HILBERT SPACE FRAMES, RESTRICTED INVERTIBILITY AND THE PAVING CONJECTURE

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ABSTRACT. We show that the conjectured generalization of the Bourgain-Tzafriri *restricted-invertibility theorem* is equivalent to the conjecture of Feichtinger concerning bounded Hilbert space frames: Can every bounded frame be written as a finite union of Riesz basic sequences? We further show that these two conjectures are implied by the *paving conjecture* of Bourgain-Tzafriri (which in turn is known to be equivalent to the *Kadison-Singer conjecture*). We also extend in several ways the Bourgain-Tzafriri restricted-invertibility principle, showing in particular that it holds on *random* subspaces, which shows that slightly weaker versions of the conjectures hold. Finally, we show that Weyl-Heisenberg frames over rational lattices are finite unions of Riesz basic sequences.

1. INTRODUCTION

A *frame* for a Hilbert space $\mathcal{H}$ is a family of vectors $\{f_i\}_{i\in I}$ in $\mathcal{H}$ so that there are constants $A, B > 0$ satisfying:

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \quad \text{for all } f \in \mathcal{H}.$$

If $A = B$ we call this an *A-tight frame*. For any frame $\{f_i\}_{i\in I}$ it is immediate that

$$\sup_{i\in I} \|f_i\| < \infty.$$

For the basic properties of frames we refer the reader to [Ca, Ch, Y]. The frame is *bounded* if

$$\inf_{i\in I} \|f_i\| > 0.$$

The *analysis operator* associated to a sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is $T : \mathcal{H} \to \ell^2(I)$ given by:

$$Tf = (\langle f, f_i\rangle)_{i\in I}\,.$$

The *frame operator* is $S = T^*T$ and the *Grammian matrix* is the matrix of TT^*. A direct calculation shows that this is the matrix $(\langle f_i, f_j\rangle)_{i,j\in I}$. A

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sequence $\{f_i\}_{i\in I}$ is a *Bessel sequence* for $\mathcal{H}$ with *Bessel constant* B if the operator $T^*:\ell^2(I)\to\mathcal{H}$ given by

$$T^*(\{a_i\}_{i\in I}) = \sum_{i\in I} a_i f_i,$$

satisfies: $\|T^*\| \le B$. Or, equivalently,

$$\|\sum_{i\in I} a_i f_i\|^2 \le B \sum_{i\in I} |a_i|^2, \quad \text{for all} \quad \{a_i\}_{i\in I}\in \ell^2(I).$$

Now,

$$Sf := T^*Tf = \sum_{i\in I} \langle f, f_i\rangle f_i,$$

is a positive, self-adjoint invertable operator on $\mathcal{H}$ called the *frame operator* for $\{f_i\}_{i\in I}$. If dim $\mathcal{H}$ = N, $\{e_i\}_{i=1}^N$ is an orthonormal basis of $\mathcal{H}$ and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$ are the eigenvalues of S then

$$\sum_{i=1}^N \lambda_i = \sum_{i=1}^N \langle Se_i, e_i\rangle = \sum_{i=1}^N \|Tf_i\|^2.$$

Also, $\lambda_1 \le \|S\| \le \|T\|^2 = B$.

A bounded unconditional basis for $\mathcal{H}$ is called a *Riesz basis* for $\mathcal{H}$. That is, $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$ if $\{f_i\}_{i\in I}$ is complete in $\mathcal{H}$,

$$0 < \inf_{i\in I} \|f_i\| \le \sup_{i\in I} \|f_i\| < \infty,$$

and there are constants $K_1, K_2 > 0$ so that for all families of scalars $\{a_i\}_{i\in I}$ we have

$$K_1 \sum_{i\in I} |a_i|^2 \le \|\sum_{i\in I} a_i f_i\|^2 \le K_2 \sum_{i\in I} |a_i|^2.$$

In this case we call $\sqrt{K_1}$ the *lower Riesz basis bound* of $\{f_i\}_{i\in I}$ and $\sqrt{K_2}$ the *upper Riesz basis bound.* If $\{f_i\}_{i\in I}$ is a Riesz basis for its closed linear span we call it a *Riesz basic sequence.*

Notation: Throughout the paper $\{e_i\}$ will denote an orthonormal basis for whatever Hilbert space we are working in.

Feichtinger has made the following conjecture concerning bounded frames for a Hilbert space $\mathcal{H}$.

Conjecture 1.1 (Feichtinger)**.** *Every bounded frame can be written as a finite union of Riesz basic sequences.*

In 1987, Bourgain and Tzafriri [BT] proved the following fundamental result known as the *Restricted-Invertibility Theorem*:

Theorem 1.2 (Bourgain-Tzafriri). *There is a universal constant $c > 0$ so that whenever $T : \ell_2^n \to \ell_2^n$ is a linear operator for which $\|Te_i\| = 1$, for $1 \le i \le n$, then there exists a subset $\sigma \subset \{1, 2, \cdots, n\}$ of cardinality $|\sigma| \ge \frac{cn}{\|T\|^2}$ so that*

$$\|\sum_{j\in\sigma} a_j Te_j\|^2 \ge c \sum_{j\in\sigma} |a_j|^2,$$

for all choices of scalars $\{a_j\}_{j\in\sigma}$.

In the course of proving our results, we will extend Theorem 1.2 in several ways showing in particular that it holds on *random* subspaces.

