EXISTENCE AND CONSTRUCTION OF FINITE TIGHT FRAMES

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Abstract. The space of finite tight frames of M vectors in $\mathbb{R}^N$ with prescribed norms $\{b_j\}_{j=1}^M$ and frame constant A corresponds to the first N columns of matrices in $\boldsymbol{O}(M)$ (the orthogonal group) with the property that the norms of the first N elements of their rows equal the values $a_j = b_j/\sqrt{A}$. Then $a_j^2 \leq 1$ and $\sum\limits_{j=1}^{j=M} a_j^2 = N$.

For any sequence $\{a_j\}_{j=1}^M$ of positive real numbers such that $a_j^2 \leq 1$ and $\sum\limits_{j=1}^{j=M} a_j^2 = N$, we prove existence of such frames. The proof is constructive, giving an embedding of $\boldsymbol{O}(N) \times \boldsymbol{O}(M)$ into the space of such frames. In addition, for any such $\{a_j\}_{j=1}^M$, all solutions can be factorised into $\frac{M*(M-1)}{2} - \frac{N*(N-1)}{2}$ parameters and one matrix $R_N \in \boldsymbol{O}(N)$, and each different solution provides a different embedding of $\boldsymbol{O}(N)$ into the space of such frames. In particular, Parseval frames correspond to sequences $\{a_j\}_{j=1}^M$ such that $a_j = \sqrt{\frac{N}{M}}$. The results provide an independent proof of some of the existence results in [CKLT02]. All proofs are constructive. A MATLAB toolbox implementing all results is freely distributed by the authors.

1. Introduction

Frames for Hilbert spaces play a fundamental role in a variety of importatnt areas including signal/image processing [Cas00], multiple antenna coding [GKV98], perfect reconstruction filter banks [DGK02], quantum theory [EF01], [Š] and much more. For applications, tight frames are preferred since they alow simple reconstruction formulas. A common problem in applications of frames is to find a tight frame $\{\varphi_j\}_{j=1}^M$ for an N-dimensional Hilbert space H_N with $\{\|\varphi_j\|\}_{j=1}^M$ prescribed in advance. In [CKLT02] the authors give necessary and sifficient conditions on $\{a_j\}_{j=1}^M$ so that there exists a tight frame $\{\varphi_j\}_{j=1}^M$ for H_N with $\|\varphi_j\| = a_j$, for all $j = 1, \ldots, M$. The main tool used in [CKLT02] is the notion of *frame potentials* introduced by Benedetto and Fickus [BF02]. However, this is an existence proof while for applications we need an exact representation for the frame. In this paper we will give an algorithm for constructing a family of tight frame vectors $\{\varphi_j\}_{j=1}^M$ for H_N with $\|\varphi_j\| = a_j$, for all $j = 1, \ldots, M$. This provides an independent proof of some of the results in [CKLT02] while at the same time allowing exact constructions for the necessary frames. A MATLAB toolbox implementing all these results is freely distributed by the authors.

2. Definitions

Througout the paper we let $N < M$ be fixed positive integers. First we recall the well known fact that finite normalized tight frames correspond to the first N columns of matrices in $\boldsymbol{O}(M)$,

Date: April 15, 2002.

Key words and phrases. Finite tight frame, orthogonal matrix.

P.G. Casazza and M. T. Leon were supported by NSF DMS 0102686.

the $M \times M$ orthonormal matrices. Background definitions and a detailed proof of this fact is found in [GK01] and [BF02].

