} \int_0^1 |f(\xi)|^2 \left| \hat{\phi}\left(\frac{n + \xi}{b}\right) \right|^2 d\xi \\ &= \frac{1}{b} \int_0^1 |f(\xi)|^2 \Phi_b(\xi) d\xi. \end{aligned}$$

This immediately implies the theorem.  $\square$

Theorem 2.1 yields a generalization of a result of Benedetto and Li [1] which is part (3) of the next theorem (note that an unnecessary hypothesis in [1] is also eliminated).

**Theorem 2.2.** *If  $\phi \in L^2(\mathbb{R})$ , and  $b > 0$  then:*

(1)  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is an orthonormal sequence if and only if

$$\Phi_b(\gamma) = b \quad \text{a.e.}$$

(2)  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is an exact frame sequence with frame bounds  $A, B$  if and only if

$$bA \leq \Phi_b(\gamma) \leq bB \quad \text{a.e.}$$

(3)  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is a frame sequence with frame bounds  $A, B$  if and only if

$$(2.1) \quad bA \leq \Phi_b(\gamma) \leq bB \quad \text{a.e.}$$

on  $\mathbb{T} \setminus N_b$  where  $N_b = \{\xi \in \mathbb{T} : \Phi_b(\xi) = 0\}$ .

*Proof.* Note that (1) follows easily from the fact that  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is orthonormal if and only if  $TU$  is unitary. (2) is immediate from (3). For (3) we note that if  $\Lambda = \mathbb{Z}$  then  $H_\Lambda = L^2(\mathbb{T})$  and  $E_\Lambda = L^2(N_b)$ . Hence  $d(f, E_\Lambda)^2 = \int_{\mathbb{T} \setminus N_b} |f|^2 d\xi$ . The proof is then immediate.  $\square$

The next theorem addresses the relationship between frame properties for two families  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  and  $(\tau_{na}\phi)_{n \in \mathbb{Z}}$ . Observe that we can assume  $0 < b < a$  without loss of generality.

**Theorem 2.3.** *Let  $a, b \in \mathbb{R}$  with  $0 < b < a$ .*

(1) *There exists a function  $\phi \in L^2(\mathbb{R})$  so that  $(\tau_{na}\phi)_{n \in \mathbb{Z}}$  is a frame sequence but  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is not a frame sequence.*

(2) *If  $\frac{a}{b} \in \mathbb{Z}$  and  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is a frame sequence, then  $(\tau_{na}\phi)_{n \in \mathbb{Z}}$  is a frame sequence.*

(3) *If  $\frac{a}{b} \notin \mathbb{Z}$  then there is a function  $\phi \in L^2(\mathbb{R})$  so that  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is a frame sequence, but  $(\tau_{na}\phi)_{n \in \mathbb{Z}}$  is not a frame sequence.*

*Proof.* (1) Define a function  $\phi \in L^2(\mathbb{R})$  so that

$$\hat{\phi}(\xi) = 1, \quad \text{if } 0 \leq \xi \leq \frac{1}{a},$$

$$\hat{\phi}(\xi) = \frac{ab}{b-a} \left(x - \frac{1}{b}\right) \quad \text{if } \frac{1}{a} \leq \xi \leq \frac{1}{b},$$

and

$$\hat{\phi}(\xi) = 0, \quad \text{otherwise.}$$

Then we can easily check that

$$\Phi_b(\xi) = \hat{\phi}\left(\frac{\xi}{b}\right)^2,$$

for all  $\xi \in \mathbb{T}$ . Hence,  $\Phi_b(\xi)$  is not bounded below on  $\mathbb{T}$  and so  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is not a frame sequence. On the other hand, if  $\xi \in \mathbb{T}$  then  $\frac{\xi}{a} \in [0, \frac{1}{a}]$ , so that  $\Phi_a(\xi) \geq 1$ . Therefore,  $\Phi_a(\xi)$  is bounded below on  $\mathbb{T}$  and so  $(\tau_{na}\phi)_{n \in \mathbb{Z}}$  is a frame sequence.

(2) By our assumption, there is a natural number  $m$  so that  $a = mb$ . Hence,

$$\begin{aligned} \Phi_a(\xi) &= \sum_n \left| \hat{\phi}\left(\frac{\xi+n}{a}\right) \right|^2 = \sum_n \left| \hat{\phi}\left(\frac{\xi+n}{mb}\right) \right|^2 \\ &= \sum_{k=0}^{m-1} \left| \hat{\phi}\left(\frac{\xi}{m} + \frac{k}{m} + n\right) \right|^2 = \sum_{k=0}^{m-1} \Phi_b\left(\frac{\xi+k}{m}\right). \end{aligned}$$

It follows that if  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is a frame sequence, then either  $\Phi_a(\xi) = 0$  or  $A \leq \Phi_a(\xi) \leq Bm$ . Hence,  $(\tau_{na}\phi)_{n \in \mathbb{Z}}$  is also a frame sequence.

(3) Since  $\frac{a}{b} \notin \mathbb{Z}$ , there is an  $0 < \epsilon$  so that

$$\frac{\xi+n}{a} \notin \left[\frac{1}{b}, \frac{1}{b} + \epsilon\right], \quad \forall n \in \mathbb{Z}, \quad \forall 0 < \xi < \epsilon.$$

Define a function  $\phi \in L^2(\mathbb{R})$  so that,

$$\begin{aligned} \hat{\phi}(\xi) &= \xi, \quad \forall 0 \leq \xi \leq \frac{1}{a}, \\ \hat{\phi}(\xi) &= 1, \quad \forall \xi \in \left[\frac{1}{b}, \frac{1}{b} + \epsilon\right], \end{aligned}$$

and  $\hat{\phi}(\xi) = 0$  otherwise. Then,

$$\Phi_a(\xi) = \hat{\phi}\left(\frac{\xi}{a}\right)^2, \quad \forall 0 \leq \xi \leq \epsilon,$$

and so  $\Phi_a(\xi)$  is not bounded below on  $\mathbb{T}$ . Therefore,  $(\tau_{na}\phi)_{n \in \mathbb{Z}}$  is not a frame sequence. On the other hand,

$$\Phi_b(\xi) \geq 1 \quad \text{on } [0, c],$$

where  $c = \min\{\epsilon, \frac{\epsilon}{b}\}$ , and for  $\xi \in [c, 1]$ , either  $\Phi_b(\xi) = 0$  or  $\Phi_b(\xi) \geq c^2$ . It follows that where  $\Phi_b(\xi)$  is non-zero, it is bounded above and below and hence  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is a frame sequence.  $\square$

Notice that Theorem 2.3(2) implies that if  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is a frame sequence than  $(\tau_{nb}\phi)_{n \in \Lambda}$  is a frame sequence whenever  $\Lambda$  is a *subgroup* of  $\mathbb{Z}$ . It is also clear that any subsequence of an exact frame sequence remains an exact frame sequence. The following result is a converse of this.

