PROJECTIONS OF FRAMES

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Abstract. Let $\{\phi_m\}_{m=1}^M$ be a frame for an N-dimensional Hilbert space H_N and let P be a rank k orthogonal projection on H_N. We give a simple classification of the eigenvalues and eigenvectors for the frame operator for $\{P\phi_m\}_{m=1}^M$. Using this we derive several results on finite frames. For example, if $\{\phi_m\}_{m=1}^M$ is a frame for H_{2N} (respectively, H_{2N+1}) then there is a rank N (respectively, $N+1$) orthogonal projection P on H_{2N} (respectively, H_{2N+1}) so that $\{P\phi_m\}_{m=1}^M$ is a tight frame. MATLAB functions implementing all constructive results are available from the authors.

1. Introduction

Let H_N be an N-dimensional Hilbert space. A family $\{\phi_m\}_{m=1}^M$ of elements of H_N is a *frame* for H_N if there are constants $A, B > 0$ so that for all $f \in H_N$,

$$A\|f\|^2 \leq \sum_{m=1}^{M} |\langle f, \phi_m\rangle|^2 \leq B\|f\|^2$$

We call A, B the *lower* (*upper*) *frame bounds* respectively. If $A = B$, it is an *A-tight frame* . If $A = B = 1$ it is a *Parseval frame* . If $\|\phi_m\| = \|\phi_k\|$ for all $1 \leq m, k \leq M$, it is an *equal norm frame* and if $\|\phi_m\| = 1$ for all $1 \leq m \leq M$ it is a *unit norm frame*. For finite dimensional Hilbert spaces a frame is just a spanning set.

If $\{e_n\}_{n=1}^M$ is the canonical basis in l_2^M, the *analysis operator* for the frame is the operator $T : H_N \to l_2^M$ given by

$$Tf = \sum_{m=1}^{M} \langle f, \phi_n\rangle e_n$$

Then $T^* : l_2^M \to H_N$ is given by

$$T(\{a_n\}_{n=1}^M) = \sum_{n=1}^{M} a_n \phi_n$$

and is called the *synthesis operator*. The frame operator $S = T^*T : H_N \to H_N$ is the *frame operator*.

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The frame operator is a positive, self-adjoint invertible operator on H_N satisfying $AI \leq S \leq BI$. So $\{\phi_n\}_{n=1}^M$ is an A-tight frame if and only if $S = AI$. *Reconstruction* is achieved by:

$$f = \sum_{n=1}^{M} \langle f, S^{-1}\phi_n\rangle \phi_n = \sum_{n=1}^{M} \langle f, S^{-\frac{1}{2}}\phi_n\rangle S^{-\frac{1}{2}}\phi_n.$$

For applications we prefer A-tight frames since in this case $S^{-1} = \frac{1}{A}I$.

The point of this paper is to show that we can improve the properties of a frame by projecting it onto a carefully chosen subspace. For a comprehensive treatment of the eigenvector representation of frames we refer the reader to [**?**].

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2. EIGENVECTORS FOR A FRAME OPERATOR

First we identify the eigenvectors for a frame operator.

Proposition 2.1. *Let $\{\phi_m\}_{m=1}^M$ be a frame for H_N with frame operator S. Let $e_1 \in H_N$, $\|e_1\| = 1$ and let P be the orthogonal projection of H_N onto span e_1. The following are equivalent:*

(1) *e_1 is an eigenvector for S with eigenvalue λ_1.*
(2) *For every $f_1 \in (I - P)H$,*

$$\text{(a)} \quad \sum_{n=1}^{M} \langle \phi_n, e_1\rangle \overline{\langle \phi_n, f_1\rangle} = 0$$

$$\text{(b)} \quad \sum_{n=1}^{M} |\langle \phi_n, e_1\rangle|^2 = \lambda_1$$

Proof. (1) ⇒ (2) Let $\{e_i\}_{i=1}^N$ be an orthonormal basis for H_N consisting of eigenvectors for S. By our assumption (1),

$$\lambda_1 e_1 = Se_1 = \sum_{n=1}^{M} \langle e_1, \phi_n\rangle \phi_n = \sum_{n=1}^{M} \langle e_1, \sum_{i=1}^{N} \langle \phi_n, e_i\rangle e_i\rangle \sum_{j=1}^{N} \langle \phi_n, e_j\rangle e_j =$$

