

ANALYZING THE WEYL-HEISENBERG FRAME IDENTITY

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ABSTRACT. In 1990, Daubechies proved a fundamental identity for Weyl-Heisenberg systems which is now called the Weyl-Heisenberg Frame Identity. WH-Frame Identity: If $g \in W(L^\infty, L^1)$, then for all continuous, compactly supported functions f we have:

$$\sum_{m,n} |\langle f, E_{mb}T_{na}g \rangle|^2 = \frac{1}{b} \sum_k \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) \sum_n g(t - na) \overline{g(t - na - k/b)} dt.$$

It has been folklore that the identity will not hold universally. We make a detailed study of the WH-Frame Identity and show: (1) The identity does not require any assumptions on ab (such as the requirement that $ab \leq 1$ to have a frame); (2) As stated above, the identity holds for all $f \in L^2(\mathbb{R})$; (3) The identity holds for all bounded, compactly supported functions if and only if $g \in L^2(\mathbb{R})$; (4) The identity holds for all compactly supported functions if and only if $\sum_n |g(x - na)|^2 \leq B$ a.e.; Moreover, in (2)-(4) above, the series on the right converges unconditionally. We will also see that in general, symmetric, norm and unconditional convergence of the series in the WH-frame identity are all different.

1. INTRODUCTION

The WH-frame identity has been extensively used in Weyl-Heisenberg frame theory and has gone through some small improvements over time. It has been part of the folklore that the identity does not hold universally. But, until now, it has been a little mysterious as to exactly where and when one can be sure the identity holds. In this paper we give a detailed analysis of the WH-frame Identity.

In 1952, Duffin and Schaeffer [5] defined:

Definition 1.1. *A sequence $(f_n)_{n \in \mathbb{Z}}$ of elements of a Hilbert space H is called a **frame** if there are constants $A, B > 0$ such that*

$$(1.1) \quad A\|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in H.$$

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The numbers A, B are called the **lower and upper frame bounds** respectively.

We will work here with a particular class of frames called Weyl-Heisenberg frames. For a function f on \mathbb{R} we define the operators:

$$\begin{aligned} \text{Translation: } T_a f(x) &= f(x - a), \quad a \in \mathbb{R} \\ \text{Modulation: } E_a f(x) &= e^{2\pi i a x} f(x), \quad a \in \mathbb{R} \end{aligned}$$

We also use the symbol E_a to denote the **exponential function** $E_a(x) = e^{2\pi i a x}$. Each of the operators T_a, E_a are unitary operators on $L^2(\mathbb{R})$.

In 1946 Gabor [6] formulated a fundamental approach to signal decomposition in terms of elementary signals. This method resulted in **Gabor frames** or as they are often called today **Weyl-Heisenberg frames**.

Definition 1.2. *If $a, b \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$ we call $(E_{mb}T_{na}g)_{m,n \in \mathbb{Z}}$ a **Weyl-Heisenberg system (WH-system for short)** and denote it by (g, a, b) . We call g the **window function**.*

For any $a, b \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$ for all $k \in \mathbb{Z}$ we let,

$$G_k(t) = \sum_{n \in \mathbb{Z}} g(t - na) \overline{g(t - na - k/b)}.$$

We will be analyzing the WH-frame identity. The form we give here is due to Walnut [8] (see also Heil and Walnut [7]).

Theorem 1.3. (WH-Frame Identity.) *If $g \in W(L^\infty, L^1)$ and $f \in L^2(\mathbb{R})$ is continuous and compactly supported, then*

$$\begin{aligned} \sum_{n,m \in \mathbb{Z}} |\langle f, E_{mb}T_{na}g \rangle|^2 &= \frac{1}{b} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt \\ &= b^{-1} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt \end{aligned}$$

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2. BOUNDED, COMPACTLY SUPPORTED FUNCTIONS AND THE WH-FRAME IDENTITY

We start with a simple observation.

Proposition 2.1. *If $0 < a, b \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$ is bounded and compactly supported, then (g, a, b) has a finite upper frame bound.*

Proof. First, assume that g is supported on $[0, a]$. Since g is bounded above and compactly supported, there is a constant B so that

$$(2.1) \quad \sum_{k \in \mathbb{Z}} |g(t - k/b)|^2 \leq B, \quad \text{a.e. } t.$$