Theorem 1.2 gave rise to the following conjecture which is still open today:

Conjecture 1.3. *For every $B > 0$ there is a natural number $M = M(B)$ and a $A = A(B) > 0$ so that if $T : \ell_2^n \to \ell_2^n$ is a linear operator for which $\|Te_i\| = 1$, for all $1 \le i \le n$ and $\|T\| \le B$, then there is a partition $\{I_j\}_{j=1}^M$ of $\{1, 2, \cdots, n\}$ so that for each $1 \le j \le M$ and all choices of scalars $\{a_i\}_{i\in I_j}$ we have:*

$$\|\sum_{i\in I_j} a_i Te_i\|^2 \ge A \sum_{i\in I_j} |a_i|^2.$$

We also have a finite form of the Feichtinger conjecture.

Conjecture 1.4 (Finite Feichtinger Conjecture). *For every $B, C > 0$ there is a natural number $M = M(B, C)$ and an $A = A(B, C) > 0$ so that whenever $\{f_i\}_{i\in I}$ is a frame for ℓ_2^N with upper frame bound B and $\|f_i\| \ge C$ for all $i \in I$, then I can be partitioned into $\{I_j\}_{j=1}^M$ so that for each $1 \le j \le M$, $\{f_i\}_{i\in I_j}$ is a Riesz basic sequence with lower Riesz basis bound A and upper Riesz basis bound $\sqrt{B}$.*

We will show that Conjectures 1.1, 1.3 and 1.4 are equivalent in the sense that all three have positive answers or all three have negative answers. We will also show that these conjectures are equivalent to the corresponding conjectures about Bessel sequences and that all of these are true if the well known *Paving Conjecture* holds. Given a subset I of the integers, we denote by P_I the orthogonal projection in ℓ_2 onto the subspace spanned by e_i, $i \in I$.

Conjecture 1.5 (The Paving Conjecture [B-Tz]). *For $\epsilon > 0$, there is a constant $M = M(\epsilon)$ such that for every integer n and every linear operator S on l_2^n whose matrix has zero diagonal, one can find a partition $\{\sigma_j\}_{j=1}^M$ of $\{1, \cdots, n\}$, such that*

$$\|P_{\sigma_j} S P_{\sigma_j}\| \le \varepsilon \|S\| \quad \text{for all } j = 1, 2, \cdots, M.$$

In an interesting paper [W], Weaver gives several reformulations of the Kadison-Singer conjecture. One of these, in terms of frames, is the following:

Conjecture 1.6. *There exists a universal constant $B \ge 2$ and a natural number M such that the following holds. Let $\{f_i\}_{i=1}^N$ is a B-tight frame for ℓ_2^n*

with $\|f_i\| \leq 1$*, for all* $i = 1, 2, \cdots, N$*. Then there is a partition* $\{I_j\}_{j=1}^M$ *of* $\{1, 2, \cdots N\}$ *such that for all* $1 \leq j \leq M$ *we have*

$$\sum_{i \in I_j} |\langle f, f_i \rangle|^2 \leq (B-1)\|f\|^2, \quad \textit{for all } f \in \ell_2^n.$$

It is possible that all these conjectures have negative answers in general. In this case it becomes important to know the strongest results available. In this direction we will show that these conjectures are true - up to a logarithmic factor. We also show that the finite forms of these conjectures holds for $\{f_i\}_{i=1}^n$ if the off-diagonal elements of the Grammian matrix satisfies for some $\gamma > 0$:

$$|\langle f_i, f_j \rangle| \leq \frac{1}{log^{1+\gamma} n}, \quad \text{for all } i \neq j.$$

We also give a positive result for Weyl-Heisenberg frames over rational lattices. If $g \in L^2(\mathbb{R})$, $a, b > 0$ we define for all $m, n \in \mathbb{Z}$:

$$E_{mb}g(t) = e^{2\pi imbt}g(t)$$

and

$$T_{na}g(t) = g(t - na).$$

If $\{E_{mb}T_{na}g\}_{n,m\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, we call this a *Weyl-Heisenberg frame.* In Section 5 we show that whenever ab is rational, the Weyl-Heisenberg frame can be written as a finite union of Riesz basic sequences. In [G1], Gröchenig shows that frames with a certain "localization property" can always be written as finite unions of Riesz basic sequences. This includes the case of Weyl-Heisenberg frames for ab irrational when g lies in a certain *modulation space.* The latter assumption is not required in our approach.

2. Restricted Invertibility and Bounded Frames

Now we state two other conjectures which we will see later are also equivalent to the two conjectures above.

Conjecture 2.1. *Every bounded Bessel sequence can be written as a finite union of Riesz basic sequences.*

Conjecture 2.2. *For every* $B > 0$ *there exists a natural number* $M = M(B)$ *and an* $A = A(B)$ *so that every Bessel sequence* $\{f_i\}_{i=1}^n$ *with Bessel constant* $B > 0$ *and* $\|f_i\| = 1$*, for all* $1 \leq i \leq n$ *can be written as a union of* M *Riesz basic sequences each with lower Riesz basis bound* A*.*

To simplify the proof of the main result of this section, we first prove an elementary proposition.