For the purposes of this paper it suffices to recall that $\{\varphi_j\}_{j=1}^M$ is a (finite) tight frame in $\mathbb{R}^N$ with frame constant A if for every vector $y \in \mathbb{R}^N$

$$Sy = \sum_{j=1}^{j=M} \langle y, \varphi_j \rangle \varphi_j = Ay$$

The "analysis frame operator" L of the frame is given by

$$Ly = \{\langle y, \varphi_j \rangle\}_{j=1}^{j=M}$$

with adjoint operator L^* given by

$$\{a_n\}_{j=1}^{j=M} \longrightarrow \sum_{j=1}^{j=M} a_j \varphi_j \ .$$

We have the diagram

$$\mathbb{R}^N \xrightarrow{L} l^2(M) \xrightarrow{L^*} \mathbb{R}^M \xrightarrow{L} l^2(M)$$

Then the frame operator S and grammian operator G equal $S = L^*L$ and $G = LL^*$, where L is an $M \times N$ matrix and L^* is an $N \times M$ matrix given by

$$L = \begin{pmatrix} \varphi_1^* \\ \vdots \\ \varphi_M^* \end{pmatrix}, \quad L^* = \left(\varphi_1 | \dots | \varphi_M\right)$$

where φ_j^* are row vectors and φ_j are column vectors. The tight frame condition gives $L^*L = AI_{N\times N}$ and the diagonal of the grammian is $\{\|\varphi_1\|^2, \dots, \|\varphi_M\|^2\}$.

Now extend $\frac{L}{\sqrt{A}}$ to a matrix $U \in \boldsymbol{O}(M)$,

$$U = \begin{pmatrix} \frac{\varphi_1}{\sqrt{A}} & | \cdot \\ \vdots & | \vdots \\ \frac{\varphi_M}{\sqrt{A}} & | \cdot \end{pmatrix}$$

In [CKLT02] it is proved that for $\{a_j \geq 0\}_{j=1}^{j=M}$ the condition $\sum_{j=1}^{j=M} a_j^2 \geq N a_1^2$ (suppose a_1 is the largest a_j) is sufficient for the existence of a tight frame $\{\varphi_j\}_{j=1}^{j=M}$ with $\|\varphi_j\| = a_j$. The associated matrix $U \in \boldsymbol{O}(M)$ satisfies the condition

$$N = \sum_{j=1}^{j=M} \left\| \frac{\varphi_j}{\sqrt{A}} \right\|^2, \quad \text{and} \quad \left\| \frac{\phi_j}{\sqrt{A}} \right\|^2 \leq 1,$$

where A is the tight frame bound of $\{\varphi_j\}_{j=1}^{j=M}$.

The main result in this paper is then:

Theorem 2.1. *For a sequence* $\{a_j\}_{j=1}^{M}$ *of positive real numbers such that* $a_j^2 \leq 1$ *and* $\sum_{j=1}^{j=M} a_j^2 = N$ *a matrix* $U \in \boldsymbol{O}(M)$ *exists such that, in squared norms, the first* N *columns are*

$$\begin{pmatrix} a_1^2 & | \;\cdot \\ \vdots & | \;\vdots \\ a_M^2 & | \;\cdot \end{pmatrix} .$$

We proceed to obtain such matrices. Their construction provides an independent proof of some of the results on existence of tight frames in [CKLT02].

Any sequence $\{a_j\}_{j=1}^{M}$ of positive real numbers such that $a_j^2 \leq 1$ and $\sum_{j=1}^{j=M} a_j^2 = N$ will be called *admissible*. For $R_M \in \boldsymbol{O}(M)$ let $r_{left,j}$ denote the first N terms of its jth row. Then $\sum_{j=1}^{j=M} \| r_{left,j}\|^2 = N$. We will say that R_M is a *solution* for the *admissible* sequence $\{a_j\}_{j=1}^{M}$ if $\|r_{left,j}\| = a_j$ for $j = 1, \ldots, M$.

3. Factorisation of $\boldsymbol{O}(M)$

The following factorisation is similar, with minor changes, to that described in [Vai93], p. 747. Every matrix in $\boldsymbol{O}(M)$ is obtained as a product of Givens rotations $\theta(t, j, k) \in \boldsymbol{O}(M)$, $j < k$, where

$$\theta(t,j,k) = \begin{pmatrix} I_{j-1,j-1} & 0 & 0 & 0 & 0 \\ 0 & \cos(t) & 0 & \sin(t) & 0 \\ 0 & 0 & I_{M-j-k-2,M-j-k-2} & 0 & 0 \\ 0 & -\sin(t) & 0 & \cos(t) & 0 \\ 0 & 0 & 0 & 0 & I_{k-1,k-1} \end{pmatrix}$$

It is clear that

$$\theta(t,j,k)^{-1} = \theta(-t,j,k)$$

To begin with, every $R_2 \in \boldsymbol{O}(2)$ is obtained as

$$R_2 = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \quad , \mu = \pm 1$$

and thus can be stored with one parameter t and a sign ± 1 (taking account of the determinant).