We define the preframe operator $L : \ell_2 \otimes \ell_2 \rightarrow L^2(\mathbb{R})$ by

$$L\left(\sum_{m,n \in \mathbb{Z}} a_{mn} e_{mn}\right) = \sum_{m,n \in \mathbb{Z}} a_{mn} E_{mb} T_{na} g,$$

where (e_{mn}) is the natural orthonormal basis of $\ell_2 \otimes \ell_2$. We need to show that L is a bounded operator. By our assumption on the support of g , we see that $(T_{na}g)_{n \in \mathbb{Z}}$ are disjointly supported functions. Hence,

$$(2.2) \quad \left\| L\left(\sum_{m,n \in \mathbb{Z}} a_{mn} e_{mn}\right) \right\|^2 = \sum_{n \in \mathbb{Z}} \left\| \sum_{m \in \mathbb{Z}} a_{mn} E_{mb} T_{na} g \right\|^2.$$

Applying inequality 2.1 above at the appropriate step, we have

$$\begin{aligned} \left\| \sum_{m \in \mathbb{Z}} a_{mn} E_{mb} T_{na} g \right\|^2 &= \int_{\mathbb{R}} \left| \sum_{m \in \mathbb{Z}} a_{mn} E_{mb} T_{na} g(t) \right|^2 dt \\ &= \int_0^{1/b} \left| \sum_{m \in \mathbb{Z}} a_{mn} E_{mb} \right|^2 \sum_{k \in \mathbb{Z}} |g(t - k/b - na)|^2 dt \\ &\leq B \int_0^{1/b} \left| \sum_{m \in \mathbb{Z}} a_{mn} E_{mb} \right|^2 dt \\ &= B \sum_{m \in \mathbb{Z}} |a_{mn}|^2. \end{aligned}$$

It follows from equation (2.2) that

$$\left\| \sum_{m \in \mathbb{Z}} a_{mn} E_{mb} T_{na} g \right\|^2 \leq \sum_{mn \in \mathbb{Z}} |a_{mn}|^2.$$

Hence, L is a bounded operator.

For the general case, we observe that g can be written as a finite sum, say k , of translates of functions supported on $[0, a]$ and so the preframe operator is also bounded in this case by $k\|L\|$. \square

Corollary 2.2. *If $g \in L^2(\mathbb{R})$, then for every bounded, compactly supported function f on \mathbb{R} , we have*

$$\sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 < \infty.$$

Proof. By Proposition 2.1, if f is bounded and compactly supported then $(E_{mb} T_{na} f)_{m,n \in \mathbb{Z}}$ has a finite upper frame bound, say B . Now

$$\begin{aligned} \sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 &= \sum_{m,n \in \mathbb{Z}} |\langle T_{-na} E_{-mb} f, g \rangle|^2 = \\ \sum_{m,n \in \mathbb{Z}} |e^{-2\pi i m b(x-na)} \langle E_{mb} T_{na} f, g \rangle|^2 &= \sum_{m,n \in \mathbb{Z}} |\langle E_{mb} T_{na} f, g \rangle|^2 \leq B. \end{aligned}$$

□

We now present the main result of this section.

Theorem 2.3. *Let g be a measurable function on \mathbb{R} . The following are equivalent:*

- (1) $g \in L^2(\mathbb{R})$.
- (2) *The WH-frame Identity holds for all bounded, compactly supported functions f on \mathbb{R} and the series converges unconditionally.*

Proof. (1) \Rightarrow (2): We assume that f is supported on $[-N, N]$ and bounded above by B . For a fixed $n \in \mathbb{Z}$ we consider the $1/b$ -periodic function

$$H_n(t) = \sum_{k \in \mathbb{Z}} f(t - k/b) \overline{g(t - na - k/b)}.$$

Now, the above sum only has $2N$ non-zero terms for each $t \in \mathbb{R}$. This justifies interchanging the (now finite) sums and integrals below.

$$\begin{aligned} \sum_{m, n \in \mathbb{Z}} | \langle f, E_{mb} T_{na} g \rangle |^2 &= \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \left| \int_{\mathbb{R}} f(t) \overline{g(t - na)} e^{-2\pi i m b t} dt \right|^2 \\ &= \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \left| \int_0^{1/b} \sum_{k \in \mathbb{Z}} f(t - k/b) \overline{g(t - na - k/b)} dt \right|^2 \\ &= b^{-1} \sum_n \int_0^{1/b} \sum_{\ell} \overline{f(t - \ell/b) g(t - na - \ell/b)} \cdot \sum_k f(t - k/b) \overline{g(t - na - k/b)} dt \\ &= b^{-1} \sum_n \sum_{\ell} \int_0^{1/b} \overline{f(t - \ell/b) g(t - na - \ell/b)} \cdot \sum_k f(t - k/b) \overline{g(t - na - k/b)} dt \\ &= \frac{1}{b} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \overline{f(t) g(t - na)} \cdot \sum_{k \in \mathbb{Z}} f(t - k/b) \overline{g(t - na - k/b)} \\ &= \frac{1}{b} \sum_{n \in \mathbb{Z}} \int_{-N}^N \overline{f(t) g(t - na)} \cdot \sum_{k \in \mathbb{Z}} f(t - k/b) \overline{g(t - na - k/b)} \end{aligned}$$