Proposition 2.3. *Fix a natural number* M *and assume for every natural number* n *we have a partition* $\{I_i^n\}_{i=1}^M$ *of* $\{1, 2, \cdots, n\}$*. Then there are natural* *numbers* $\{n_1 < n_2 < \cdots\}$ *so that if* $j \in I_i^{n_j}$, *then* $j \in I_i^{n_k}$, *for all* $k \geq j$. *Hence, if* $I_i = \{j | j \in I_i^{n_j}\}$ *then*

(1) $\{I_i\}_{i=1}^K$ *is a partition of* $\mathbb{N}$.

(2) If $I_i = \{j_1 < j_2 < \cdots\}$ *then for every natural number* k *we have* $\{j_1, j_2, \cdots, j_k\} \subset I_i^{n_k}$.

Proof: For each natural number n, 1 is in one of the sets $\{I_i^n\}_{i=1}^M$. Hence, there are natural numbers $n_1^1 < n_2^1 < n_3^1 < \cdots$ so that $1 \in I_i^{n_j^1}$, for all $j \in \mathbb{N}$. Now, for every natural number n_j^1, 2 is in one of the sets $\{I_i^{n_j^1}\}_{i=1}^M$. Hence, there is a subsequence $\{n_j^2\}$ of $\{n_j^1\}$ and an $1 \leq i \leq M$ so that $2 \in I_i^{n_j^2}$, for all $j \in \mathbb{N}$. Continuing by induction, we get a subsequence $\{n_j^{\ell+1}\}_{j=1}^\infty$ of $\{n_j^\ell\}_{j=1}^\infty$ and an $1 \leq i \leq M$ so that $\ell + 1 \in I_i^{n_j^{\ell+1}}$, for all $j \in \mathbb{N}$. Letting $\{n_j\}_{j=1}^\infty$ be $\{n_j^j\}_{j=1}^\infty$ gives the conclusion of the proposition. □

Theorem 2.4. *Conjectures 1.1, 1.3, 2.1, and 2.2 are all equivalent in the sense that either all four of these conjectures have positive answers or all four have negative answers.*

Proof: Conjecture 2.1 ⇒ Conjecture 1.1: This is obvious.

Conjecture 1.1 ⇒ Conjecture 2.2: We will prove the contrapositive. So we assume that Conjecture 2.2 fails. Then there is a constant $B > 0$ so that for every $M \in \mathbb{N}$ and for every $A > 0$ there is a $n = n(M, A) \in \mathbb{N}$, a finite dimensional Hilbert space H and a Bessel sequence $\{f_i\}_{i=1}^n$ in H with Bessel constant B and $\|f_i\| = 1$, for all $1 \leq i \leq n$ and whenever we partition $\{1, 2, \cdots, n\}$ into sets $\{I_j\}_{j=1}^M$, then there exists some $1 \leq \ell \leq M$ and a set of scalars $\{a_i\}_{i \in I_\ell}$ with

$$\|\sum_{i \in I_\ell} a_i f_i\|^2 \leq A \sum_{i \in I_\ell} |a_i|^2.$$

Now, for each $k \in \mathbb{N}$, we can choose a finite dimensional Hilbert space H_k and letting $M = k$ and $A = 1/k$ above we can choose $n_k = n(k, 1/k)$ and $\{f_i^k\}_{i=1}^{n_k}$ satisfying the above conditions. Let $H = (\sum \oplus H_k)_{\ell_2}$ and consider $\{f_i^k\}_{i=1,k=1}^{n_k,\infty}$ as elements of H. This family is now a Bessel sequence with Bessel bound B and $\|f_i^k\| = 1$, for all $1 \leq i \leq n_k, k \in \mathbb{N}$. Assume we can partition $\{f_i^k\}_{i=1,k=1}^{n_k,\infty}$ into M sets of Riesz basic sequences each with lower Riesz basis constant A. But, for all k with $k \geq M$ and $1/k \leq A$, $\{f_i^k\}_{i=1}^{n_k}$ cannot be partitioned into M sets each with lower Riesz basis constant $\geq A$, and hence $\{f_i^k\}_{i=1,k=1}^{n_k,\infty}$ cannot be partitioned this way. This shows that Conjecture 1.1 fails.

Conjecture 2.2 ⇒ Conjecture 1.4. Given $\{f_i\}$ as in Conjecture 1.4, the sequence $\{\frac{f_i}{\|f_i\|}\}$ is a Bessel sequence in ℓ_2^N with Bessel bound $\frac{B}{C}$. So Conjecture 1.4 follows from conjecture 2.2.

Conjecture 1.4 $\Rightarrow$ Conjecture 1.3. This is obvious.