**Theorem 2.4.** *Suppose  $\phi \in L^2(\mathbb{R})$  and  $b > 0$ . Then the sequence  $(\tau_{nb}\phi)_{n=1}^\infty$  is a frame sequence if and only if  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is an exact frame sequence.*

*Proof.* Assume  $(\tau_{nb}\phi)_{n \in \mathbb{N}}$  is a frame sequence. Then (see for example [7]) if  $0 \neq f \in H_{\mathbb{N}}$  we have that  $\log |f| \in L^1$  and so, in particular,  $|f| > 0$  a.e. This implies that  $E_{\mathbb{N}} = \{0\}$ . It follows that if  $A, B$  are the frame bounds for  $(\tau_{nb}\phi)_{n \in \mathbb{N}}$  then for every trigonometric polynomial in  $H_{\mathbb{N}}$  we have

$$A\|f\|^2 \leq \int |f(\xi)|^2 \Phi_b(\xi) d\xi \leq B\|f\|^2.$$

Now suppose  $f$  is any trigonometric polynomial. Then for large enough  $n$  we have that  $e^{2\pi ni\xi} f \in H_{\mathbb{N}}$ . Thus the same inequality follows trivially for all trigonometric polynomials in  $L^2(\mathbb{T})$ . This implies the theorem.  $\square$

*Remark.* If  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is a frame sequence but not an exact sequence frame, then of course  $(\tau_{nb}\phi)_{n=1}^\infty$  cannot be a frame sequence. The set  $(\tau_{nb}\phi)_{n \in \mathbb{Z}}$  is linearly independent, but it follows easily that the lower frame bound for  $(\tau_{nb}\phi)_{n=1}^N$  must converge to 0 as  $N \rightarrow \infty$ .

*Remark.* Note that if  $\Lambda \subset \mathbb{N}$  then the argument of the above theorem shows that  $(\tau_{nb}\phi)_{n \in \Lambda}$  is a frame sequence if and only if it is also an exact frame sequence.

### 3. UPPER FRAME BOUNDS FOR SUBSEQUENCES

In this section we give some criteria for the existence of an upper frame bound for a sequence  $(\tau_{nb}\phi)_{n \in \Lambda}$ .

First we introduce some notation. If  $\Lambda$  is a countable subset of  $\mathbb{R}$  we define for  $x > 0$  the function

$$D_\Lambda(x) = \sup_{t \in \mathbb{R}} |\Lambda \cap [t, t+x]|.$$

Our first result concerns general conditions for an upper frame bound to exist for a sequence of translates (which need not in this case be regularly spaced).

**Theorem 3.1.** *Let  $F : (0, \infty) \rightarrow (0, \infty)$  be a monotone-decreasing function such that  $F \in L_2$ . Define*

$$(3.1) \quad G(x) = F(x) \int_0^x F(t) dt + \int_x^\infty F(t)^2 dt.$$

*Let  $\Lambda = (\lambda_n)$  be a countable subset of  $\mathbb{R}$ . Then:*

(1) *Suppose  $\phi \in L^2(\mathbb{R})$  is such that for some  $X$  and every  $x$  with  $|x| \geq X$  we*

have  $|\phi(x)| \leq F(|x|)$ . Then in order that  $(\tau_{\lambda_n} \phi)$  have an upper frame bound it is sufficient that

$$\int_1^\infty G(x) D_\Lambda(x) \frac{dx}{x} < \infty.$$

(2) If  $\psi(x) = F(|x|)$  for  $x \neq 0$  then a necessary condition for  $(\tau_{\lambda_n} \psi)$  to have an upper frame bound is that

$$\sup_{x>1} G(x) D_\Lambda(x) < \infty.$$

*Remark.* Note that  $G$  is continuous, decreasing and bounded by  $G(0) = \lim_{x \rightarrow 0} G(x) = \int_0^\infty F(x)^2 dx$ .

*Proof.* Suppose  $t > 0$ . Note that

$$\int_{-\infty}^\infty \psi(x+t)\psi(x-t)dx = 2 \int_0^{2t} \psi(x+t)\psi(x-t)dx + 2 \int_{2t}^\infty \psi(x+t)\psi(x-t)dx.$$

Now

$$2F(3t) \int_0^t F(x)dx \leq \int_0^{2t} \psi(x+t)\psi(x-t)dx \leq 2F(t) \int_0^t F(x)dx$$

and

$$\int_{3t}^\infty F(x)^2 dx \leq \int_{2t}^\infty \psi(x+t)\psi(x-t)dx \leq \int_t^\infty F(x)^2 dx.$$

We now argue that

$$\frac{4}{3}G(3t) \leq \int_{-\infty}^\infty \psi(x+t)\psi(x-t)dx \leq 4G(t).$$

Indeed we have

$$\begin{aligned} \frac{4}{3}G(3t) &= \frac{4}{3}F(3t) \int_0^{3t} F(x)dx + \frac{4}{3} \int_{3t}^\infty F(x)^2 dx \\ &\leq 4F(3t) \int_0^t F(x)dx + \frac{4}{3} \int_{2t}^\infty \psi(x+t)\psi(x-t)dx \\ &\leq 2 \int_0^{2t} \psi(x+t)\psi(x-t)dx + \frac{4}{3} \int_{2t}^\infty \psi(x+t)\psi(x-t)dx \\ &\leq \int_{-\infty}^\infty \psi(x+t)\psi(x-t)dx \\ &= 2 \int_0^{2t} \psi(x+t)\psi(x-t)dx + 2 \int_{2t}^\infty \psi(x+t)\psi(x-t)dx \\ &\leq 4F(t) \int_0^t F(x)dx + 2 \int_t^\infty F(x)^2 dx \\ &\leq 4G(t). \end{aligned}$$

Let us now prove (1). We first estimate  $|\phi| \leq |\psi| + |\phi\chi_{[-X,X]}|$ . It now follows that if  $t \geq X$

$$\int_{-\infty}^{\infty} |\phi(x+t)| |\phi(x-t)| dx \leq 4G(t) + CF(2t - X)$$

for a suitable constant  $C$ . It follows (noting that  $G$  is bounded) that for a suitable constant  $C$  we have that

$$\langle \phi, \tau_a \phi \rangle \leq CG\left(\frac{1}{2}a\right)$$

for all  $a > 0$ .