Let $R_M \in \boldsymbol{O}(M)$, and we will see how to factor R_M. Let

$$t_M^{M-1} = \pi/2 \ \text{ if } \ R_M(M,M) = 0,$$

and

$$t_M^{M-1} = \arg(R_M(M,M) - iR_M(M-1,M)) \ \text{ otherwise.} \tag{3.1}$$

Then, if $T=\theta(t_M^{M-1},M-1,M)R_M$, we have that $T(M-1,M)=0$ and $T(M,M)\geq 0$. That is,

$$T=\begin{pmatrix} \dots & | & T(1,M) \\ \dots & | & \dots \\ \dots & | & T(M-2,M) \\ \dots & | & 0 \\ \dots & | & T(M,M) \end{pmatrix}$$

Now repeat the process on the $M-2$ term of Mth column of T, thus obtaining t_M^{M-2}. The matrix

$$S=\theta(t_M^{M-2},M-2,M)T=\theta(t_M^{M-2},M-2,M)\theta(t_M^{M-1},M-1,M)R_M$$

is such that $S(M-2,M)=S(M-1,M)=0$ and $S(M,M)\geq 0$.

$$S=\begin{pmatrix} \dots & | & S(1,M) \\ \dots & | & \dots \\ \dots & | & S(M-3,M) \\ \dots & | & 0 \\ \dots & | & 0 \\ \dots & | & S(M,M) \end{pmatrix}$$

Iterating the process we obtain $t_M^1,\dots,t_M^{M-1}$ such that if

$$Q_M=\theta(t_M^1,1,M)\theta(t_M^2,2,M)\dots\theta(t_M^{M-2},M-2,M)\theta(t_M^{M-1},M-1,M)R_M$$

then

$$Q_M(1,M)=Q_M(2,M)=\dots=Q_M(M-1,M)=0$$

and $Q_M(M,M)\geq 0$. Since $Q_M\in \boldsymbol{O}(M)$, we now have $Q_M(M,M)=1$. Also and since $Q_M(M,M)=1$, it follows that

$$Q_M(M,1)=Q_M(M,2)=\dots=Q_M(M,M-1)=0$$

This means that

$$Q_M=\begin{pmatrix} R_{M-1} & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2}$$

for some matrix R_{M-1}, and since $Q_M\in O(M)$, we have $R_{M-1}\in \boldsymbol{O}(M-1)$.

Therefore,

$$R_M=\theta(-t_M^{M-1},M-1,M)\theta(-t_M^{M-2},M-1,M)\dots\theta(-t_M^2,2,M)\theta(-t_M^1,1,M)Q_M$$

for some matrix Q_M as in (3.2).

Any choice of $M-1$ parameters $t_M^1,\dots,t_M^{M-1}$ will induce embeddings

$$\theta_{t_M^1,\dots,t_M^{M-1}}:\boldsymbol{O}(M-1)\longrightarrow \boldsymbol{O}(M) \tag{3.3}$$

taking $R_{M-1}\in \boldsymbol{O}(M-1)$ into

$$\theta(t_M^{M-1},M-1,M)\theta(t_M^{M-2},M-1,M)\dots\theta(t_M^2,2,M)\theta(t_M^1,1,M)Q_M$$

for Q_M as in (3.2). Multiplication by any of the matrices $\theta(t,j,k)$ will only introduce changes in rows j and k. Parameters defined in (3.1) are defined up to integer multiples of 2π. Therefore for different (up to integer multiples of 2π) $M-1$ parameters $s_M^1, \ldots, s_M^{M-1}$ we obtain different embeddings $\theta_{s_M^1,\ldots,s_M^{M-1}}$.