Conjecture 1.3 $\Rightarrow$ Conjecture 2.1. Let $\{f_i\}_{i=1}^{\infty}$ be a bounded Bessel sequence for an infinite dimensional Hilbert space H with Bessel bound B. Without loss of generaltiy we may assume that $\|f_i\| = 1$, for all $1 \leq i < \infty$ (Since $\{f_i\}$ is a bounded Bessel sequence if and only if $\{\frac{f_i}{\|f_i\|}\}$ is a bounded Bessel sequence). For each n, choose an n-dimensional Hilbert space H_n containing the span of $\{f_i\}_{i=1}^{n}$ and let $\{e_i^n\}_{i=1}^{n}$ be an orthonormal basis for H_n. Define $T_n : H_n \to H_n$ by $Te_i^n = f_i$. Then $\|T_n\| \leq \sqrt{B}$ and so by assuming that Conjecture 1.3 has a positive answer, we can find a partition $\{I_j^n\}_{j=1}^{M}$, $M = M(B)$, of $\{1, 2, \cdots, n\}$ so that for every $1 \leq j \leq k$, $\{f_i\}_{i \in I_j^n}$ is a Riesz basic sequence with lower Riesz basis bound A. By Proposition 2.3, we can partition $\mathbb{N}$ into sets $\{I_i\}_{i=1}^{M}$ and choose an $A = A(B)$ so that if $I_i = \{j_1 < j_2 < \cdots\}$, then for every natural number k we have that $\{j_1, j_2, \cdots, j_k\} \subset I_i^{n_k}$. It follows that $\{f_{j_\ell}\}_{\ell=1}^{k}$ is a Riesz basic sequence with the same lower Riesz basis bound A for all $k \in \mathbb{N}$. Hence, $\{f_j\}_{j \in I_i}$ has lower Riesz basis bound A, for all $1 \leq i \leq M$. Also, $\sqrt{B}$ is a upper Riesz basis bound for all these sets. This shows that Conjecture 2.1 has a positive answer. □

3. Random Restricted Invertibility

Remark 3.1. *Throughout this section, c will denote a universal constant.*

Let T be a bounded linear operator on l_2. We will be interested in the coordinate subspaces $\mathbb{R}^{\sigma}$, $d\sigma \subset \mathbb{N}$, on which T is an isomorphism. Precisely, we look at the family $K(T)$ of all finite subsets σ of the integers so that the equivalence

$$\frac{1}{4}\sum_{i\in\sigma}\|a_iTe_i\|^2 \leq \|\sum_{i\in\sigma}a_iTe_i\|^2 \leq 4\sum_{i\in\sigma}\|a_iTe_i\|^2 \tag{3.1}$$

holds for any choice of scalars a_i. There is nothing special about the constant 4 here, it can be replaced by any constant larger than 1 in all the results below.

Our first impression is that the family $K(T)$ is small. Indeed, for an even integer n, consider the linear operator T on l_2^n defined by $Te_i = \frac{1}{\sqrt{2}}e_{\lceil i/2\rceil}$. Every subset $\sigma \in K(T)$ contains no pairs of the form $\{2i-1, 2i\}$, hence $|K(T)| = 3^{n/2-1} \ll 2^n$. However, the subsets $\{1, 3, 5, \cdots, n-1\}$ and $\{2, 4, 6, \cdots, n\}$ both belong to $K(T)$, and the average of the characteristic functions of these subsets is a half of the characteristic function of the whole interval $\{1, 2, \cdots, n\}$.

The main result of this section gives an asymptotically precise lower bound on the average of the characteristic functions of the sets $\sigma \in K(T)$. This extends in several ways the Bourgain-Tzafriri's principle of the restricted invertibility, as we will see shortly.

Theorem 3.2. *Let T be a norm one linear operator on l_2. Then there exists a probability measure ν on $K(T)$ such that*

$$\nu\{\sigma |\, i \in \sigma\} \geq c\|Te_i\|^2 \quad \textit{for all } i. \tag{3.2}$$

Note that the left hand side in (3.2) clearly equals $\int_{K(T)} \chi_\sigma(i)\, d\nu(\sigma)$.

The next two results follow immediately from Theorem 3.1. However, as we will see, they also imply Theorem 3.1 and so we will actually prove them independently later and use them to prove Theorem 3.1.

Let μ be a measure on $\mathbb{N}$; for simplicity we will denote $\mu(\{i\})$ by $\mu(i)$. Summing over i with weights $\mu(i)$in (3.2), we obtain

$$\int_{K(T)} \mu(\sigma)\, d\nu(\sigma) = \sum_i \mu(i) \int_{K(T)} \chi_\sigma(i)\, d\nu(\sigma) \geq c \sum_i \mu(i)\|Te_i\|^2.$$

This proves the following corollary.

Corollary 3.3. *Let T be a norm one linear operator on l_2, and let μ be a measure on $\mathbb{N}$. Then there exists a set $\sigma \in K(T)$ such that*

$$\mu(\sigma) \geq c \sum_i \mu(i)\|Te_i\|^2. \tag{3.3}$$

Corollary 3.2 was essentially proved by S.Szarek [Sz] with only the *upper* bound in (3.1).

In particular, for the counting measure on $\mathbb{N}$, Corollary 3.3 proves the existence of a set $\sigma \in K(T)$ with cardinality $|\sigma| \geq c\|T\|_{\mathrm{HS}}^2$ (where $\|T\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm of T). It is known [V 1] that the constant c in this estimate can be improved to $1 - \varepsilon$ for any $\varepsilon > 0$ at the cost of replacing 4 in the definition of $K(T)$ by a number depending on ε.

An immediate consequence of Corollary 3.3 is

Corollary 3.4. *Let T be a linear operator on l_2 with $\|Te_i\| = 1$ for all i, and let μ be a measure on $\mathbb{N}$. Then there exits a subset σ of the integers such that $\mu(\sigma) \geq c/\|T\|^2$ and*

$$\frac{1}{2}\|x\| \leq \|Tx\| \leq 2\|x\| \quad \textit{for } x \in \mathbb{R}^\sigma. \tag{3.4}$$

This result contains the Bourgain-Tzafriri restricted-invertibility principle, Theorem 1.2, as is seen by considering the uniform measure on $\{1, \cdots, n\}$.