$$\sum_{n=1}^{M} \overline{\langle \phi_n, e_1\rangle} \langle e_1, e_1\rangle \langle \phi_n, e_1\rangle e_1 + \sum_{n=1}^{M} \sum_{j\neq 1} \overline{\langle \phi_n, e_1\rangle} \langle \phi_n, e_j\rangle e_j =$$

$$\left[\sum_{n=1}^{M} |\langle \phi_n, e_1\rangle|^2\right] e_1 + \sum_{n=1}^{M} \sum_{j\neq 1} \overline{\langle \phi_n, e_1\rangle} \langle \phi_n, e_j\rangle e_j =$$

$$\sum_{n=1}^{M} |\langle \phi_n, e_1\rangle|^2 e_1 + \sum_{j\neq 1} \left[\sum_{n=1}^{M} \overline{\langle \phi_n, e_1\rangle} \langle \phi_n, e_j\rangle\right] e_j.$$

Hence, $\sum\limits_{n=1}^{M} |\langle \phi_n, e_1\rangle|^2 = \lambda_1$ and

$$\sum_{n=1}^{M} \langle \phi_n, e_1\rangle \overline{\langle \phi_n, e_j\rangle} = \overline{\sum_{n=1}^{M} \overline{\langle \phi_n, e_1\rangle} \langle \phi_n, e_j\rangle} = \overline{0} = 0$$

for all $2 \leq j \leq N$.

If $f_1 = \sum_{j=2}^{N} a_j e_j$ then

$$\sum_{n=1}^{M} \langle \phi_n, e_1 \rangle \overline{\langle \phi_n, f_1 \rangle} = \sum_{j=2}^{N} a_j \sum_{n=1}^{M} \langle \phi_n, e_1 \rangle \overline{\langle \phi_n, e_j \rangle} = 0.$$

(2) ⇒ (1) Given (2), choose $\{e_i\}_{i=2}^{N}$ so that $\{e_i\}_{i=1}^{N}$ is an orthonormal basis for H_N. Using (2) with $f_1 = e_i$ repeatedly for $i = 2, 3, \ldots, N$ yields

$$Se_1 = \left[\sum_{n=1}^{M} |\langle \phi_n, e_1 \rangle|^2 \right] e_1 + \sum_{j=2}^{N} \left[\langle \phi_n, e_1 \rangle \overline{\langle \phi_n, e_j \rangle} \right] e_j = \lambda_1 e_1 + 0$$

where the first equality takes place as in (1) ⇒ (2) and the second by (2). □

We now need a proposition so that we can generalize these results.

Proposition 2.2. *Let* $\{\phi_n\}_{n=1}^{M}$ *be a frame for* H_N *with frame operator* S*. Let* $\{e_i\}_{i=1}^{N}$ *be an orthonormal basis for* H_N *of eigenvectors for* S *with eigenvalues* $\{\lambda_i\}_{i=1}^{N}$ *respectively. For any* $\phi \in H_N$ *we have*

$$\sum_{n=1}^{M} |\langle \phi, \phi_n \rangle|^2 = \sum_{i=1}^{N} \lambda_i \, |\langle \phi, e_i \rangle|^2$$

Proof.

$$\sum_{n=1}^{M} |\langle \phi, \phi_n \rangle|^2 = \sum_{n=1}^{M} \langle \langle \phi, \phi_n \rangle \phi_n, \phi \rangle = \langle \sum_{n=1}^{M} \langle \phi, \phi_n \rangle \phi_n, \phi \rangle = \langle S\phi, \phi \rangle =$$

$$\langle \sum_{i=1}^{N} \lambda_i \langle \phi, e_i \rangle e_i, \sum_{i=1}^{N} \langle \phi, e_i \rangle e_i \rangle = \sum_{i,j=1}^{N} \lambda_i \langle \phi, e_i \rangle \overline{\langle \phi, e_j \rangle} \langle e_i, e_j \rangle = \sum_{i=1}^{N} \lambda_i \, |\langle \phi, e_i \rangle|^2 .$$

□

Next we give a simple classification of the eigenvalues and eigenvectors for the frame operator of $\{P\phi_m\}_{m=1}^{M}$ where $\{\phi_m\}_{m=1}^{M}$ is a frame for H_N and P is an orthogonal projection on H_N.