The process can be iterated for R_{M-1}. And composing such embeddings (3.3) we obtain embeddings

$$\boldsymbol{O}(N) \longrightarrow \boldsymbol{O}(N+1) \longrightarrow \ldots \boldsymbol{O}(M-1) \longrightarrow \boldsymbol{O}(M)$$

The resulting parametrisation and the embeddings of $\boldsymbol{O}(N)$ into $\boldsymbol{O}(M)$ will play a significant role and we record them as follows.

Lemma 3.1. *If $N < M$, then*

(a) Every matrix in $\boldsymbol{O}(M)$ is parametrised by $(M-1)+(M-2)+\cdots+2+1 = \frac{M(M-1)}{2}$ parameters and one sign.
(b) Every matrix in $\boldsymbol{O}(M)$ is parametrised by $(M-1)+(M-2)+\cdots+(N+1) = \frac{M(M-1)}{2} - \frac{N(N-1)}{2}$ parameters and a matrix $R_N \in \boldsymbol{O}(N)$.
(c) Both of these parametrisations are unique (up to integer multiples of 2π for the real parameters).

Proof. Parts (a) and (b) are the well known representation of matrices in $\boldsymbol{O}(M)$ described above. For part (c), the sign of the decomposition is the determinant of the matrix. The real parameters t in (3.3) are defined up to integer multiples of 2π. □

The factorisation reformulates, for storage and computation purposes, the well known fact that as a Lie Group, $\boldsymbol{O}(M)$ has dimension $\frac{M(M-1)}{2}$ and two components, corresponding to the two possible values of the determinant.

Whenever we refer to a matrix in $\boldsymbol{O}(M)$ by giving the $M*(M-1)/2$ parameters and a sign we refer to the decomposition of the Lemma 3.1 . For later reference we now record both factorisations of arbitrary $R_M \in \boldsymbol{O}(M)$.

$$\begin{aligned} R_M = {} & \theta(t_M^{M-1}, M-1, M)\theta(t_M^{M-2}, M-1, M)\ldots\theta(t_M^2, 2, M)\theta(t_M^1, 1, M) \\ & \theta(t_{M-1}^{M-2}, M-2, M-1)\theta(t_{M-1}^{M-3}, M-2, M-1)\ldots\theta(t_{M-1}^2, 2, M-1)\theta(t_{M-1}^1, 1, M-1) \\ & \ldots\theta(t_3^2, 2, 3)\theta(t_3^1, 1, 3)\begin{pmatrix}\cos(t_2^1) & \sin(t_2^1) & \\ -\sin(t_2^1) & \cos(t_2^1) & \\ & & I_{M-2,M-2}\end{pmatrix}\begin{pmatrix}\mu & 0 & \\ 0 & 1 & \\ & & I_{M-2,M-2}\end{pmatrix}, \mu = \pm 1 \end{aligned} \tag{3.4}$$

and if any $Q_N \in \boldsymbol{O}(M)$ is given by

$$\begin{aligned} Q_N = {} & \theta(t_N^{N-1}, N-1, N)\theta(t_N^{N-2}, N-2, N)\ldots\theta(t_N^2, 2, N)\theta(t_N^1, 1, N-1) \\ & \ldots\theta(t_3^2, 2, 3)\theta(t_3^1, 1, 3)\begin{pmatrix}\cos(t_2^1) & \sin(t_2^1) & \\ -\sin(t_2^1) & \cos(t_2^1) & \\ & & I_{M-2,M-2}\end{pmatrix}\begin{pmatrix}\mu & 0 & \\ 0 & 1 & \\ & & I_{M-2,M-2}\end{pmatrix}, \mu = \pm 1 \end{aligned}$$

then

$$Q_N = \begin{pmatrix} R_N & 0 \\ 0 & I_{M-N,M-N}\end{pmatrix} \tag{3.5}$$

where $R_N \in \boldsymbol{O}(N)$, and

$$R_M = \theta(t_M^{M-1}, M-1, M)\theta(t_M^{M-2}, M-1, M)\dots\theta(t_M^2, 2, M)\theta(t_M^1, 1, M)\dots \\ \theta(t_{N+1}^N, N, N+1)\theta(t_{N+1}^{N-1}, N-1, N+1)\dots\theta(t_{N+1}^1, 1, N+1)Q_N = \Psi Q_N \tag{3.6}$$