We pass now to the proof of Theorem 3.2, which relies on the methods of Bourgain-Tzafriri [B-Tz] and of [V 2].

In [B-Tz], a suppression analogue of Theorem 3.2 is proved. By P_σ we denote the orthogonal projection in $\mathbb{R}^n$ onto $\mathbb{R}^\sigma$, where σ is a subset of $\{1, \cdots, n\}$.

Theorem 3.5 (Bourgain-Tzafriri)**.** *Let S be a linear operator on l_2 whose matrix relative to the unit vector basis has zero diagonal. For an $\varepsilon > 0$, denote by $K'(S,\varepsilon)$ the family of all subsets σ of the integers such that $\|P_\sigma S P_\sigma\| \le \varepsilon \|S\|$. Then there exists a probability measure ν' on $K'(S,\varepsilon)$ such that*

$$\nu'\{\sigma \,|\, i \in \sigma\} \ge c\varepsilon^2 \quad \textit{for all } i. \tag{3.5}$$

This implies a weaker version of Corollary 3.4. Indeed, under the assumptions of Corollary 3.4 we automatically have $\|T\| \ge 1$; let $S = T^*T - I$ and $\varepsilon = \frac{1}{2\|S\|} \ge \frac{1}{4\|T\|^2}$. Then for every $\sigma \in K'(S,\varepsilon)$ and every $x \in \mathbb{R}^\sigma$, $\|x\| = 1$, we have

$$\big| \|Tx\|^2 - \|x\|^2 \big| \le \varepsilon \|S\| = \frac{1}{2}.$$

Hence (3.4) holds. Now, summing over i in (3.5), we obtain

$$\begin{aligned} \int_{K'(S,\varepsilon)} \mu(\sigma)\, d\nu'(\sigma) &= \sum_i \mu(i) \int_{K'(S,\varepsilon)} \chi_\sigma(i)\, d\nu'(\sigma) \\ &\ge \Big(\sum_i \mu(i) \Big) c\varepsilon^2 \gtrsim 1/\|T\|^4. \end{aligned}$$

Replacing the integral by the maximum, we conclude:

$$\text{Corollary 3.4 holds with } \mu(\sigma) \ge c/\|T\|^4. \tag{3.6}$$

We will use this weaker estimate in the proof of the actual Corollary 3.4. One way to do this is to apply the weaker form of Corollary 3.3 due to Szarek (see the remark above) and then apply (3.6) to the operator $T_1 : l_2^\sigma \to l_2$ that sends e_i to $\frac{Te_i}{\|Te_i\|}$, $i \in \sigma$. Since $\|T_1\| \le 2$, Corollary 3.4 will follow.

We will chose another way, without appealing to Szarek's result. The proof of Theorem 3.2 will be organized backwards: Corollary 3.4 $\Longrightarrow$ Corollary 3.3 $\Longrightarrow$ Theorem 3.2. Corollary 3.4 itself will be a consequence of the following suppression result, which is a "weighted" variant of an unpublished theorem of Kashin and Tzafriri [K-Tz] (see [V 2]) and a slight improvement of a result of Szarek [Sz].

Theorem 3.6. *Let T be a linear operator on l_2, and μ be a probability measure on $\mathbb{N}$. Then for any $0 < \delta < 1/4$ there exists a subset σ of the integers with $\mu(\sigma) \ge \delta$ and such that*

$$\|TP_\sigma\| \le c\sqrt{\delta}\|T\| + c\Big(\sum_i \mu(i) \|Te_i\|^2 \Big)^{1/2}.$$

The proof is by now standard, through a random selection followed by the Grothendieck factorization. The random selection is done in the next lemma. Let $0 < \delta < 1$. Consider random selectors δ_i, i.e. independent $\{0,1\}$-valued random variables with $\mathbb{E}\delta_i = \delta$. Then the linear operator $P_\delta := \sum_i \delta_i \, e_i \otimes e_i$ is a random orthogonal projection in l_2.

Lemma 3.7. *Let T be a linear operator on l_2^n and let μ be a probability measure on $\{1,\cdots,n\}$. Then for $0<\delta<1/4$ a random coordinate projection $P_\delta := \sum_{i=1}^n \delta_i\, e_i \otimes e_i$ satisfies*

$$\mathbb{E}\|P_\delta T^*\|_{l_2^n \to L_1^n(\sqrt{\mu})} \le \delta\|T\| + 2\sqrt{\delta}\Big(\sum_{i=1}^n \mu(i)\|Te_i\|^2\Big)^{1/2},$$

where the space $L_1^n(\sqrt{\mu})$ is $\mathbb{R}^n$ equipped with the norm

$$\|x\|_{L_1^n(\sqrt{\mu})} = \sum_{i=1}^n \sqrt{\mu(i)}|x(i)|.$$

Proof. This is the Gine-Zinn's symmetrization scheme,

$$\begin{aligned}(3.7)\quad \mathbb{E}\|P_\delta T^*\|_{l_2^n \to L_1^n(\sqrt{\mu})} &= \mathbb{E}\sup_{x\in B(l_2^n)} \sum_{i=1}^n \delta_i\sqrt{\mu(i)}|\langle Te_i, x\rangle| \\ &\le \delta \sup_{x\in B(l_2^n)} \sum_{i=1}^n \sqrt{\mu(i)}|\langle Te_i,x\rangle| + \mathbb{E}\sup_{x\in B(l_2^n)} \sum_{i=1}^n (\delta_i-\delta)\sqrt{\mu(i)}|\langle Te_i,x\rangle|.\end{aligned}$$