Theorem 2.3. *Let* $\{\phi_m\}_{m=1}^{M}$ *be a frame for* H_N *with frame operator* S*. Let* $\{e_n\}_{n=1}^{N}$ *be an orthonormal basis for* H_N *consisting of eigenvectors for* S *with respective eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$*. Fix* $1 \leq k \leq N$ *and let* P *be a rank* k *orthogonal projection of* H_N *onto a subspace of* H*. Let* S_1 *be the frame operator for* $\{P\phi_m\}_{m=1}^{M}$ *and choose norm one vectors* $f_i \in H$*,* $1 \leq i \leq k$*. The following are equivalent:*

(1) $\{f_i\}_{i=1}^{k}$ *is an orthogonal family of eigenvectors for* S_1 *with eigenvalues* $\mu_1, \mu_2, \ldots, \mu_k$*.*

(2) *We have*

$$(a)\quad \sum_{n=1}^{N}\langle f_i, e_n\rangle\overline{\langle f_j, e_n\rangle} = 0 \ \ \textit{for all} \ \ i \neq j.$$

$$(b)\quad \sum_{n=1}^{N}\lambda_n\langle f_i, e_n\rangle\overline{\langle f_j, e_n\rangle} = 0 \ \ \textit{for all} \ \ i \neq j$$

$$(c)\quad \sum_{n=1}^{N}\lambda_n\left|\overline{\langle f_i, e_n\rangle}\right|^2 = \mu_i \ \ \textit{for all} \ \ 1 \leq i \leq k.$$

Proof. (1) $\Rightarrow$ (2): Since $f_i \perp f_j$ for all $i \neq j$ we have

$$\langle f_i, f_j\rangle = \sum_{n=1}^{N}\langle f_i, e_n\rangle\overline{\langle f_i, e_n\rangle} = 0.$$

Next, by our assumption (1), since $\{f_i\}_{i=1}^k$ is an orthonormal basis of eigenvectors for S_1, by Theorem 2.3 we have for $i \neq j$,

$$0 = \sum_{n=1}^{M}\langle f_i, P\phi_n\rangle\overline{\langle f_j, P\phi_n\rangle} = \sum_{n=1}^{M}\langle f_i, \phi_n\rangle\overline{\langle f_j, \phi_n\rangle} = \sum_{n=1}^{M}\langle f_i, \sum_{l=1}^{N}\langle\phi_n, e_l\rangle e_l\rangle\overline{\langle f_j, \sum_{l=1}^{N}\langle\phi_n, e_l\rangle e_l\rangle} =$$
$$\sum_{n=1}^{M}\sum_{l=1}^{N}\sum_{m=1}^{N}\overline{\langle\phi_n, e_l\rangle}\langle\phi_n, e_m\rangle\langle f_i, e_l\rangle\overline{\langle f_j, e_m\rangle} = \sum_{l=1}^{N}\sum_{m=1}^{N}\langle f_i, e_l\rangle\overline{\langle f_j, e_m\rangle}\sum_{n=1}^{M}\overline{\langle\phi_n, e_l\rangle}\langle\phi_n, e_m\rangle =$$
$$\sum_{l=1}^{M}\langle f_i, e_l\rangle\overline{\langle f_j, e_l\rangle}\sum_{n=1}^{M}\left|\overline{\langle\phi_n, e_l\rangle}\right|^2 + \sum_{m=1, m\neq l}^{N}\langle f_i, e_l\rangle\overline{\langle f_j, e_m\rangle}\sum_{n=1}^{M}\overline{\langle\phi_n, e_l\rangle}\langle\phi_n, e_m\rangle \overset{*}{=}$$
$$\sum_{l=1}^{M}\langle f_i, e_l\rangle\overline{\langle f_j, e_l\rangle}\lambda_l$$

where equality (*) holds because $\sum_{n=1}^{M}\overline{\langle\phi_n, e_l\rangle}\langle\phi_n, e_m\rangle = 0$ by Proposition 2.1 .