Consider any finitely nonzero sequence  $(c_n)$ . Then

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} c_n \tau_{\lambda_n} \phi \right\|^2 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \overline{c_m} c_n \langle \phi, \tau_{\lambda_n - \lambda_m} \phi \rangle \\ &\leq C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |c_m| |c_n| G\left(\frac{1}{2}|\lambda_m - \lambda_n|\right) \\ &\leq \frac{1}{2}C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (|c_m|^2 + |c_n|^2) G\left(\frac{1}{2}|\lambda_m - \lambda_n|\right) \\ &\leq C \sum_{m=1}^{\infty} |c_m|^2 \sum_{n=1}^{\infty} G\left(\frac{1}{2}|\lambda_m - \lambda_n|\right). \end{aligned}$$

Now

$$\begin{aligned} \sum_{n=1}^{\infty} G\left(\frac{1}{2}|\lambda_m - \lambda_n|\right) &\leq 2 \sum_{k=0}^{\infty} \sum_{2^{k-1} < |\lambda_m - \lambda_n| \leq 2^k} G(2^{k-1}) + \sum_{|\lambda_m - \lambda_n| \leq 1/2} G(0) \\ &\leq 4 \sum_{k=0}^{\infty} D_{\Lambda}(2^{k-1}) G(2^{k-1}) + D_{\Lambda}(1) G(0) \\ &\leq 8 \sum_{k=0}^{\infty} D_{\Lambda}(2^{k-2}) G(2^{k-1}) + D_{\Lambda}(1) G(0) \\ &\leq 8(\log 2)^{-1} \int_{1/4}^{\infty} D_{\Lambda}(x) G(x) \frac{dx}{x} + D_{\Lambda}(1) G(0) \\ &\leq 8(\log 2)^{-1} \int_1^{\infty} D_{\Lambda}(x) G(x) \frac{dx}{x} + (1 + \log 4) D_{\Lambda}(1) G(0). \end{aligned}$$

This establishes (1).

To prove (2) fix  $x > 1$  and  $t \in \mathbb{R}$ , and let  $\mathbb{A} = \{n : t \leq \lambda_n \leq t + x\}$ . Then

$$\begin{aligned} \left\| \sum_{n \in \mathbb{A}} \tau_{\lambda_n} \phi \right\|^2 &\geq \frac{4}{3} \sum_{m \in \mathbb{A}} \sum_{n \in \mathbb{A}} G\left(\frac{3}{2}|\lambda_m - \lambda_n|\right) \\ &\geq \frac{4}{3} |\mathbb{A}|^2 G\left(\frac{3}{2}x\right) \end{aligned}$$

so that if  $B$  is the upper frame bound,

$$G\left(\frac{3}{2}x\right)|\mathbb{A}| \leq \frac{3}{4}B.$$

Hence

$$G\left(\frac{3}{2}x\right)D_\Lambda(x) \leq \frac{3}{4}B.$$

Now this implies that

$$G(x)D_\Lambda(x) \leq 2G(x)D_\Lambda\left(\frac{1}{2}x\right) \leq \frac{3}{2}B.$$

This completes the proof.  $\square$

**Corollary 3.2.** *Under the hypotheses of the theorem if  $F \in L^1(\mathbb{R})$  then a necessary and sufficient condition for  $(\tau_{\lambda_n} \phi)$  to have an upper frame bound is that  $D_\Lambda(1) < \infty$ .*

*Proof.* In this case we clearly have an estimate  $G(x) \leq CF(x)$ . The condition  $D_\Lambda(1) < \infty$  is equivalent to  $D_\Lambda(x) \leq Cx$  for  $x \geq 1$ .

**Lemma 3.3.** *Suppose that  $F : (0, \infty) \rightarrow (0, \infty)$  satisfies the conditions of Theorem 3.1 and is such that for some  $\epsilon > 0$  we have that  $x^{1-\epsilon}F(x)$  is increasing for  $x \geq 1$  and  $x^{1+\epsilon}F(x)^2$  is decreasing for  $x \geq 1$ . Then there is a constant  $C$  so that  $C^{-1}xF(x)^2 \leq G(x) \leq CxF(x)^2$ , for  $x \geq 1$ , where  $G$  is defined in (3.1).*

*Proof.* This is an immediate calculation from (3.1).

*Remark.* Note that in the theorem if  $F(x) = \min(1, x^{-a})$  where  $\frac{1}{2} < a < 1$  then  $G(x) \approx \min(1, x^{1-2a})$ . Hence a necessary condition for  $(\tau_{\lambda_n}(\phi))$  to have an upper-frame bound is that  $D_\Lambda(x) \leq Cx^{2a-1}$  for  $x \geq 1$  and a sufficient condition is that  $D_\Lambda(x) \leq Cx^{2a-1-\epsilon}$  for  $x \geq 1$ , for some  $\epsilon > 0$ . If  $F(x) = \min(1, x^{-1})$  then  $G(x) \approx x^{-1}(1 + \log x)$  for  $x \geq 1$ .

We now prove a more precise result for the case when  $F$  is of the form given in Lemma 3.3, by giving a condition which is close to necessary and sufficient. This result will not be used later and the reader who is only interested in later results may therefore omit it. The argument is simply an adaptation of an argument used for a similar result in [9] and derives from more general results in [10].

**Theorem 3.4.** *Suppose  $F : (0, \infty) \rightarrow (0, \infty)$  is a monotone decreasing function with the property that for some  $\epsilon > 0$  we have  $x^{1-\epsilon}F(x)$  is increasing for  $x \geq 1$  but  $x^{1+\epsilon}F(x)$  is decreasing for  $x \geq 1$ . Let  $\Lambda = (\lambda_n)$  be a countable subset of  $\mathbb{R}$ . Then:*

(1) *Suppose  $\phi \in L^2(\mathbb{R})$  is such that for some  $X$  and every  $x \geq X$  we have  $|\phi(x)| \leq F(|x|)$ . Then in order that  $(\tau_{\lambda_n}\phi)$  has an upper frame bound it is necessary that there is a constant  $C$  so that for every finite interval  $I$  we have*

$$(3.2) \quad \sum_{\lambda_m, \lambda_n \in I} G(|\lambda_m - \lambda_n|) \leq C|\Lambda \cap [t, t+x]|$$

where  $G$  is defined by (3.1).