By Hölder's inequality, the first sumand is bounded by

$$\delta \sup_{x\in B(l_2^n)} \Big(\sum_{i=1}^n |\langle Te_i,x\rangle|^2\Big)^{1/2} = \delta\|T\|.$$

To bound the second summand in (3.7), let δ_i' be independent copies of δ_i. Then $(\delta_i-\delta)$ can be replaced by $(\delta_i-\delta_i')$, which (by symmetry) has the same distribution as $\varepsilon_i(\delta_i-\delta_i')$, where ε_i denote Rademacher random variables (independend random variables takig values -1 and 1 with probability $1/2$). So $(\delta_i-\delta)$ in (3.7) can be replaced by $2\varepsilon_i\delta_i$, which can further be replaced (by the standard comparison inequality) by $g_i\delta_i$, where g_i are independent normalized Gaussian random variables. These probabilistic techniques, as well as Slepian's inequality below, can be found in [L-T] §3, §6. Hence

$$\mathbb{E}\|P_\delta T^*\|_{l_2^n \to L_1^n(\sqrt{\mu})} \le \delta\|T\| + 2\mathbb{E}\sup_{x\in B(l_2^n)} \sum_{i=1}^n g_i\delta_i\sqrt{\mu(i)}|\langle Te_i,x\rangle|.$$

By Slepian's inequality, $|\langle Te_i, x\rangle|$ can be replaced by $\langle Te_i, x\rangle$, and we continue the estimate as

$$\begin{aligned}
&\le \delta\|T\| + 2\mathbb{E}\Big\| \sum_{i=1}^{n} g_i \delta_i \sqrt{\mu(i)} T e_i \Big\| \\
&\le \delta\|T\| + 2\Big(\mathbb{E} \sum_{i=1}^{n} \delta_i \mu(i) \|Te_i\|^2\Big)^{1/2} \\
&= \delta\|T\| + 2\sqrt{\delta}\Big(\sum_{i=1}^{n} \mu(i)\|Te_i\|^2\Big)^{1/2}.
\end{aligned}$$

The proof is complete. ■

The Grothendieck factorization is done in the following lemma.

Lemma 3.8. *Let $T : l_2^n \to l_2$ be a linear operator and let μ be a measure on $\{1, \cdots, n\}$ of total mass m. Then there exists a subset $\sigma \in \{1, \cdots, n\}$ such that $\mu(\sigma) \ge m/2$ and*

$$\|TP_\sigma\|_{l_2^n \to l_2} \le \frac{c}{\sqrt{m}} \|T\|_{L_\infty^n(\sqrt{\mu}) \to l_2},$$

where the space $L_\infty^n(\sqrt{\mu})$ is $\mathbb{R}^n$ equipped with the norm

$$\|x\|_{L_\infty^n(\sqrt{\mu})} = \max_{i \le n} \frac{|x(i)|}{\sqrt{\mu(i)}}.$$

Proof. Consider the isometry $\Delta : L_\infty^n(\sqrt{\mu}) \to L_\infty^n$ defined as $(\Delta x)(i) = \frac{|x(i)|}{\sqrt{\mu(i)}}$. By the Grothendieck theorem (see [TJ] Corollary 10.10), the operator $T\Delta^{-1} : L_\infty^n \to l_2$ is 2-summing, and its 2-summing norm is bounded as

$$\pi_2(T\Delta^{-1}) \le c\|T\Delta^{-1}\|_{L_\infty^n \to l_2} = c\|T\|_{L_\infty^n(\sqrt{\mu}) \to l_2} =: M.$$

By Pietsch's Theorem (see [TJ] Corollary 9.4), there exists a probability measure λ on $[n] = \{1, \cdots, n\}$ so that

$$\|T\Delta^{-1}\|_{L_2([n],\lambda) \to l_2} \le M.$$

Hence for every $x \in \mathbb{R}^n$,

$$\|T\Delta^{-1}x\|_{l_2} \le M\Big(\sum_{i=1}^{n} \lambda(i)|x(i)|^2\Big)^{1/2},$$

and thus

$$\|Tx\|_{l_2} \le M\Big(\sum_{i=1}^{n} \frac{\lambda(i)}{\mu(i)} |x(i)|^2\Big)^{1/2}$$

Since $\frac{1}{m}\int_{[n]} \frac{\lambda(i)}{\mu(i)}\, d\mu(i) = \frac{1}{m}\sum_{i=1}^n \lambda(i) = \frac{1}{m}$, Chebyshev's inequality gives

$$\frac{1}{m}\mu\Big\{i \;\Big|\; \frac{\lambda(i)}{\mu(i)} \leq \frac{2}{m}\Big\} \geq 1/2.$$

Hence there exists a subset $\sigma \subset [n]$ with $\mu(\sigma) \geq m/2$ and such that

$$\frac{\lambda(i)}{\mu(i)} \leq \frac{2}{m} \quad \text{for all } i \in \sigma.$$

Thus for every $x \in \mathbb{R}^\sigma$

$$\|Tx\|_{l_2} \leq \sqrt{\frac{2}{m}} M \Big(\sum_{i=1}^n |x(i)|^2\Big)^{1/2}.$$