where

$$\Psi = \theta(t_M^{M-1}, M-1, M)\theta(t_M^{M-2}, M-2, M)\dots\theta(t_M^2, 2, M)\theta(t_M^1, 1, M) \\ \dots\theta(t_{N+1}^N, N, N+1)\theta(t_{N+1}^{N-1}, N-1, N+1)\dots\theta(t_{N+1}^1, 1, N+1)$$

4. Existence and Construction Results

Lemma 4.1. *Let $R_M \in \boldsymbol{O}(M)$ be a solution for the admissible sequence $\{a_j\}_{j=1}^M$, and let R_M be factorised as in (3.6) by means of $\frac{M(M-1)}{2} - \frac{N(N-1)}{2}$ parameters and a matrix $R_N \in \boldsymbol{O}(N)$ as in (3.5). For any matrix $S_N \in \boldsymbol{O}(N)$, let*

$$P_N = \begin{pmatrix} S_N & 0 \\ 0 & I_{M-N,M-N} \end{pmatrix}$$

Then

$$S_M = \theta(t_M^{M-1}, M-1, M)\theta(t_M^{M-2}, M-2, M)\dots\theta(t_M^2, 2, M)\theta(t_M^1, 1, M)\dots \\ \theta(t_{N+1}^N, N, N+1)\theta(t_{N+1}^{N-1}, N-1, N+1)\dots\theta(t_{N+1}^1, 1, N+1)P_N = \Psi P_N$$

is a solution for the admissible sequence $\{a_j\}_{j=1}^M$. This provides an embedding of $\boldsymbol{O}(N)$ into the space of solutions for the admissible sequence $\{a_j\}_{j=1}^M$.

Proof. We will show that the values $\{a_j\}_{j=1}^M$ depend on the parameters

$$t_M^{M-1}, t_M^{M-2}, \dots t_{N+1}^2, t_{N+1}^1 \tag{4.1}$$

and are independent of the particular choice of $R_M \in \boldsymbol{O}(N)$.

Let r_l be the lth row of R_N for $l = 1, \dots, N$. Then $\langle r_l, r_k \rangle = \delta_{l,k}$. Let x_j be the first N terms of the jth row of R_M for $j = 1, \dots, M$. Multiplication by a Givens rotation will replace the first N terms of any row by a linear combination of $\{r_l\}_{l=1}^M$. At the end of the process

$$x_j = \sum_{l=1}^{l=N} \lambda_l^j \; r_l$$

where the coefficients λ_l depend only on the $\frac{M(M-1)}{2} - \frac{N(N-1)}{2}$ parameters in (4.1) and are independent of the r_l given by R_N. Since the vectors r_l are orthonormal,

$$\|x_j\|^2 = \langle x_j, x_j \rangle = \langle \sum_{l=1}^{l=N} \lambda_l^j \; r_l, \sum_{l=1}^{l=N} \lambda_l^j \; r_l \rangle = \sum_{l=1}^{l=N} (\lambda_l^j)^2$$

Thus under replacement of R_N by another matrix in $O(N)$ the values of $\|x_j\|^2$ do not change. Again from (3.6) it follows that the parameters in (4.1) induce an embedding of $\boldsymbol{O}(N)$ into the space of *solutions* for the *admissible* sequence $\{a_j\}_{j=1}^M$. □

Remark 4.2. The equations giving λ_l^j in terms of (4.1) are in practical terms unmanageable. They are given in terms of the $\frac{M(M-1)}{2} - \frac{N(N-1)}{2}$ parameters.