(2) *If  $\psi(x) = F(|x|)$  then (3.2) is also necessary for  $(\tau_{\lambda_n}\phi)$  to have an upper frame-bound.*

*Proof.* In this proof, we will use  $C$  for a constant depending only on  $F$  which may vary from line to line. Note that we have estimates of the form  $F(x/2) \leq CF(x)$  and  $G(x/2) \leq CG(x)$ . It follows therefore that

$$\frac{1}{C}G(|a|) \leq \langle \psi, \tau_a\phi \rangle \leq CG(|a|).$$

Let us first prove (2) (necessity). Indeed for any bounded interval  $I$ , if  $A = \{n : \lambda_n \in I\}$ , then

$$\left\| \sum_{n \in A} \tau_{\lambda_n} \psi \right\|_2^2 \geq \frac{1}{C} \sum_{m, n \in A} G(|\lambda_m - \lambda_n|)$$

and so the conclusion is immediate.

The other direction (1) is harder. It will be convenient to assume  $F(x) = 1$  for  $0 \leq x \leq 1$ . Clearly if we prove the result under this assumption the general case will follow.

Let us start by observing the estimate from (3.2) that if  $A = \{n : \lambda_n \in I\}$  then  $G(x)|A|^2 \leq C|A|$  so that

$$(3.3) \quad |\Lambda \cap I| \leq CG(x)^{-1}$$

Now fix  $t \in \mathbb{R}$  and suppose  $x > 1$ . We estimate, using Lemma 3.3:

$$\begin{aligned}
\sum_{|t-\lambda_n| \geq x} F(|t-\lambda_n|) &\leq C \sum_{k=1}^{\infty} F(2^{k-1}x) |\{n : 2^{k-1}x \leq |t-\lambda_n| \leq 2^k x\}| \\
&\leq C \sum_{k=0}^{\infty} F(2^k x) G(2^k x)^{-1} \\
&\leq C \sum_{k=0}^{\infty} 2^{-k} x^{-1} F(2^k x)^{-1} \\
&\leq C \sum_{k=0}^{\infty} 2^{-\epsilon k} (2^{\epsilon-1k} x^{-1} F(2^k x)^{-1}) \\
&\leq C x^{-1} F(x)^{-1}.
\end{aligned}$$

In general for all  $x > 0$  we conclude an estimate of the type:

$$(3.4) \quad \sum_{|t-\lambda_n| \geq x} F(|t-\lambda_n|) \leq C \min(1, x^{-1}) F(x)^{-1}.$$

We now introduce two further functions. Let  $N(t) = |\Lambda \cap [t-1, t+1]|$  and let  $H(t) = \sum_n F(|t-\lambda_n|)$ . Our assumptions give us that  $N(t) \leq C$  and  $N(t) \leq H(t)$ . By (3.4) we have  $H(t) \leq N(t) + C$ .

Suppose we have an interval  $I = [x - h/2, x + h/2]$  of length  $h > 1$ . Let  $J = [x - h, x + h]$ . Then we can write  $H(t) = H_1(t) + H_2(t)$  where  $H_1(t) = \sum_{\lambda_n \in J} F(|t-\lambda_n|)$  and  $H_2(t) = H(t) - H_1(t)$ . Using (3.4) we have, if  $|t-x| \leq \frac{1}{2}h$ ,

$$\begin{aligned}
H_2(t)^2 &\leq 2 \sum_{\lambda_m \notin J} \sum_{|\lambda_n - t| \geq |\lambda_m - t|} F(|t-\lambda_m|) F(|t-\lambda_n|) \\
&\leq C \sum_{\lambda_m \notin J} |t-\lambda_m|^{-1}.
\end{aligned}$$

On the other hand:

$$\int_{-\infty}^{\infty} H_1(t)^2 dt \leq C \sum_{\lambda_m, \lambda_n \in J} G(|\lambda_m - \lambda_n|) \leq C \sum_{\lambda_m \in J} 1.$$

Combining we have

$$\int_I H(t)^2 dt \leq C \sum_m \min(1, h|x-\lambda_m|^{-1}).$$

We use this inequality to estimate

$$\begin{aligned}
\int_{-\infty}^{\infty} F(|t-x|)H(x)^2 dx &\leq \int_{t-1}^{t+1} H(x)^2 dx + C \sum_{k=1}^{\infty} F(2^k) \int_{2^{k-1} < |t-x| < 2^k} H(x)^2 dx \\
&\leq C \sum_{k=0}^{\infty} F(2^k) \sum_m \min(1, 2^k |t - \lambda_m|^{-1}) \\
&= C \sum_m \sum_{k=0}^{\infty} F(2^k) \min(1, 2^k |t - \lambda_m|^{-1}) \\
&\leq C \sum_m \sum_{2^k < |x - \lambda_m|} 2^k F(2^k) |t - \lambda_m| + C \sum_m \sum_{2^k \geq |x - \lambda_m|} F(2^k) \\
(3.5) \quad &\leq C \sum_m F(|t - \lambda_m|) = CH(t)
\end{aligned}$$

For each  $n$  let  $E_n = (\lambda_n - \frac{1}{2}, \lambda_n + \frac{1}{2})$ . For any finitely nonzero sequence  $(a_n)$ , let  $f = \sum_n |a_n| \chi_{E_n}$ . Then since  $f^2 \leq N(t) \sum_n |a_n|^2 \chi_{E_n}$  we have  $\|f\|^2 \leq C(\sum_n |a_n|^2)$ . We also have:

$$\begin{aligned}
\| \sum_n a_n \tau_{\lambda_n} \phi \|_2^2 &\leq C \sum_{m,n} |a_m| |a_n| \int_{-\infty}^{\infty} F(|t - \lambda_m|) F(|t - \lambda_n|) dt \\
&\leq C \sum_{m,n} |a_m| |a_n| \int_{-\infty}^{\infty} \int_{E_n} F(|t-x|) dx \int_{E_m} F(|t-y|) dy dt \\
(3.6) \quad &\leq C \int_{-\infty}^{\infty} (\tilde{f}(t))^2 dt,
\end{aligned}$$

where

$$\tilde{f}(t) = \int_{-\infty}^{\infty} F(|t-s|) f(s) ds.$$

Now suppose  $g$  is any nonnegative  $L^2$ -function of compact support with  $\|g\| = 1$  and suppose  $\tilde{g}(t) = \int_{-\infty}^{\infty} F(|t-s|) g(s) ds$ . Then  $\tilde{g} \in L_2$  and

$$\begin{aligned}
\int_{-\infty}^{\infty} g(t) \tilde{f}(t) dt &= \int_{-\infty}^{\infty} \tilde{g}(t) f(t) dt \\
&\leq C \int_{-\infty}^{\infty} \tilde{g}(t) f(t) H(t)^2 dt \\
(3.7) \quad &\leq C \left( \int_{-\infty}^{\infty} \tilde{g}(t)^2 H(t)^2 dt \right)^{1/2} \left( \sum_n |a_n|^2 \right)^{1/2}.
\end{aligned}$$