This completes the proof. ∎

Proof of Theorem 3.6. By approximation, we can assume that T acts on l_2^n and $\delta > C/n$ with a sufficiently large absolute constant C. Then we apply Lemma 3.7. Since $\mathbb{E}\mu(\sigma) = \mathbb{E}\sum_{i=1}^n \delta_i \mu(i) = \delta$, the classical estimates on the binomial tails imply that $\mu(\sigma) \geq \delta/2$ with probability greater than $1/2$. Then there exists a subset $\sigma \subset \{1, \cdots, n\}$ such that $\mu(\sigma) \geq \delta/2$ and, by duality,

$$\|TP_\sigma\|_{L_\infty^n(\sqrt{\mu}) \to l_2^n} \leq 2\delta\|T\| + 4\sqrt{\delta}\Big(\sum_{i=1}^n \mu(i)\|Te_i\|^2\Big)^{1/2}.$$

Next we apply Lemma 3.8 for the operator $TP_\sigma : L_\infty^\sigma(\sqrt{\mu}) \to l_2^n$ with $m = \mu(\sigma) \geq \delta/2$. There exists a subset $\sigma' \subset \sigma$ with $\mu(\sigma') \geq \delta/4$ and such that

$$\begin{aligned} \|TP_{\sigma'}\|_{l_2^\sigma \to l_2^n} &\leq \frac{c}{\sqrt{\delta}}\|T\|_{L_\infty^\sigma(\sqrt{\mu}) \to l_2^n} \\ &\leq c\sqrt{\delta}\|T\| + c\Big(\sum_i \mu(i)\|Te_i\|^2\Big)^{1/2}. \end{aligned}$$

This proves the theorem. ∎

Proof of Corollary 3.4. Applying Theorem 3.6 with $\delta = \frac{1}{4\|T\|^2}$, we find a subset $\sigma \subset \mathbb{N}$ with $\mu(\sigma) \geq \frac{1}{4\|T\|^2}$ and such that

$$\|TP_\sigma\| \leq c.$$

Next, we apply Theorem 3.5; more precisely, its consequence (3.6) for the operator $TP_\sigma : l_2 \to l_2$ and for the probability measure μ conditioned on σ, i.e. for μ' defined as $\mu'(\eta) = \mu(\eta \cap \sigma)/\mu(\sigma)$, $\eta \subset \mathbb{N}$. There exists a subset $\sigma' \subset \sigma$ with $\mu(\sigma') \geq c\mu(\sigma) \geq c/\|T\|^2$ and such that (3.4) holds for all $x \in \mathbb{R}^{\sigma'}$. This completes the proof. ∎

To prove Corollary 3.3, we introduce a splitting procedure. Consider a family η_k, $k = 1, \cdots, n$ of disjoint subsets of the integers. This family defines a splitting of a probability measure λ on $\{1, \cdots, n\}$ and of any sequence $(x_i)_{i \le n}$ in l_2. Namely, put

$$\eta = \bigcup_{i \le n} \eta_i, \quad N = |\eta|, \quad \text{and } N_i = |\eta_i|.$$

Then the splitted probability measure λ' on η and the splitted sequence $(x'_k)_{k \in \eta}$ are defined as

$$\lambda'(k) = \frac{\lambda(i)}{N_i}, \quad x'_k = \frac{x_i}{\sqrt{N_i}} \quad \text{for } k \in \eta_i.$$

Splitting will be used to make the norms $\|x_i\|$ almost identical. Namely, one can easily construct a splitting such that

$$0.9\|x'_l\| \le \|x'_k\| \le 1.1\|x'_l\| \quad \text{for all } k, l \in \eta.$$

Since $\sum_{k \in \eta} \|x'_k\|^2 = \sum_{i=1}^n \|x_i\|^2 =: h$, we have a postiori:

$$\|x'_k\| \sim \sqrt{\frac{h}{N}} \quad \text{for all } k \in \eta,$$

where $a \sim b$ means $\frac{1}{2}a \le b \le 2a$. Moreover, since $\|x'_k\| = \frac{\|x_i\|}{\sqrt{N_i}}$ for $k \in \eta_i$, we also have

$$\frac{1}{N_i} = \frac{\|x'_k\|^2}{\|x_i\|^2} \sim \frac{h}{N} \cdot \frac{1}{\|x_i\|^2},$$

hence

$$\lambda'(k) = \frac{\lambda(i)}{N_i} \sim \frac{h}{N} \cdot \frac{\lambda(i)}{\|x_i\|^2} \quad \text{for } k \in \eta_i.$$

Let $T : l_2^n \to l_2$ be a linear operator defined as $Te_i = x_i$, $i = 1, \cdots, n$. The splitting of T is defined as the linear operator $T' : l_2^\eta \to l_2$ acting as $T'e_k = x'_k$, $k \in \eta$. An easily checked but important property is

$$\|T'\| \le \|T\|.$$

Proof of Corollary 3.3. By approximation, we can assume that T is an operator from l_2^n into l_2. Then, by the discussion above, for any probability measure λ on $\{1, \cdots, n\}$ there exists a splitting $\eta = \bigcup_{i \le n} \eta_i$, $|\eta| = N$, such that that the splitted measure λ' on η and the splitted operator $T' : l_2^\eta \to l_2$ satisfy:

- **:** (1) $\|T'\| \le \|T\| \le 1$,
- **:** (2) For $k \in \eta$, $\|T'e_k\| \sim \sqrt{\frac{h}{N}}$, where $h = \|T\|_{\mathrm{HS}}^2$,
- **:** (3) For $k \in \eta_i$, $\lambda'(k) \sim \frac{h}{N} \cdot \frac{\lambda(i)}{\|Te_i\|^2}$.