To finish the identity, we just need to justify interchanging the infinite sum over n with the finite sum over k . To justify this we observe that:

$$\sum_{n, k} |f(t) g(t - na)| |f(t - k/b)| |f(t - na - k/b)| \leq$$

$$B^2 \sum_{k=-N}^N \sum_{n \in \mathbb{Z}} |g(t - na) g(t - na - k/b)|.$$

By Hölder's inequality, we have that

$$\sum_{n \in \mathbb{Z}} |g(t - na)g(t - na - k/b)| \in L^1[0, a],$$

and hence

$$\sum_{k=-N}^N \sum_{n \in \mathbb{Z}} |g(t - na)g(t - na - k/b)| \in L^1[-N, N].$$

Therefore, we justify the needed interchange of sums and sums with integrals by the Lebesgue Dominated Convergence Theorem.

(2) \Rightarrow (1): We do this by contradiction. If g is not square integrable on \mathbb{R} , then

$$\|g\|^2 = \int_0^a \sum_{n \in \mathbb{Z}} |g(t - na)|^2 dt = \infty.$$

Hence, there is some interval I of length c with $0 < c < 1/b$ so that

$$\int_I \sum_{n \in \mathbb{Z}} |g(t - na)|^2 dt = \infty.$$

If we let $f = \chi_I$, then the right hand side of the WH-frame Identity becomes

$$\int_{\mathbb{R}} |f(t)|^2 \sum_{n \in \mathbb{Z}} |g(t - na)|^2 dt = \int_I \sum_{n \in \mathbb{Z}} |g(t - na)|^2 dt = \infty.$$

So the right hand side of the WH-frame identity is not a finite unconditionally convergent series. i.e. The WH-frame Identity fails. \square

3. COMPACTLY SUPPORTED FUNCTIONS AND THE WH-FRAME IDENTITY

In this section we will drop the hypotheses that our function f has to be bounded and discover necessary and sufficient conditions for the WH-frame Identity to hold. The conditions are a little stronger than those required for bounded, compactly supported functions.

Theorem 3.1. *Let g be a measurable function on \mathbb{R} . The following are equivalent:*

(1) *There is a constant $B > 0$ so that*

$$\sum_{n \in \mathbb{Z}} |g(t - na)|^2 \leq B, \quad \text{a.e. } t$$

(2) *The WH-frame Identity holds for all compactly supported functions f on \mathbb{R} and the series converges unconditionally.*

Proof. (1) \Rightarrow (2): If f is compactly supported, we see immediately that the sum over k in the right hand side of the WH-frame identity is a finite sum. So let $f_\ell(t) = f(t)$ if $|f(t)| \leq \ell$ and zero otherwise. Now, by Theorem 2.3 the WH-frame identity holds for all f_ℓ . That is, for all $\ell \in \mathbb{Z}$ we have

$$\begin{aligned} & \sum_{m,n \in \mathbb{Z}} | \langle f_\ell, E_{mb} T_{na} g \rangle |^2 = \\ & \frac{1}{b} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \overline{f_\ell(t)} f_\ell(t - k/b) G_k(t) dt \end{aligned}$$

Now we will finish the proof in three steps.

Step 1: We show that

$$\begin{aligned} & \sum_{m,n} | \langle f, E_{mb} T_{na} g \rangle |^2 = \\ & = b^{-1} \sum_n \int_0^{1/b} \left| \sum_k f(t - k/b) \overline{g(t - na - k/b)} \right|^2 dt. \end{aligned}$$

Step 1 follows from the beginning of the proof of (1) \Rightarrow (2) of Theorem 2.3.

Step 2: We show,

$$\begin{aligned} & \lim_{\ell \rightarrow \infty} \sum_n \int_0^{1/b} \left| \sum_k f_\ell(t - k/b) g(t - na - k/b) \right|^2 = \\ & = b^{-1} \sum_n \int_0^{1/b} \left| \sum_k f(t - k/b) \overline{g(t - na - k/b)} \right|^2 dt = \sum_{m,n} | \langle f, E_{mb} T_{na} g \rangle |^2. \end{aligned}$$

For step 2, choose an N so that for all $t \in [0, 1/b]$, $f(t - k/b) = 0$ for all $|k| > N$. Hence, for all $t \in [0, 1/b]$ we have

$$\begin{aligned} & \sum_n \left| \sum_k f_\ell(t - k/b) g(t - na - k/b) \right|^2 \leq \\ & \sum_{k=-N}^N |f_\ell(t - k/b)|^2 \sum_n \sum_{k=-N}^N |g(t - na - k/b)|^2 \\ & \leq \sum_{k=-N}^N |f(t - k/b)|^2 2NB^2 \in L^1[0, 1/b]. \end{aligned}$$

So, Step 2 follows by the Lebesgue Dominated convergence Theorem. The following step will complete the proof.