Now

$$\begin{aligned}
\tilde{g}(t)^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(|t-x|)F(|t-y|)g(x)g(y)dx dy \\
&\leq C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(|y-x|)(F(|t-x|) + F(|t-y|))g(x)g(y)dx dy \\
&\leq C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(|y-x|)F(|t-x|)g(x)g(y)dx dy \\
(3.8) \quad &\leq C \int_{-\infty}^{\infty} F(|t-x|)g(x)\tilde{g}(x)dx.
\end{aligned}$$

Hence, using (3.5),

$$\begin{aligned}
\int_{-\infty}^{\infty} \tilde{g}(t)^2 H(t)dt &\leq C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(|t-x|)g(x)\tilde{g}(x)H(t)^2 dt dx \\
&\leq \int_{-\infty}^{\infty} g(x)\tilde{g}(x) \int_{-\infty}^{\infty} F(|x-t|)H(t)^2 dt \\
&\leq C \int_{-\infty}^{\infty} g(x)\tilde{g}(x)H(x)dx \\
&\leq C \left( \int_{-\infty}^{\infty} \tilde{g}(x)^2 H(x)^2 dx \right)^{1/2}
\end{aligned}$$

which implies

$$\int_{-\infty}^{\infty} \tilde{g}(x)^2 H(x)^2 dx \leq C.$$

Now (3.7) implies

$$\int_{-\infty}^{\infty} g(t)\tilde{f}(t)dt \leq C \left( \sum_m |a_m|^2 \right)^{1/2}$$

and as  $g$  is arbitrary (subject to being of compact support and norm one) this implies that  $\|\tilde{f}\|^2 \leq C \sum_m |a_m|^2$  and (3.6) then yields the theorem.  $\square$

For regularly spaced sequences we have some other criteria. We use the terminology of Section 2. Recall that a subset  $\Lambda$  of  $\mathbb{Z}$  is a  $\Lambda(p)$ -set (for  $p > 2$ ) if the  $L_2$ - and  $L_p$ -norms are equivalent on  $H_\Lambda$ . See [13] and [3]

**Theorem 3.5.** *Suppose  $b \in \mathbb{R}$  and that  $\phi \in L^2(\mathbb{R})$ . Suppose  $\Lambda$  is a  $\Lambda(p)$ -set where  $p > 2$  and that  $\Phi_b \in L_r(\mathbb{T})$  where  $\frac{1}{r} + \frac{2}{p} = 1$ . Then  $(\tau_{nb}\phi)_{n \in \Lambda}$  has an upper-frame bound.*

*Proof.* If  $f \in H_\Lambda$ , we observe that

$$\int_{\mathbb{T}} |f(\gamma)|^2 \Phi_b(\gamma) d\gamma \leq \|f\|_p^2 \|\Phi_b\|_r$$

and so the theorem is immediate.  $\square$

*Remark.* In order that  $\Phi_b \in L_r$  it is sufficient by the Hausdorff-Young inequality that  $\sum_{n \in \mathbb{Z}} |\langle \phi, \tau_{nb} \phi \rangle|^{p/2} < \infty$ .

#### 4. FRAME SEQUENCES WHEN $\phi \in L^1(\mathbb{R})$ .

Let us call a subset  $\Lambda$  of  $\mathbb{Z}$  *sparse* if for any  $n \in \mathbb{N}$  the set  $\Lambda \cap (\Lambda + n)$  is finite. Thus any increasing sequence  $(\lambda_n)$  is sparse if and only if  $\lim_{n \rightarrow \infty} (\lambda_n - \lambda_{n-1}) = \infty$ .

**Theorem 4.1.** *Suppose  $0 \neq \phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  and  $\Lambda$  is a sparse subset of  $\mathbb{Z}$ . Then  $(\tau_{nb}\phi)_{n \in \Lambda}$  has an upper frame bound if and only if it is a frame. In particular if  $|\phi(x)| = O(F(x))$  where  $F : (0, \infty) \rightarrow (0, \infty)$  is monotone-decreasing and integrable then  $(\tau_{nb}\phi)_{n \in \Lambda}$  is a frame.*

*Proof.* Suppose  $(\tau_{nb}\phi)_{n \in \Lambda}$  has an upper frame bound but no lower frame bound. We appeal to Theorem 2.1. There is a sequence  $(f_n) \in H_\Lambda \cap E_\Lambda^\perp$  so that

$$\lim_{n \rightarrow \infty} \int_0^1 |f_n(\xi)|^2 \Phi_b(\xi) d\xi = 0$$

but  $\|f_n\| = 1$  for all  $n$ . By passing to a subsequence, we can assume without loss of generality that  $f_n$  converges weakly to some  $f \in H_\Lambda \cap E_\Lambda^\perp$ , and even further that  $g_n = \frac{1}{n}(f_1 + \cdots + f_n)$  converges in norm to  $f$ . Since  $(g_n)$  converges to  $f$  in measure, it follows from Fatou's Lemma that  $\int_0^1 |f(\xi)|^2 \Phi_b(\xi) d\xi = 0$  and so  $f \in E_\Lambda$ . Hence  $f = 0$ , and  $f_n$  converges weakly to zero.

Now we estimate

$$\left| \int_0^1 |f_n(\xi)|^2 e^{-2\pi i k \xi} d\xi \right| \leq \sum_{j \in \mathbb{Z}} |\hat{f}_n(j)| |\hat{f}_n(k+j)|.$$

Hence

$$\left| \int_0^1 |f_n(\xi)|^2 e^{-2\pi i k \xi} d\xi \right| \leq \sum_{j \in \Lambda \cap (\Lambda - k)} |\hat{f}_n(j)| |\hat{f}_n(j+k)|.$$

Since  $f_n$  converges to 0 weakly this converges to 0 if  $k \neq 0$ . Hence the measures  $|f_n(\xi)|^2 d\xi$  converge weak\* to  $d\xi$  in  $C(\mathbb{T})^*$ .

