FRAMES WITH A GIVEN FRAME OPERATOR

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ABSTRACT. Let S be a positive self-adjoint invertible operator on an N-dimensional Hilbert space H_N and let $M \geq N$. We give necessary and sufficient conditions on real sequences $a_1 \geq a_2 \geq \cdots \geq a_M \geq 0$ so that there is a frame $\{\varphi_n\}_{n=1}^{n=M}$ for H_N with frame operator S and $\|\varphi_n\| = a_n$, for all $n = 1, 2, \ldots M$. As a consequence we see that for any frame operator S on H_N and for any $M \geq N$, there is an equal norm frame for H_N with M elements and having S as its frame operator. A MATLAB toolbox [4] implementing all results is freely distributed by the authors.

1. INTRODUCTION

A sequence $\{\varphi_n\}_{n=1}^{n=M}$ is a *frame* for an N-dimensional Hilbert space H_N if the positive self-adjoint *frame operator*

$$S = \sum_{n=1}^{n=M} \langle \varphi, \varphi_n \rangle \varphi_n$$

is a bounded, invertible operator on H_N. A frame $\{\varphi_n\}_{n=1}^{n=M}$ is a λ-*tight frame* if $S = \lambda I$ and if $\lambda = 1$, it is a *Parseval frame* . Moreover, $Tr\ S = \sum_{n=1}^{n=M} \|\varphi_n\|^2$. Hilbert space frames have played a fundamental role in signal/image processing since the seminal work of Gabor [7]. The tools introduced by Gabor where formalized into the notion of *frames* by Duffin and Schaeffer [5]. Recently, frames have been applied in a wide variety of areas from the Internet [8] and [9], multiple antenna coding [10], quantum theory [6], and [11] and more. Each application of frame theory requires a new class of frames designed for the specific application. This often involves having to find frames with (prescribed in advance) norms for the frame vectors. In [2] there is given necessary and sufficient conditions on real sequences $a_1 \geq a_2 \geq \cdots \geq a_M > 0$ so that there exists a tight frame $\{\varphi_n\}_{n=1}^{n=M}$ for H_N with $\|\varphi_n\| = a_n$, for all $n = 1, 2, \ldots M$. The condition for the existence of a λ-tight frame given in [2] is that

$$\lambda = \sum_{n=1}^{n=M} a_n^2 \geq N a_1^2.$$

One interpretation of this result is that it gives necessary and sufficient conditions on $\|\varphi_n\|$ for $\{\varphi_n\}_{n=1}^{n=M}$ to form a frame for H_N with frame operator $S = \lambda I$. An alternative proof of this result appears in [3] where an algorithm is given for this construction which runs very efficiently in MATLAB .

Date: April 15, 2002.

1991 *Mathematics Subject Classification.* 42C15.

Key words and phrases. Finite tight frame, orthogonal matrix.

P.G. Casazza and M. T. Leon were supported by NSF DMS 0102686.

In this paper we generalize these results to the case where λI is replaced by any positive self-adjoint invertible operator S on H_N. That is, for a given S and $M \geq N$, we give necessary and sufficient conditions on $a_1 \geq a_2 \geq \cdots \geq a_M > 0$ so that there is a frame $\{\varphi_n\}_{n=1}^{n=M}$ for H_N with frame operator S and satisfying: $\|\varphi_n\| = a_n$, for all $n = 1, 2, \ldots M$.

2. MAIN RESULT

The main result in this paper is:

Theorem 2.1. *Let S be a positive self-adjoint operator on a N-dimensional Hilbert space H_N. Let $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_N$ be the eigenvalues of S. Fix $M \geq N$ and real numbers $a_1 \geq a_2 \geq \ldots a_M > 0$. The following are equivalent:*

1. *There is a frame $\{\varphi_j\}_{j=1}^{j=M}$ for H_N with frame operator S and $\|\varphi_j\| = a_j$, for all $j = 1, 2, \ldots, M$.*
2. *For every $1 \leq k \leq N$,*

$$\sum_{i=1}^{i=k} a_i^2 \leq \sum_{i=1}^{i=k} \lambda_i, \quad \text{and} \quad \sum_{i=1}^{i=M} a_i^2 = \sum_{i=1}^{i=N} \lambda_i. \tag{2.1}$$

It is well known (see e.g. [1]) that there are equal norm Parseval frames with M-elements in H_N for all $M \geq N$. As a consequence of Theorem (2.1), we see that there are equal norm tight frames for any prescribed frame operator.