(b) By Proposition 2.1 for $1 \leq i \leq k$,

$$\mu_i = \sum_{n=1}^{M}|\langle f_i, P\phi_n\rangle|^2 = \sum_{n=1}^{M}|\langle f_i, \phi_n\rangle|^2 = \sum_{l=1}^{M}\lambda_l\,|\langle f_i, e_l\rangle|^2$$

(2) $\Rightarrow$ (1): By the above argument on (2) (a), for $i \neq j$,

$$\sum_{n=1}^{M}\langle f_i, P\phi_n\rangle\overline{\langle f_j, P\phi_n\rangle} = \sum_{l=1}^{M}\langle f_i, e_l\rangle\overline{\langle f_j, e_l\rangle}\lambda_l = 0.$$

This plus (2) (b) implies by Proposition 2.1 that $\{f_i\}_{i=1}^k$ is an orthonormal basis of eigenvectors with eigenvalues $\{\mu_i\}_{i=1}^k$ □

3. Applications

Now we examine some consequences of Theorem 2.3 .

Theorem 3.1. *Let $\{\phi_m\}_{m=1}^M$ be a frame for H_N with frame operator S. Let $\{e_i\}_{i=1}^N$ be an orthonormal basis for H_N consisting of eigenvectors for S with eigenvalues $\{\lambda_i\}_{i=1}^N$ respectively. Let $\{\sigma_j\}_{j=1}^k$ be a partition of $\{1,2,\dots,N\}$ and for every $1 \le k$ let $f_j = \sum_{i\in\sigma_j}^M a_i e_i$. Assume $\|f_j\| = 1 = \sum_{i\in\sigma_j}^M |a_i|^2$. Let P be the orthogonal projection of H_N onto $H = \underset{1\le j\le k}{span} f_j$ and let S_0 be the frame operator for $\{P\phi_n\}_{n=1}^M$. Then $\{f_j\}_{j=1}^k$ is an orthonormal basis for H consisting of eigenvectors for S_0 with eigenvalues, respectively, $\mu_j = \sum_{i\in\sigma_j} \lambda_i a_i^2$.*

Proof. We will verify the conditions of Theorem 2.3 (2) for each f_i . By symmetry, we can do it for f_1. For (2) (b),

$$\sum_{n=1}^M |\langle P\phi_n, e_i\rangle|^2 = \sum_{n=1}^M |\langle \phi_n, e_i\rangle|^2 = \sum_{n=1}^M \left|\sum_{i\in\sigma_1}\langle \phi_n, e_i\rangle a_i\right|^2 =$$

$$\sum_{n=1}^M (\sum_{i\in\sigma_1}\langle \phi_n, e_i\rangle a_i)\sum_{m\in\sigma_1}\overline{\langle \phi_n, e_m\rangle a_m} = \sum_{i\in\sigma_1}|a_i|^2\sum_{n=1}^M |\langle \phi_n, e_i\rangle|^2 + \sum_{m\ne i} a_i\overline{a_m}\sum_{n=1}^M \langle \phi_n, e_i\rangle\overline{\langle \phi_n, e_m\rangle} =$$

$$\sum_{i\in\sigma_1}|a_i|^2\lambda_i + \sum_{m\ne i} a_i\overline{a_m}\cdot 0 = \sum_{i\in\sigma_1}|a_i|^2\lambda_i$$

where $\sum_{n=1}^M \langle \phi_n, e_i\rangle\overline{\langle \phi_n, e_m\rangle} = 0$ since $\{e_i\}_{i=1}^N$ is a basis of eigenvectors for S.

For (a), it suffices to verify it for $e_1 = f_1$ and f_i, $1 \le i \le k$, and so

$$\sum_{n=1}^M \langle \phi_n, f_1\rangle\overline{\langle \phi_n, f_i\rangle} = \sum_{n=1}^M (\sum_{l\in\sigma_1}\langle \phi_n, e_l\rangle\overline{a_l})(\sum_{m\in\sigma_i}\overline{\langle \phi_n, e_m\rangle} a_m) =$$

$$\sum_{l\in\sigma_1}\sum_{m\in\sigma_i}\overline{a_l}a_m\sum_{n=1}^M \langle \phi_n, e_l\rangle\overline{\langle \phi_n, e_m\rangle} = 0,$$

where the last equality holds since $l \ne m$ and $\{e_i\}$ is a basis of eigenvectors for S. □

Now we will examine some consequences of Theorem 2.3 . The first is a partial converse to the Poincaré Separation Theorem (see Theorem 3.6 below). Despite an exhaustive research, we could not find a full or partial inverse to this theorem in the literature.