Now since  $\phi \in L^1$  the function  $\hat{\phi}$  is continuous and so  $\Phi_b(t)$  is lower-semicontinuous and  $2\pi$ -periodic when regarded as a function on  $\mathbb{R}$ . In particular

$\Phi_b$  is lower-semi-continuous on  $\mathbb{T}$  and hence there is a sequence of functions  $\psi_n \in C(\mathbb{T})$  such that  $0 \leq \psi_n \uparrow \Phi_b$  pointwise.

Clearly

$$\begin{aligned} \int_0^1 \psi_k(\xi) d\xi &= \lim_{n \rightarrow \infty} \int_0^1 \psi_k(\xi) |f_n(\xi)|^2 d\xi \\ &\leq \limsup_{n \rightarrow \infty} \int_0^1 \Phi_b(\xi) |f_n(\xi)|^2 d\xi \\ &= 0. \end{aligned}$$

Hence  $\int_0^1 \Phi_b(\xi) d\xi = 0$  which is a contradiction.  $\square$

If  $\phi$  decays rapidly enough at  $\infty$  we can achieve a stronger result. We will need the following lemma.

**Lemma 4.2.** *Suppose  $\Lambda \subset \mathbb{Z}$ , and suppose  $f \in H_\Lambda$ , with  $\|f\| = 1$ . Let  $F = |f|^2$ . If  $\mathbb{J}$  is a finite interval in  $\mathbb{Z}$  then*

$$\sum_{n \in \mathbb{J}} |\hat{F}(n)| \leq D_\Lambda(|\mathbb{J}|)$$

*Proof.* We have

$$|\hat{F}(n)| \leq \sum_{k \in \mathbb{Z}} \chi_\Lambda(k) \chi_\Lambda(n-k) |\hat{f}(k)| |\hat{f}(n-k)|.$$

Making the estimate  $|\hat{f}(k)| |\hat{f}(n-k)| \leq \frac{1}{2} (|\hat{f}(k)|^2 + |\hat{f}(n-k)|^2)$  we obtain

$$|\hat{F}(n)| \leq \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 \chi_\Lambda(n-k)$$

so that

$$\sum_{n \in \mathbb{J}} |\hat{F}(n)| \leq D_\Lambda(|\mathbb{J}|) \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2$$

which immediately yields the lemma.  $\square$

**Theorem 4.3.** *Suppose  $\phi \in L^2(\mathbb{R})$  and that  $b \in \mathbb{R}$ . Assume  $\Phi_b$  is continuous on  $\mathbb{T}$  and has exactly  $N$  zeros. Suppose  $\Lambda$  is any set satisfying the density condition*

$$\lim_{x \rightarrow \infty} \frac{D_\Lambda(x)}{x} < \frac{1}{N}.$$

Then  $(\tau_{nb}\phi)_{n \in \Lambda}$  is a frame sequence.

*Proof.* Since  $\Phi_b$  is bounded the upper frame bound is trivial. Assume there is no lower frame bound. Then there is a sequence  $f_n \in H_\Lambda$  with  $\|f_n\| = 1$  and

$$\lim_{n \rightarrow \infty} \int \Phi_b(x) |f(x)|^2 = 0.$$

Without loss of generality we can suppose the measures  $|f_n(x)|^2 dx$  converge weak\* to a probability measure  $d\mu$ . Since  $\Phi_b$  is continuous (lower semi-continuity would suffice!) we have

$$\int \Phi_b(x) d\mu(x) = 0$$

so that  $\mu$  can be written in the form  $\mu = \sum_{k=1}^N a_k \delta_{t_k}$  where  $t_1, \dots, t_N$  are the zeros of  $\Phi_b$  and  $0 \leq a_k$  with  $\sum a_k = 1$ .

Now if  $F_n = |f_n|^2$  we have, applying Lemma 4.2, for every natural number  $m$  that

$$\sum_{k=-m}^m |\hat{F}_n(k)|^2 \leq D_\Lambda(2m+1).$$

Hence

$$\sum_{k=-m}^m |\hat{\mu}(k)|^2 \leq D_\Lambda(2m+1).$$

It follows from a theorem of Wiener [11] that

$$\sum_{k=1}^N a_k^2 \leq \lim_{m \rightarrow \infty} \frac{D_\Lambda(2m+1)}{2m+1} < \frac{1}{N}.$$

But then this contradicts the Cauchy-Schwartz inequality since  $\sum_{k=1}^N a_k = 1$ .  $\square$

**Theorem 4.4.** *Suppose  $h : (0, \infty) \rightarrow (0, \infty)$  is an increasing continuous function such that there exist constants  $1 < c_1 < c_2 < \infty$  with  $c_1 h(x) \leq h(2x) \leq c_2 h(x)$  for all  $x$ , and  $\int_1^\infty t^{-2} h(t) dt = \infty$ . Suppose  $\phi \in L_2(\mathbb{R})$  satisfies the condition that  $\phi(x) = O(e^{-\delta h(x)})$  as  $x \rightarrow \infty$ , where  $\delta > 0$ . Then, for any  $b \in \mathbb{R}$ , there exists an integer  $N$  so that if  $\Lambda \subset \mathbb{Z}$  with  $\lim_{x \rightarrow \infty} \frac{D_\Lambda(x)}{x} < \frac{1}{N}$  then  $(\tau_{nb}\phi)_{n \in \Lambda}$  is a frame sequence.*

*Proof.* The idea of the proof is to show that the function  $\Phi_b$  is  $C^\infty$  on  $\mathbb{T}$  and is *quasi-analytic* (see [11]) so that there is no  $\xi \in [0, 1)$  for which all derivatives  $\Phi_b^{(n)}(\xi) = 0$  for  $n \geq 0$ . It then follows from Taylor's theorem that the zeros of  $\Phi_b$

are isolated and hence finite. Theorem 4.3 will then complete the proof. To show  $\Phi_b$  is quasi-analytic we need to show that if  $M_n = \|\Phi_b^{(n)}\|_2$  and  $\tau(r) = \inf_n (M_n r^n)$  then

$$(4.1) \quad \int_0^\infty \frac{\log \tau(r)}{1+r^2} dr = -\infty.$$

Throughout the proof  $C$  will denote a constant, depending on  $h$  and  $\phi$ , which may vary from line to line but is independent of  $x, m, n, k$  etc. We begin with the observation that there exists a constant  $C$  so that

$$\frac{1}{C} \int_0^x h(t) \frac{dt}{t} \leq h(x) \leq C \int_0^x h(t) \frac{dt}{t}$$