Corollary 2.2. *If S is a positive self-adjoint operator on H_N then for every $M \geq N$ there is an equal norn frame $\{\varphi_n\}_{n=1}^{n=M}$ for H_N whose frame operator equals S.*

Proof. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ be the eigenvalues of S and let $\lambda = \sum_{i=1}^{i=N} \lambda_i$. Fix $M \geq N$. If $a_n = \sqrt{\lambda/M}$ for every n then $\sum_{i=1}^{i=M} a_n^2 = \lambda$, and for all $1 \leq k \leq N$,

$$\sum_{i=1}^{i=k} a_n^2 = \frac{k}{M}\lambda = \frac{k}{M}\sum_{i=1}^{i=N}\lambda_i = k\frac{N}{M}\frac{1}{N}\sum_{i=1}^{i=N}\lambda_i \leq k\frac{N}{M}\frac{1}{k}\sum_{i=1}^{i=k}\lambda_i \leq \sum_{i=1}^{i=k}\lambda_i.$$

In the next to last inequality above we have used the fact that deleting some of the smallest numbers from a set of numbers will increase the average of the numbers.

Hence, by Theorem (2.1), there is a frame $\{\varphi_n\}_{n=1}^{n=M}$ with $\|\varphi_n\| = \sqrt{\lambda/M}$ for all $n = 1, 2, \ldots, M$ having frame operator S. □

To show that (1) implies (2) in the theorem we will actually prove a more general result.

Theorem 2.3. *Let $\{\varphi_j\}_{j=1}^{j=M}$ be a frame for H_N with frame operator S having eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. If P is an orthogonal projection of H_N onto a k-dimensional subspace, $1 \leq k \leq N$, then*

$$\sum_{j=N-k+1}^{j=N} \lambda_j \leq \sum_{j=1}^{j=M} \|P\varphi_j\|^2 \leq \sum_{j=1}^{j=k} \lambda_j.$$

Proof. Let $\{e_n\}_{n=1}^{n=N}$ be an orthonormal basis for H_N with $Se_n = \lambda_n e_n$, $n = 1, 2, \ldots, N$. Let P be a rank k orthogonal projection on H_N and let $\{\psi_i\}_{i=1}^{i=k}$ be an orthonormal basis for PH_N. It is known (see e.g. [2]),that

(1) $\sum_{n=1}^{n=M} |\langle \varphi_n, e_m\rangle|^2 = \lambda_m$, for all $1 \le m \le N$

(2) $\sum_{n=1}^{n=M} \langle \varphi_n, e_l\rangle\overline{\langle \varphi_n, e_m\rangle} = 0$ for all $1 \le l \ne m \le N$.

Now we compute using (1) and (2) above.

$$\sum_{n=1}^{n=M} \|P\varphi_n\|^2 = \sum_{n=1}^{n=M}\sum_{i=1}^{i=k} |\langle \psi_i, P\varphi_n\rangle|^2 = \sum_{n=1}^{n=M}\sum_{i=1}^{i=k} |\langle \psi_i, \varphi_n\rangle|^2 =$$

$$\sum_{n=1}^{n=M}\sum_{i=1}^{i=k}\left|\sum_{m=1}^{m=N}\langle \psi_i, e_m\rangle\overline{\langle \varphi_n, e_m\rangle}\right|^2 = \sum_{n=1}^{n=M}\sum_{i=1}^{i=k}\sum_{m=1}^{m=N}\sum_{l=1}^{l=N}\langle \psi_i, e_m\rangle\overline{\langle \varphi_n, e_m\rangle\langle \psi_i, e_l\rangle}\langle \varphi_n, e_l\rangle =$$

$$\sum_{m=1}^{m=N}\sum_{n=1}^{n=M}\sum_{i=1}^{i=k} |\langle \psi_i, e_m\rangle|^2 |\langle \varphi_n, e_m\rangle|^2 + \sum_{i=1}^{i=k}\sum_{m\ne l}\langle \psi_i, e_m\rangle\overline{\langle \psi_i, e_l\rangle}\sum_{n=1}^{n=N}\langle \varphi_n, e_l\rangle\overline{\langle \varphi_n, e_m\rangle} =$$

$$\sum_{m=1}^{m=N}\sum_{i=1}^{i=k} |\langle \psi_i, e_m\rangle|^2 \sum_{n=1}^{n=M} |\langle \varphi_n, e_m\rangle|^2 = \sum_{m=1}^{m=N}\sum_{i=1}^{i=k} |\langle \psi_i, e_m\rangle|^2 \lambda_m.$$