Corollary 3.2. *Let $\{\phi_n\}_{n=1}^M$ be a frame for H_{2N} with frame operator S. Let $\{e_i\}_{i=1}^{2N}$ be an orthonormal basis for H_{2N} of eigenvectors for S with eigenvalues $\lambda_1 \ge \lambda_2, \cdots \ge \lambda_{2N}$, respectively. For $\lambda_{2N-i+1} \le \mu_i \le \lambda_i$ there exists a subspace $H \subseteq H_{2N}$ with dim $H = N$ so that if P is the orthogonal projection of H_{2N} onto H then $\{P\phi_n\}_{n=1}^M$ is a frame for H with frame operator S_1 having eigenvalues $\mu_1, \dots, \mu_n$.*

Proof. For $1 \leq i \leq N$ choose $0 < \epsilon_i < 1$ so that $\lambda_i\epsilon_i^2 + \lambda_{2N-i+1}(1-\epsilon_i^2) = \mu_i$ and let $f_i = \epsilon_i e_i + \sqrt{1-\epsilon_i^2}e_{2N-i+1}$. Clearly, $\{f_i\}_{i=1}^N$ is an orthonormal sequence in H_{2N} which by Theorem 2.3 consists of eigenvectors for $\{P\phi_n\}$ where P is the orthogonal projection of H_{2N} onto $H = \underset{1\leq i\leq N}{\text{span}} f_i$. Also by Theorem 2.3 , the eigenvalues for these eigenvectors are given by

$$\lambda_i\epsilon_i^2 + \lambda_{2N-i+1}(1-\epsilon_i^2) = \mu_i$$

This completes the proof. □

Remark 3.3. *The corollary holds for* H_{2N+1}*,* $\mu_{N+1} = \lambda_{N+1}$ *and* $\lambda_{2N+1-i+1} \leq \mu_i \leq \lambda_i$*, taking* $f_{N+1} = e_{N+1}$*, choosing* $0 < \epsilon_i < 1$ *so that* $\lambda_i\epsilon_i^2 + \lambda_{2N+1-i+1}(1-\epsilon_i^2) = \mu_i$ *and* $f_i = \epsilon_i e_i + \sqrt{1-\epsilon_i^2}e_{2N+1-i+1}$ *for* $1 \leq i \leq N$*.*

Now we will prove the result announced in the abstract. Later, using the Poincaré Separation Theorem we will see that this result is best possible.

Proposition 3.4. *If* $\{\phi_n\}_{n=1}^M$ *is a frame for* H_{2N} *(respectively for* H_{2N+1} *) then there is a rank* N *(resp. rank* $N+1$ *) orthogonal projection* P *on* H_{2N} *(resp. on* H_{2N+1} *) so that* $\{P\phi_n\}_{n=1}^M$ *is a tight frame for* PH_{2N} *(resp.* PH_{2N+1} *).*

Proof. For the H_{2N} case let $\mu_1 = \mu_2 = \cdots = \mu_N = \lambda$ where $\lambda_{N+1} \geq \lambda \geq \lambda_N$ in Corollary 3.2 . Then there is a subspace $H \subset N_{2N}$ with $\dim H = N$ so that $\{P_H\phi_n\}_{n=1}^M$ is a frame for H whose frame operator has eigenvalues $\{\mu_i\}_{i=1}^M$, i.e., $\{P_H\phi_n\}_{n=1}^M$ is a λ-tight frame.

For the H_{2N+1} case, let $\mu_1 = \mu_2 = \cdots = \mu_{N+1} = \lambda_{N+1}$ and proceed as in Remark 3.3 . □

There is an alternative method for capturing Proposition 3.4 using a well known result concerning quadratic forms.