so that by replacing  $h$  with  $H(x) = \int_0^x h(t) \frac{dt}{t}$  we can suppose  $h$  is continuously differentiable and that there is a constant  $C$  so that

$$\frac{1}{C} \leq \frac{xh'(x)}{h(x)} \leq C$$

for all  $x > 0$ . If we let  $v = h^{-1}$  be the inverse function then we also have

$$\frac{1}{C} \leq \frac{xv'(x)}{v(x)} \leq C.$$

Let  $m \in \mathbb{N}$ . Then  $x^m e^{-h(x)}$  attains a maximum at a point  $x_m$  where  $x_m h'(x_m) = m$  and so  $C^{-1}m \leq h(x_m) \leq Cm$ . Hence also  $C^{-1}v(m) \leq x_m \leq Cv(m)$ . Combining we have

$$x^m e^{-h(x)} \leq (Cv(m))^m.$$

Let  $N$  be an integer greater than  $2/\delta$ . It follows that

$$x^{Nm} e^{-h(x)} \leq (C(v(Nm)))^{Nm}$$

and so

$$x^m e^{-\delta h(x)/2} \leq (Cv(m))^m.$$

From this we obtain

$$\int_0^\infty x^m e^{-\delta h(x)} dx \leq (Cv(m))^m \int_0^\infty e^{-\delta h(x)/2} dx \leq (Cv(m))^m.$$

We now use the argument of Theorem 3.1 to deduce that

$$|\langle \phi, \tau_n b \phi \rangle| \leq C \min(1, e^{-\delta h(\frac{1}{2}|n|b)})$$

for some suitable constant  $C$  and all  $n \in \mathbb{Z}$ . It follows easily that

$$\Phi_b(\xi) = \sum_{n \in \mathbb{Z}} \langle \phi, \tau_{nb} \phi \rangle e^{2\pi i n \xi}$$

is  $C^\infty$ . Furthermore

$$M_n = \|\Phi_b^{(n)}\| \leq C \sum_{k \in \mathbb{Z}} |k|^n e^{-\delta h(\frac{1}{2}|k|b)}$$

for  $n \in \mathbb{N}$ . Hence

$$M_n \leq C^n \int_0^\infty x^n e^{-\delta h(x)} dx \leq (Cv(n))^n.$$

It follows that

$$\tau(r) \leq \inf_{n \geq 1} (Cv(n)r)^n.$$

For given  $r < 1$  choose  $n = [Ch(r^{-1})/e]$ . Then  $Cv(n)r \leq 1/e$  and  $n \geq Ch(r^{-1})/e - 1$ . Hence

$$\log \tau(r) \leq -Ch(r^{-1})$$

for small enough  $r$ . Now

$$\int_0^1 \frac{h(r^{-1})}{1+r^2} dr = \int_1^\infty \frac{h(r)}{1+r^2} dr = \infty$$

so that (4.1) holds and the proof is complete.  $\square$

## 5. FRACTIONAL DIMENSION OF THE ZERO-SET AND FRAMES

For our next lemma (which is probably known) we let  $\ell(I)$  be the length of an interval  $I \subset \mathbb{T}$ .

**Lemma 5.1.** *Suppose  $\Lambda \subset \mathbb{Z}$ , and suppose  $f \in H_\Lambda$ , with  $\|f\| = 1$ . Then there is a constant  $C$  so that if  $I$  is an interval contained in  $\mathbb{T}$  then*

$$\int_I |f(\xi)|^2 dx \leq C \ell(I) D_\Lambda(\ell(I)^{-1}).$$

*Proof.* Let  $I$  be the interval  $[t-h, t+h]$ . Set  $r = 1-h$  and then let

$$\psi(x) = \frac{1-r^2}{1-2r \cos(2\pi(x-t)) + r^2} = \sum_{k \in \mathbb{Z}} r^{|k|} e^{2\pi i k(t-x)}.$$

Then for some absolute constant  $C$  independent of  $t, h$  we have  $\chi_I \leq Ch^{-1}\psi$ .  
Hence

$$\begin{aligned} \int_I |f(x)|^2 dx &\leq Ch^{-1} \int_0^1 \psi(x)F(x)dx \\ &\leq Ch^{-1} \sum_{k \in \mathbb{Z}} r^{|k|} |\hat{F}(n-k)| \\ &\leq 2Ch^{-1} \sum_{k=0}^{\infty} r^{mk} D_{\Lambda}(m) \end{aligned}$$

for any integer  $m$ . Let  $m$  be chosen so that  $\ell(I)^{-1} \leq m \leq 2\ell(I)^{-1}$ . Then  $r^m \leq (1-h)^{1/(2h)} \leq c < 1$  for some absolute constant  $c$ . We thus quickly obtain the result since  $D_{\Lambda}(2x) \leq 2D_{\Lambda}(x)$  for any  $x$ .  $\square$

Now if  $0 < \alpha < 1$  let us define the *essential  $\alpha$ -Hausdorff measure* denoted  $H_{\alpha}(E)$  of a subset  $E$  of  $\mathbb{T}$  to be the infimum of  $\sum_{n=1}^{\infty} \ell(I_n)^{\alpha}$  over all collections of intervals  $(I_n)_{n=1}^{\infty}$  so that  $E \subset F \cup \bigcup_{n=1}^{\infty} I_n$  where  $F$  has measure zero.

**Theorem 5.2.** *Suppose  $\phi \in L^2(\mathbb{R})$ ,  $b > 0$  and  $\Lambda \subset \mathbb{Z}$  are such that  $(\tau_{nb}\phi)_{n \in \Lambda}$  has an upper frame bound. Suppose further that  $0 < \alpha < 1$  and  $\lim_{\epsilon \rightarrow 0} H_{\alpha}(\Phi_b > \epsilon) = 0$  and that  $D_{\Lambda}(x) \leq Cx^{1-\alpha}$  for some  $C$  and all  $x$ . Then  $(\tau_{nb}\phi)_{n \in \Lambda}$  is an exact frame sequence.*

*Proof.* If  $\epsilon > 0$  then pick intervals  $I_n$  so that  $\{\Phi_b > \epsilon\} \subset \bigcup_{n=1}^{\infty} I_n$  up to a set of measure zero and  $\sum_{n=1}^{\infty} \ell(I_n)^{\alpha} < 2H_{\alpha}(\Phi_b > \epsilon)$ . Then

$$\int_0^1 \Phi_b(\xi) |f(\xi)|^2 \geq \epsilon (\|f\|^2 - \sum_{n=1}^{\infty} \int_{I_n} |f(\xi)|^2 d\xi).$$