Step 3:

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \frac{1}{b} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \overline{f_{\ell}(t)} f_{\ell}(t - k/b) G_k(t) dt = \\ \frac{1}{b} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt \end{aligned}$$

For Step 3, note that $\text{support } f_{\ell} \subset \text{support } f_{\ell+1} \subset \text{support } f$. Hence, for k fixed we have:

$$|f_{\ell}(t)| |T_{k/b} f_{\ell}(t)| |G_k(t)| \uparrow |f(t)| |T_{k/b} f(t)| |G_k(t)|.$$

Also, by assumption $|G_k(t)| \leq B^2$. Since $f \in L^2(\mathbb{R})$ this implies

$$|f(t)| |T_{k/b} f(t)| |G_k(t)| \in L^1(\mathbb{R}).$$

Hence, by the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \frac{1}{b} \int_{\mathbb{R}} \overline{f_{\ell}(t)} f_{\ell}(t - k/b) G_k(t) dt = \\ \frac{1}{b} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt. \end{aligned}$$

Finally, since the right hand side of the WH-frame identity has only a finite number of non-zero k 's (and the same ones for f and all f_{ℓ}), we have the equality in Step 3 and unconditional convergence in the right hand side of the Identity.

(2) \Rightarrow (1): For any f supported on an interval of length $1/b$, we are assuming the WH-frame Identity holds. But, for all such f we have for all $0 \neq k \in \mathbb{Z}$,

$$\frac{1}{b} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt = 0.$$

Hence, by the WH-frame identity,

$$\sum_{m, n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 = \frac{1}{b} \int_{\mathbb{R}} |f(t)|^2 G_0(t) dt < \infty.$$

This implies that G_0 is bounded. To see this, let $I = [c, d]$ be any interval of length $< 1/b$. It suffices to show that G_0 is bounded on I . Let

$$A_n = \{t \in I : |G_0(t)| \leq n\}.$$

Let $T_n : L^2[c, d] \rightarrow L^2[c, d]$ be given by $T_n f = \chi_{A_n} f \cdot \sqrt{G_0}$. The T_n are bounded linear operators and the family is pointwise bounded by the above. Hence they are uniformly bounded and so

$$Tf = f \cdot G_0$$

is a bounded linear operator. But the norm of this ‘‘multiplication’’ operator is $\text{ess sup } |G_0(t)|$.

□

We remark that we could simplify the proof of Theorem 3.1 if (g, a, b) has a finite upper frame bound. For in this case we use the frame operator S to get some of the needed convergence. For example, in this case we would observe:

$$\sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 = \langle S f, f \rangle,$$

while

$$\lim_{\ell \rightarrow \infty} \langle S f_\ell, f_\ell \rangle = \langle S f, f \rangle.$$

4. TYPES OF CONVERGENCE OF THE WH-FRAME IDENTITY

For the general case of the WH-frame identity, the situation becomes much more complicated. Casazza, Christensen and Janssen [3] made a detailed study of the convergence properties of the Walnut representation of the frame operator for WH-systems. Proposition 2.4 of [3] shows that the convergence of the Walnut representation of the frame operator is the same as convergence of the series on the right hand side of the WH-frame identity. In [3] there are complete classifications of when the Walnut representation (and hence when the WH-frame identity) converges symmetrically, in norm or unconditionally. Each of these cases is different and there are WH-frames for which these modes of convergence are different.

In [3] Corollary 6.5 it is shown that there is a case where the Walnut representation of the frame operator converges unconditionally for all $f \in L^2(\mathbb{R})$. This is when g satisfies the CC-condition.

Definition 4.1. *We say that $g \in L^2(\mathbb{R})$ satisfies the **CC-condition** if there is a constant $0 < B$ so that*

$$\sum_{k \in \mathbb{Z}} |G_k(t)| \leq B \quad \text{a.e. } t.$$

The CC-condition holds if g is in the **Wiener amalgam space** $W(L^\infty, \ell^1)$:

$$W(L^\infty, \ell^1) = \{f \in L^2(\mathbb{R}) : \sum_{n \in \mathbb{Z}} \|\chi_{[n, n+1)} \cdot g\|_\infty < \infty\}.$$

This gives cases where the WH-frame identity holds for all functions and the series converges unconditionally.

Theorem 4.2. *If $g \in L^2(\mathbb{R})$ satisfies the CC-condition (In particular, if $g \in W(L^\infty, \ell^1)$) then the WH-frame identity holds for all $f \in L^2(\mathbb{R})$ and the series converges unconditionally.*

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