Theorem 4.3. *Let $N < M$ be fixed. If $\{a_j\}_{j=1}^M$ is an admissible sequence, then*

(a) There is a solution for $\{a_j\}_{j=1}^M$
(b) Moreover, there is an embedding from $\boldsymbol{O}(N) \times \boldsymbol{O}(M-N)$ into the space of all solutions for $\{a_j\}_{j=1}^M$. The embedding will take the pair of matrices $R_N \in \boldsymbol{O}(N)$ and $R_{M-N} \in \boldsymbol{O}(M-M)$ into

$$\Psi \begin{pmatrix} R_N & 0 \\ 0 & R_{M-N,M-N} \end{pmatrix}$$

where the matrix Ψ is determined by the *admissible* sequence $\{a_j\}_{j=1}^M$.

Proof. The proof is just an algorithm. Let $\Psi = I_{M,M}$ We start with the matrix $O \in \boldsymbol{O}(M)$ given by

$$O = \begin{pmatrix} R_N & 0 \\ 0 & R_{M-N} \end{pmatrix}$$

where R_N and R_{M-N} are arbitrary matrices in $\boldsymbol{O}(N)$ and $\boldsymbol{O}(M-N)$ respectively. The fact that there is no other requirement on R_N and R_{M-N} will provide the clue for the proof of the second claim of the lemma. We will construct matrices $O^n \in \boldsymbol{O}(M)$ for $n = 1, \dots, M-1$. We will denote

$$\begin{aligned} o^n_{left,j} &= \text{first } N \text{ terms of } j\text{th row of } O^n \text{ and} \\ o^n_{right,j} &= \text{last } M-N \text{ terms of } j\text{th row of } O^n \end{aligned}$$

Once the values $a_1^2, \dots, a_{M-1}^2$ have been attained the problem is solved since each O^n is an orthogonal matrix. The *solution* will be O^{M-1}. Let

$$t_1 = \arg(a_1 + i\sqrt{1-a_1^2})$$

and rename

$$\Psi = \theta(t_1, 1, M)\Psi.$$

Let

$$O^1 = \theta(t_1, 1, M)O$$

Then $O^1 = \Psi O$, and

$$\left\| o^1_{left,1} \right\|^2 = a_1^2, \quad \text{and} \quad \left\| o^1_{left,M} \right\|^2 = 1 - a_1^2.$$

Also $o^1_{left,M}$ remains orthogonal to $o^1_{left,j}$ for $j \geq 2$ and $o^1_{right,M}$ remains orthogonal to $o^1_{right,j}$ for $j \leq M-1$. The values of the squared norms of $o^1_{left,j}$ and $o^1_{right,j}$ are described by the diagram:

$$\begin{pmatrix} a_1^2 & | & 1-a_1^2 \\ 1 & | & 0 \\ 1 & | & 0 \\ \vdots & | & \vdots \\ 1 & | & 0 \\ --- & | & ---- \\ 0 & | & 1 \\ \vdots & | & \vdots \\ 0 & | & 1 \\ 1-a_1^2 & | & a_1^2 \end{pmatrix}$$

Now one of the following applies:

(CASE I) $1 \leq a_1^2 + a_2^2$, or
(CASE II) $1 > a_1^2 + a_2^2$

In CASE I, take t_2 a solution of

$$\sin(t_2)^2 = \frac{1-a_2^2}{a_1^2} \quad \text{and} \quad O^2 = \theta(t_2, 2, M)O^1$$

Then

$$\left\|o^2_{left,2}\right\|^2 = \left\|\cos(t_2)o^1_{left,2} + \sin(t_2)o^1_{left,M}\right\|^2 \stackrel{(1)}{=} \left\|\cos(t_2)o^1_{left,2}\right\|^2 + \left\|\sin(t_2)o^1_{left,M}\right\|^2 =$$
$$\cos(t_2)^2\left\|o^1_{left,2}\right\|^2 + \sin(t_2)^2\left\|o^1_{left,M}\right\| \stackrel{(2)}{=} \cos(t_2)^2 + \sin(t_2)^2(1-a_1^2) =$$
$$1 - \sin(t_2)^2(a_1^2) = 1 - \frac{1-a_2^2}{a_1^2}a_1^2 = a_2^2$$