Since $\{\psi_i\}_{i=1}^{i=k}$ is an orthonormal basis for its span, we have that

$$\sum_{i=1}^{i=k} |\langle \psi_i, e_m\rangle|^2 \le 1, \quad \text{for all} \quad 1 \le i \le k, \quad 1 \le m \le N$$

and

$$\sum_{m=1}^{m=N}\sum_{i=1}^{i=k} |\langle \psi_i, e_m\rangle|^2 = \sum_{i=1}^{i=k}\sum_{m=1}^{m=N} |\langle \psi_i, e_m\rangle|^2 = \sum_{i=1}^{i=k} \|\psi\|^2 = k.$$

Combined with our calculations above, we obtain,

$$\sum_{m=1}^{m=k}\lambda_m \ge \sum_{m=1}^{m=N}\left(\sum_{i=1}^{i=k} |\langle \psi_i, e_m\rangle|^2\right)\lambda_m = \sum_{m=1}^{m=N} \|P\varphi_m\|^2 \ge \sum_{m=N-k}^{m=N} \lambda_m.$$

□

We now give two corollaries. The first is one of the implications of Theorem (2.1).

Corollary 2.4. *Let $\{\varphi_j\}_{j=1}^{M}$ be a frame for H_N with frame operator S having eigenvalues $\lambda_1 \ge \lambda_2 \ge \dots \lambda_N > 0$. If $\|\varphi_1\| \ge \|\varphi_2\| \ge \cdots \ge \|\varphi_M\|$, then for every $1 \le k \le N$,*

$$\sum_{j=1}^{j=k} \|\varphi_j\|^2 \le \sum_{j=1}^{j=k}\lambda_j$$

Proof. Given k, let P be an orthogonal projection of rank k on H_N so that $\varphi_j \in PH_N$, for all $1 \le j \le k$. By Theorem (2.3) we have :

$$\sum_{j=1}^{j=k} \|\varphi_j\|^2 = \sum_{j=1}^{j=k} \|P\varphi_j\|^2 \le \sum_{j=1}^{j=M} \|P\varphi_j\|^2 \le \sum_{j=1}^{j=k}\lambda_j.$$

□

The next corollary is well-known in many areas of mathematics. For example, in PDE's this is a consequence of the Rayleigh min/max principle. In stochastic processes this is the Karhunen-Loéwe theorem.

Corollary 2.5. *Let* S *be a positive self-adjoint operator on* H_N *with eigenvalues* $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_N > 0$. *If* P *is a rank* k *orthogonal projection on* H_N *then*

$$Tr(PSP) \leq \sum_{j=1}^{j=k} \lambda_j.$$

Proof. If $\{e_j\}_{j=1}^{N}$ is an orthonormal sequence in H_N with $Se_j = \lambda_j e_j$, then $\{\varphi_j = \sqrt{\lambda_j} e_j\}_{j=1}^{N}$ is a frame for H_N with frame operator S. Hence, PSP is the frame operator for $\{P\varphi_j\}_{j=1}^{N}$. Applying Theorem (2.3), for every $1 \leq k \leq N$, we have

$$Tr(PSP) = \sum_{j=1}^{j=N} \|P\varphi_j\|^2 \leq \sum_{j=1}^{j=k} \lambda_j.$$

□

Now we complete the proof of the main result by showing that (2) implies (1). We'll start with a frame $\{\varphi_j\}_{j=1}^{j=M}$ with frame operator S. The vectors used in Corollary (2.5) can be extended to a frame on H_N with frame operator S. More generally, since S is symmetric,

$$S = V\Lambda V^*$$

where V is an orthonormal matrix and Λ is a diagonal matrix with $diag(\Lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_N)$. Let $\Delta_{M\times N}$ be such that its top N rows equal $\Lambda^{\frac{1}{2}}$ and all remaining entries are zero. Let $W_{M\times M}$ be orthonormal (or unitary), and let $\varphi_j = j-$th row of $F = W\Delta V^*$. Then

$$F^*F = (W\Delta V^*)^* W\Delta V^* = V\Lambda V^* = S.$$

The Gram operator is given by $G = FF^*$, and

$$diag(G) = (\|\varphi_1\|^2, \ldots, \|\varphi_M\|^2).$$

Then (Horn. p. 101) there is an orthogonal matrix $U_{M\times M}$ and an diagonal matrix Λ such that

$$G = U^*\Lambda U, \quad \text{where} \quad diag(\Lambda) = (\lambda_1, \ldots, \lambda_N, 0, \ldots, 0).$$