Proposition 3.5. *If* G *is a quadratic form on* H_{2N} *(resp.* H_{2N+1} *) there is a subspace* K *of dimension* N *(resp.* $N+1$ *) so that* $G(f) = A\|f\|^2$ *for some* A *and all* $f \in K$*.*

Proof. Let S be a symmetric matrix (self-adjoint in the complex case) so that $G(f) = \langle f, Sf\rangle$ for all $f \in H_{2N}$. Let S have eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2N}$. Choose $\lambda_N \geq A \geq \lambda_{N+1}$. Then

$$G_1(f) = \langle f, (S-AI)f\rangle$$

has eigenvalues $\mu_i = \lambda_i - A$. Thus $\mu_i \geq 0 \geq \mu_j$ for all $1 \leq i \leq N$ and all $N+1 \leq j \leq 2N$. If $\{e_i\}_{i=1}^{2N}$ are eigenvectors for the μ_i, choose a_i, b_i so that $a_i\mu_i + b_i\mu_{N+i} = 0$. Let $K = \text{span}\{a_ie_i + b_ie_{N+i}\}$. Then $G_1(f) = 0$ for all $f \in K$. Hence, $\dim K = N$ and $G(f) = A\langle f, f\rangle$ for all $f \in K$.

For the H_{2N+1} case, let $A = \lambda_{N+1}$, choose a_i, b_i so that $a_i\mu_i + b_i\mu_{N+1+i} = 0$, $f_i = a_ie_i + b_ie_{N+1+i}$ for $1 \leq i \leq N$, $f_{N+1} = e_{N+1}$ and $K = \text{span}\{f_i\}_{i=1}^{i=N+1}$.

□

If $\{\phi_m\}_{m=1}^M$ is a frame for H_{2N} then

$$G(f) = \sum_{n=1}^M |\langle f, \phi_m\rangle|^2$$

is a quadratic form which, by Proposition 3.5 , can be restricted to a subspace K of dimension N where it is constant. It follows that $\{P\phi_m\}_{m=1}^M$ is a tight frame where P is the orthogonal projection onto K.

Now we will check that these results are best possible. If $\{\phi_n\}_{n=1}^M$ is a frame for H_N with frame operator S and P is an orthogonal projection on H_N, then $PS = PSP$ on PH_N is the frame operator for $\{P\phi_n\}_{n=1}^M$. To see this we check for all $f \in PH$:

$$\sum_{n=1}^{M} \langle f, P\phi_n\rangle P\phi_n = P\sum_{n=1}^{M}\langle Pf, \phi_n\rangle \phi_n = \langle Pf, \phi_n\rangle \phi_n = PSf.$$

As a consequence of the Poincaré Separation Theorem ([?], p. 190, Corollary 4.3.16) we have:

Theorem 3.6. *Let $\{\phi_m\}_{m=1}^M$ be a frame for H_N with frame operator S and P a rank $k \leq N$ orthogonal projection on H_N. Let $\{\lambda_n\}_{n=1}^N$ be the eigenvalues for S and let $\{\mu_i\}_{i=1}^k$ be the eigenvalues for the frame operator PSP of $\{P\phi_n\}_{n=1}^M$. Then for all $1 \leq i \leq k$ we have*

$$\lambda_i \geq \mu_i \geq \lambda_{N-k+i}.$$

Now we observe that in general we may not be able to project a frame onto a subspace of dimension larger than half the dimension of the space and get a tight frame.

Example 3.7. *: For any N and any M there is a frame $\{\phi_n\}_{n=1}^M$ for H_{2N} so that for any rank $N+1$ orthogonal projection P on H_{2N}, $\{P\phi_n\}_{n=1}^M$ is not a tight frame.*

Proof. Choose a positive self-adjoint invertible operator on H_{2N} with eigenvalues $\lambda_1 = \lambda_2 = \cdots = \lambda_N = 1$ and $\lambda_{N+1} = \lambda_{N+2} = \cdots = \lambda_{2N} = 2$. It is shown in [?], there is a frame $\{\phi_n\}_{n=1}^M$ for H_{2N} whose frame operator is S. By Theorem 3.6 , if P is an orthogonal projection of rank $N+1$ on H_{2N} then PSP has 1 and 2 amongst its eigenvalues. So $\{P\phi_n\}_{n=1}^M$ is not a tight frame. ☐