where equality (1) holds because $o^1_{left,2}$ and $o^1_{left,M}$ are orthogonal and equality (2) holds because $o^1_{left,2}$ is a unitary vector (in R_N). The values of the squared norms of $o^2_{left,j}$ and $o^2_{right,j}$ are described by the diagram:

$$\begin{pmatrix} a_1^2 & | & 1-a_1^2 \\ a_2^2 & | & 1-a_2^2 \\ 1 & | & 0 \\ \vdots & | & \vdots \\ 1 & | & 0 \\ ----- & | & ------ \\ 0 & | & 1 \\ \vdots & | & \vdots \\ 0 & | & 1 \\ 2-a_1^2-a_2^2 & | & a_1^2+a_2^2-1 \end{pmatrix}$$

Now for $j = 3, \ldots, N$, we have $o^2_{left,j} = o^1_{left,j}$ (they come from R_N) so they are unitary and they remain orthogonal to $o^2_{left,M}$ since the latter has been obtained as a linear combination of $o^1_{left,j}$ for $j \leq 2$.

Similarly, for $j = N+1, \dots, M-1$, we have $o^2_{right,j} = o^1_{right,j}$ (they come from R_M) so they are unitary and remain orthogonal to $o^2_{right,M}$ since the latter is just a scalar multiple of $o^1_{right,M}$.

In CASE II, take t_2 a solution of

$$\sin(t_2)^2 = \frac{a_2^2}{1-a_1^2} \quad \text{and let} \quad O^2 = \theta(t_2, M-1, M)O^1$$

Then

$$\left\|o^2_{right,M-1}\right\|^2 = \left\|\cos(t_2)o^1_{right,M-1} + \sin(t_2)o^1_{right,M}\right\|^2 \overset{(1)}{=} \left\|\cos(t_2)o^1_{right,M-1}\right\|^2 + \left\|\sin(t_2)o^1_{right,M}\right\|^2 =$$
$$\cos(t_2)^2 \left\|o^1_{right,M-1}\right\|^2 + \sin(t_2)^2 \left\|o^1_{right,M}\right\| \overset{(2)}{=} \cos(t_2)^2 + \sin(t_2)^2 a_1^2 =$$
$$1 - \sin(t_2)^2 + \sin(t_2)^2 a_1^2 = 1 + \sin(t_2)^2(a_1^2 - 1) = 1 + \frac{a_2^2}{1-a_1^2}(a_1^2 - 1) = 1 - a_2^2$$

where equality (1) holds because $o^1_{right,M-1}$ and $o^1_{right,M}$ are orthogonal and equality (2) holds since $o^1_{right,M-1}$ is a unitary vector (in R_{M-N}). The values of the squared norms of $o^2_{left,j}$ and $o^2_{right,j}$ are described by the diagram:

$$\begin{pmatrix} a_1^2 & | & 1-a_1^2 \\ 1 & | & 0 \\ 1 & | & 0 \\ \dots & | & \dots \\ 1 & | & 0 \\ ----- & | & ----- \\ 0 & | & 1 \\ \dots & | & \dots \\ 0 & | & 1 \\ a_2^2 & | & 1-a_2^2 \\ 1-a_1^2-a_2^2 & | & a_1^2+a_2^2 \end{pmatrix}$$

Now for $j = 2, \dots, N$, we have $o^2_{left,j} = o^1_{left,j}$ (they come from R_{M-N}). Hence, they are unitary and they remain orthogonal to $o^2_{left,M}$ since the latter has been obtained as a linear combination of $o^1_{left,j}$ for $j < 2$.

Similarly, for $j = N+1, \dots, M-2$, we have $o^2_{right,j} = o^1_{right,j}$ (they come from R_M). So they are unitary and remain orthogonal to $o^2_{right,M}$ since the latter has been obtained as a linear combination of $o^1_{right,j}$ for $j > M-2$.