Let $V_{M\times M}$ be (see Proposition (3.1) in the appendix) an orthogonal matrix such that if

$$T = V\Lambda V^*, \quad \text{then,} \quad diag(T) = (a_1^2, \ldots, a_M^2).$$

Let $\psi_j = j-$th row of $H = VUF$. Then $\{\psi_j\}_{j=1}^{j=M}$ is a frame since $rank(H) = rank(F) = N$. Its frame operator is given by

$$H^*H = (VUF)^*VUF = F^*F = S,$$

and the diagonal of its Gram matrix is

$$diag(VUF(VUF)^*) = diag(VUFF^*U^*V^*) = diag(V\Lambda V^*) = (a_1^2, \ldots, a_M^2).$$

3. Appendix

Every matrix in $\boldsymbol{O}(M)$ (the orthogonal group) is obtained as a product of Givens rotations $\theta(t,j,k,M) \in \boldsymbol{O}(M), j < k$, where

$$\theta(t,j,k,M) = \begin{pmatrix} I_{j-1,j-1} & 0 & 0 & 0 & 0 \\ 0 & \cos(t) & 0 & \sin(t) & 0 \\ 0 & 0 & I_{M-j-k-2,M-j-k-2} & 0 & 0 \\ 0 & -\sin(t) & 0 & \cos(t) & 0 \\ 0 & 0 & 0 & 0 & I_{k-1,k-1} \end{pmatrix}$$

It is clear that

$$\theta(t,j,k,M)^{-1} = \theta(-t,j,k,M)$$

Proposition 3.1. *Let $\lambda_1, \ldots, \lambda_M$ and $a_1, \ldots, a_M$ be real numbers such that $a_1^2 \geq a_2^2 \geq \cdots \geq a_M^2$ and for every $1 \leq k \leq M$,*

$$\sum_{i=1}^{i=k} a_i^2 \leq \sum_{i=1}^{i=k} \lambda_i, \quad \textit{and} \quad \sum_{i=1}^{i=M} a_i^2 = \sum_{i=1}^{i=M} \lambda_i. \tag{3.1}$$

Let Λ be a diagonal matrix with $diag(\Lambda) = (\lambda_1, \ldots, \lambda_M)$. Then there is a matrix $O \in \boldsymbol{O}(M)$ such that

$$diag(O\Lambda O^*) = (a_1^2, \ldots, a_M^2).$$

Proof. We'll prove the proposition by induction on M. If $M = 2$, let $t = \arcsin(\sqrt{\lambda_1 - a_1^2/\lambda_1 - \lambda_2}$ and $O = \theta(t,1,2,2)$. Next, Assume the result holds for $M-1$. From the hypothetsis, $\lambda_1 \geq a_1^2$. Let k be such that $\lambda_j \geq a_1^2$ for $j = 1, \ldots, k-1$ and $a_1^2 \geq \lambda_k$. Let

$$t = \arcsin(\sqrt{\lambda_1 - a_1^2/\lambda_1 - \lambda_k} \quad \text{and} \quad O_1 = \theta(t,1,k,M).$$

Then

$$O_1 \Lambda O_1^* = \begin{pmatrix} a_1^2 & 0 \ldots & * \; 0 \ldots 0 \\ 0 & \ldots & \\ \vdots & \vdots & \\ * & \ldots & \\ 0 & \ldots & \\ \vdots & \vdots & \\ 0 & \ldots & \end{pmatrix}.$$

where $*$ represents a possibly nonzero entry on kth row and 1st column (or 1st row and kth column). Let Λ_1 be the $(M-1) \times (M-1)$ bottom right box of $O_1 \Lambda O_1^*$. Then, Λ_1 is a diagonal matrix and, since $Tr(\Lambda) = Tr(O_1 \Lambda O_1^*)$,

$$diag(\Lambda_1) = (\lambda_2, \ldots, \lambda_{k-1}, \lambda_k + \lambda_1 - a_1^2, \lambda_{k+1}, \ldots, \lambda_M).$$

Now we'll verify that $diag(\Lambda_1)$ and $a_2, \ldots, a_M$ meet the premises of the lemma.

If $m < k$,

$$\lambda_2 + \lambda_3 + \cdots + \lambda_m \geq (m-1) * a_1^2 \geq (m-1) * a_2^2 \geq a_2^2 + \cdots + a_m^2.$$

If $m \geq k$,

$$\lambda_2 + \lambda_3 + \cdots + \lambda_m = \lambda_2 + \lambda_3 + \cdots + \lambda_{k-1} + \lambda_k + \lambda_1 - a_1^2 + \lambda_{k+1} + \cdots + \lambda_m = \lambda_1 + \lambda_2 + \cdots + \lambda_m - a_1^2 \geq a_2^2 + \cdots + a_m^2.$$

Then

$$O = \begin{pmatrix} 1 & 0 \\ 0 & O_2 \end{pmatrix} O_1.$$

where O_2 is the solution for Λ_1, will satisfy the claim. □

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