Recall that operators S, T for N-dimensional Hilbert spaces H_N and K_N are *similar* if there exists a unitary operator $U : H_N \longrightarrow K_N$ so that $S = U^{-1}TU$. We now have:

Proposition 3.8. *If $\{\phi_n\}_{n=1}^M$ is a frame for H_{2N} then there exists a projection P of rank N on H_{2N} so that the frame operator for $\{P\phi_n\}_{n=1}^M$ and $\{(I-P)\phi_n\}_{n=1}^M$ are similar.*

Proof. Given eigenvectors $\{e_n\}_{n=1}^{2N}$ with eigenvalues $\{\lambda_n\}_{n=1}^{2N}$ for the frame operator S of $\{\phi_n\}_{n=1}^M$, let

$$\psi_n = \frac{e_{2n-1} + e_{2n}}{\sqrt{2}}$$

for $1 \leq n \leq N$. If P is the orthogonal projection of H_{2N} onto $H = \text{span } \{\psi_n\}_{n=1}^N$, then by Theorem 2.3 we have that the frame operator T of $\{\phi_n\}_{n=1}^M$ has eigenvectors $\{\psi_n\}_{n=1}^N$ with respective eigenvalues

$$\{\frac{\lambda_{2n-1} + \lambda_{2n}}{\sqrt{2}}\}_{n=1}^N.$$

If S_1 is the frame operator for $\{(I-P)\phi_n\}_{n=1}^M$ then S_1 has eigenvectors

$$\{\frac{e_{2n-1} - e_{2n}}{\sqrt{2}}\}_{n=1}^N$$

with respective eigenvalues

$$\{\frac{\lambda_{2n-1} + \lambda_{2n}}{\sqrt{2}}\}_{n=1}^N.$$

If we define

$$U(\frac{e_{2n-1}+e_{2n}}{\sqrt{2}}) = \frac{e_{2n-1}-e_{2n}}{\sqrt{2}}$$

then $S_1 = U^{-1}TU$. $\square$

We can now assert the existence of non-tight frames which are orthogonal sums of tight frames.

Example 3.9. *For every $M \geq N$ there are tight frames $\{\phi_n\}_{n=1}^M$ and $\{\psi_n\}_{n=1}^M$ for H_N and K_N so that the frame operator S for*

$$f_n = \phi_n \oplus \psi_n, \quad 1 \leq n \leq M$$

in $H_N \bigoplus K_N$ has $2N$ distinct non-zero eigenvalues. In particular, $\{f_n\}_{n=1}^M$ is a non-tight frame for $H_N \bigoplus K_N$.

Proof. Let $c_n = \frac{1}{N}$ for $1 \leq n \leq N$. Let $\{f_n\}_{n=1}^M$ be a frame for H_{2N} whose frame operator has eigenvalues $\{1+c_n, 1-c_n\}_{n=1}^N$ and eigenvectors $\{e_n\}_{n=1}^{2N}$. Let

$$\phi_n = \frac{e_n + e_{N-n+1}}{\sqrt{2}}, \quad 1 \leq n \leq N.$$

If P is the orthogonal projection of H_{2N} onto H_N =span $\{\phi_n\}_{n=1}^N$ then by Proposition 3.8 $\{Pf_n\}_{n=1}^M$ and $\{(I-P)f_n\}_{n=1}^M$ have the same eigenvalues for their frame operators:

$$\frac{\lambda_n + \lambda_{N-n+1}}{2} = \frac{1+c_n+1-c_n}{2} = 1$$

so,

$$f_n = Pf_n \oplus (I-P)f_n$$

and $\{Pf_n\}_{n=1}^M$, $\{(I-P)f_n\}_{n=1}^M$ are both Parseval frames. $\square$

At first, Example 3.9 looks like a contradiction. Isn't an orthogonal sum of tight frames a tight frame? Actually, an orthogonal sum $\{\phi_m \oplus \psi_m\}_{m=1}^M$ of tight frames is a tight frame if and only if

$$\sum_{n=1}^{M} \langle f, \phi_n \rangle \psi_n = 0 = \sum_{n=1}^{M} \langle f, \psi_n \rangle \phi_n$$

for all $f \in H \bigoplus K$.

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