If  $f \in H_{\Lambda}$  then

$$\sum_{n=1}^{\infty} \int_{I_n} |f(\xi)|^2 d\xi \leq C \|f\|^2 \sum_{n=1}^{\infty} \ell(I_n)^{\alpha} \leq 2CH_{\alpha}(\Phi_B > \epsilon) \|f\|^2.$$

If we take  $\epsilon$  small enough we have:

$$\sum_{n=1}^{\infty} \int_{I_n} |f(\xi)|^2 d\xi \leq \frac{1}{2} \|f\|^2$$

and the result follows by appealing to Theorem 2.1.  $\square$

Combining this with Theorem 3.1 gives us the following theorem:

**Theorem 5.3.** *Suppose  $\frac{1}{2} < a < 1$  and that  $\phi \in L^2(\mathbb{R})$  satisfies the conditions that  $|\phi(x)| = O(x^{-a-\epsilon})$  as  $|x| \rightarrow \infty$  where  $\epsilon > 0$  and that  $\lim_{\epsilon \rightarrow 0} H_{2a-1}(\Phi_b > \epsilon) = 0$ . Then for any subset  $\Lambda \subset \mathbb{Z}$  with  $D_\Lambda(x) \leq Cx^{2(1-a)}$  we have that  $(\tau_{nb}\phi)_{n \in \Lambda}$  is an exact frame sequence.*

Note that in the case when  $\Phi_b$  is lower-semi-continuous the condition  $\lim_{\epsilon \rightarrow 0} H_\alpha(\Phi_b > \epsilon)$  reduces to the condition that  $h_\alpha(\Phi_b = 0) = 0$  where  $h_\alpha(E)$  is the infimum of  $\sum \ell(I_n)^\alpha$  over all coverings  $E \subset \cup_{n=1}^\infty I_n$ . Thus in this case the condition is essentially a condition on the fractal dimension of the zero set of  $\Phi_b$ .

*Example.* We conclude by constructing an example where  $\Phi_b$  is bounded,  $\lim_{\epsilon \rightarrow 0} H_\alpha(\Phi_b > \epsilon) = 0$ , for every  $\epsilon > 0$  we have an estimate  $D_\Lambda(x) \leq C_\epsilon x^{1-\alpha+\epsilon}$ , but  $(\tau_{nb}\phi)_{n \in \Lambda}$  fails to be a frame sequence. We observe however we do not have any such example with  $\Phi_b$  continuous or lower-semicontinuous.

For each  $n \in \mathbb{N}$  let  $m_n = \max([\alpha n - \sqrt{n}], 0)$ . We then let  $\Lambda = \cup_{n \geq 1} \{2^n + k2^{m_n} : 1 \leq k \leq 2^{n-m_n}\}$ . Now suppose  $N = 2^p$ . Then for any interval  $\mathbb{J}$  of length  $N$  it is clear that we have

$$|\Lambda \cap \mathbb{J}| \leq \sum_{j=1}^{p-1} 2^{j-m_j} + 2|\mathbb{J}| \max_{j \geq p} 2^{-m_j}.$$

Hence

$$|\Lambda \cap \mathbb{J}| \leq 2^{p-m_p} \sum_{j=1}^{p-1} 2^{-j+m_p-m_{p-j}} + 2^{p-m_p} \max_{j \geq p} 2^{m_p-m_j}.$$

Since we have an estimate  $m_p \leq m_{p-j} + C\beta j$  where  $\alpha < \beta < 1$  and  $C$  is a constant independent of  $j, p$ , this implies that

$$D_\Lambda(2^p) \leq C2^{p-m_p}$$

which implies  $D_\Lambda(x) \leq C_\epsilon x^{1-\alpha+\epsilon}$  for all  $\epsilon > 0$ .

Now suppose

$$f_n(\xi) = 2^{(m_n-n)/2} \sum_{k=1}^{2^{n-m_n}} e^{2\pi i k 2^{m_n} \xi}$$

so that  $\|f_n\| = 1$  and  $f_n \in H_\Lambda$ . Notice that

$$|f_n(\xi)| = 2^{(m_n-n)/2} \left| \frac{\sin(2^{n-1}\xi/2)}{\sin(2^{m_n-1}\xi)} \right|.$$

Let  $E_n = \{|f_n| < 2^{\frac{1}{2}(n-m_n-\frac{1}{2}\sqrt{n})}\}$ , and let  $F_n = \mathbb{T} \setminus E_n$ . We will have for an appropriate constant  $C$  that

$$\int_{E_n} |f(\xi)|^2 d\xi \leq C2^{-\frac{1}{2}\sqrt{n}}.$$

Also  $F_n$  is a union of at most  $2^{m_n}$  equal intervals and has total measure bounded by  $C2^{m_n-n+\frac{1}{2}\sqrt{n}}$ . Hence

$$H_\alpha(F_n) \leq C2^{m_n-\alpha n+\frac{1}{2}\alpha\sqrt{n}} \leq 2^{-\frac{1}{2}\alpha\sqrt{n}}.$$

Set  $F_0 = \mathbb{T}$ . Now define  $\Phi$  on  $\mathbb{T} = [0, 1)$  by

$$\Phi(\xi) = \inf_{k \geq 0} 2^{-k} \chi_{F_k}.$$

Then  $H_\alpha(\cup_{j \geq k} F_j) \rightarrow 0$  so that  $\Phi > 0$  a.e. and indeed  $\lim_{\epsilon > 0} H_\alpha(\Phi > \epsilon) = 0$ .

We choose  $\phi \in L^2(\mathbb{R})$  so that  $\hat{\phi} = \Phi^{1/2} \chi_{[0,1)}$ . If we take  $b = 1$  then  $\Phi = \Phi_b$ .  
Now

$$\int_0^1 |f_n(\xi)|^2 \Phi(\xi) d\xi \leq 2^{-n} \int_{F_n} |f(\xi)|^2 d\xi + C \cdot 2^{-\frac{1}{2}\sqrt{n}} \rightarrow 0.$$

Hence  $(\tau_n \phi)_{n \in \Lambda}$  cannot be a frame sequence.  $\square$

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1 DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI  
65211, USA

2 MATHEMATICAL INSTITUTE, BUILDING 303, TECHNICAL UNIVERSITY OF DENMARK, 2800  
LYNGBY, DENMARK

3 DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI  
65211, USA

*E-mail address:* pete@casazza.math.missouri.edu; olechr@mat.dtu.dk; nigel@math.missouri.edu