In either case, by renaming

$$\Psi = \theta(t_2, m, M)\Psi$$

where m is given by the CASE used, we have

$$O^2 = \Psi O$$

After k applications of CASE I and l applications of CASE II we obtain the matrix O^{k+l+1}.The values of the squared norms of $o^{k+l+1}_{left,j}$ and $o^{k+l+1}_{right,j}$ are described (up to reordering of $\{a_j\}_{j=1}^M$) by the diagram:

$$\left(\begin{array}{c|c}
a_1^2 & 1-a_1^2 \\
a_2^2 & 1-a_2^2 \\
\dots & \dots \\
a_{k+1}^2 & 1-a_{k+1}^2 \\
1 & 0 \\
\dots & \dots \\
1 & 0 \\
----------- & ---------- \\
0 & 1 \\
\dots & \dots \\
0 & 1 \\
\dots & \dots \\
a_{k+l+1}^2 & 1-a_{k+l+1}^2 \\
\dots & \dots \\
a_{k+3}^2 & 1-a_{k+3}^2 \\
a_{k+2}^2 & 1-a_{k+2}^2 \\
k+1-a_1^2-\dots-a_{k+l+1}^2 & a_1^2+\dots+a_{k+l+1}^2-k
\end{array}\right)$$

Now the two cases correspond to :

(CASE I) $k+1 \leq a_1^2+\cdots+a_{k+l+1}^2+a_{k+l+2}^2$, or
(CASE II) $k+1 > a_1^2+\cdots+a_{k+l+1}^2+a_{k+l+2}^2$

In CASE I, take t_{k+l+2} a solution of

$$\sin(t_{k+l+2})^2 = \frac{1-a_{k+l+2}^2}{a_1^2+\cdots+a_{k+l+1}^2-k} \quad \text{and} \quad O^{k+l+2} = \theta(t_{k+l+2}, k+2, M)O^{k+l+1}$$

In CASE II, take t_{k+l+2} a solution of

$$\sin(t_{k+l+2})^2 = \frac{a_{k+l+2}^2}{k+1-a_1^2-\cdots-a_{k+l+1}^2} \quad \text{and} \quad O^{k+l+2} = \theta(t_{k+l+2}, M-l-1, M)O^{k+l+1}$$

In either case, by renaming

$$\Psi = \theta(t_{k+l+2}, m, M)\Psi$$

where m is given by the CASE used, we have

$$O^{t_{k+l+2}} = \Psi O.$$

Now it remains to prove that CASE I eventually takes place whenever $k < N-1$ and when $k = N-1$ only CASE II can apply.

Suppose $k < N-1$. Then $k+1 \leq N-1$. At the same time

$$a_1^2+\cdots+a_{k+l+1}^2+a_{M-1}^2 \geq N-1$$

giving

$$k+1 \leq N-1 \leq a_1^2+\cdots+a_{k+l+1}^2+a_{M-1}^2$$

so CASE I will eventually apply .

If $k = N - 1$ and CASE I applies at some step,

$$k + 1 = N \leq a_1^2 + \cdots + a_{k+l+2}^2 \leq N$$

This could only happen if $N = a_1^2 + \cdots + a_{k+l+2}^2$, so $a_{k+l+3} = \cdots = a_{M-1} = a_M = 0$. In this case O^{k+l+1} is a *solution* for $\{a_j\}_{j=1}^M$. It is clear from the construction that

$$O^{M-1} = \Psi O.$$

Finally, the parameters $t_1, \ldots, t_{M-1}$ and the matrix Ψ were determined by the norms $a_1^2, \ldots, a_{M-1}^2$ and so is Ψ. Replacement of O by any other

$$O' = \begin{pmatrix} R'_N & 0 \\ 0 & R'_{M-N} \end{pmatrix}$$

would have lead to another solution

$$O'^{M-1} = \Psi O'$$

It is clear that multiplication by a unitary matrix is an injection. The claim is thus proved. □

It must be pointed out that the embedding of $\boldsymbol{O}(N) \times \boldsymbol{O}(M - M)$ into $\boldsymbol{O}(M)$ does not fill out the space of *solutions*. Using alternative algorithms it is possible to construct *solutions* which do not belong to the image of the embedding, that is, can not be factorised as

$$\psi O = \psi \begin{pmatrix} R_N & 0 \\ 0 & R_{M-N} \end{pmatrix}$$

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