We apply Corollary 3.4 to the operator $S : l_2^\eta \to l_2$ defined as

$$Se_k = \frac{T'e_k}{\|T'e_k\|}, \quad k \in \eta.$$

Note that

$$\|S\| \leq \max_{k\in\eta} \frac{\|T\|}{\|T'e_k\|} \leq 2\sqrt{\frac{N}{h}}.$$

Therefore there exists a subset $\sigma' \subset \eta$ such that

$$\sigma' \in K(S) \quad \text{and} \quad \lambda'(\sigma') \geq c\frac{h}{N}. \tag{3.8}$$

Now the crucial fact is that for every i the set $\sigma' \cap \eta_i$ contains at most one element (denoted by k_i if it exists). This is because for a fixed i, the vectors $(Se_k, k \in \eta_i)$ are all multiples of one vector; while, since $\sigma' \in K(S)$, the set $(Se_k, k \in \sigma')$ can not contain colinear vectors, as they would fail the lower bound in (3.1).

Let $\sigma = \{i \mid \sigma' \cap \eta_i \neq \emptyset\}$. Then $\sigma \in K(T)$, and

$$\lambda'(\sigma') = \sum_{i\in\sigma} \lambda'(k_i) \sim \frac{h}{N}\sum_{i\in\sigma}\frac{\lambda(i)}{\|Te_i\|^2}.$$

This and (3.8) imply

$$\sum_{i\in\sigma}\frac{\lambda(i)}{\|Te_i\|^2} \geq c. \tag{3.9}$$

The conclusion of the Corollary follows by applying (3.9) to the probability measure λ defined as

$$\lambda(i) = \frac{\mu(i)\|Te_i\|^2}{\sum_{i=1}^n \mu(i)\|Te_i\|^2}, \quad i = 1, \cdots, n.$$

■

Actually, (3.3) and (3.9) are easily seen to be equivalent. Indeed, one can get (3.9) by applying Corollary 3.3 to the measure μ defined by $\mu(i) = \frac{\lambda(i)}{\|Te_i\|^2}$.

Proof of Theorem 3.2. This argument is a minor adaptation of [B-Tz] Corolary 1.4. $K(T)$ is a w^*-compact set. For each integer i, define a function $\pi_i \in C(K(T))$ by setting

$$\pi_i(\sigma) = \frac{\chi_\sigma(i)}{\|Te_i\|^2}, \quad \sigma \in K(T).$$

Let H be the convex hull of the functions (π_i). Fix a $\pi \in H$ and write it as a convex combination $\pi = \sum_i \lambda_i\pi_i$. By Corollary 3.3, or rather by its consequence (3.9), there exists a set $\sigma \in K(T)$ such that $\pi(\sigma) \geq c$. Looking at

σ as a point evaluation functional on $C(K)$, we conclude by the Hahn-Banach theorem that there exists a probability measure $\nu \in C(K(T))^*$ such that

$$\nu(\pi) = \int_{K(T)} \pi(\sigma)\, d\nu(\sigma) \geq c \quad \text{for all } \pi \in H.$$

Applying this estimate for $\pi = pi_i$, we obtain

$$\int_{K(T)} \chi_\sigma(i)\, d\nu(\sigma) \geq c\|Te_i\|^2,$$

which is exactly the conclusion of the theorem. ■

While Theorem 3.2 seems to be unable to yield the conjectures under discussion, it proves that every bounded frame (and actually very bounded Bessel sequence) has *many* Riesz basic subsequences.

Theorem 3.9. *If $\{f_i\}_{i=1}^\infty$ is a Bessel sequence with Bessel constant $B > 0$ and $\|f_i\| = 1$, then there exists a probability measure ν on the set K of all Riesz basic subsequences of $\{f_i\}$ with Riesz basis constant 2, such that the measure ν of the subsequences in K that contain any given element f_i is at least $b = b(B) > 0$.*

Finally, we state a conjectured "paving" analogue of Theorem 3.2 that would clearly imply Conjectures 1.1, 1.3, 2.1 and 2.2.

Conjecture 3.10. *Let T be a linear operator on l_2^n such that $\|Te_i\| = 1$ for all i. Then there exists a partition $\{\sigma_k\}_{k\leq M}$ of the set $\{1, \cdots, n\}$, where M depends only on the norm of T, and such that $\sigma_k \in K(T)$ for all k.*

To see the relation to Theorem 3.2, assume that this conjecture is true, and let ν be the probability measure on $K(T)$ that assigns each $\sigma(k)$ measure $1/M$. Then

$$\nu\{\sigma |\, i \in \sigma\} \geq 1/M \quad \text{for all } i.$$

If moreover $M \sim 1/\|T\|^2$, then this clearly implies Theorem 3.2 for operators T such that $\|Te_i\|$ have same value for all i.

4. The Paving Conjecture

Kadison and Singer raised the problem, which is still open, whether every pure state on $\mathbb{D}$, the C^*-algebra of the diagonal operators on l_2, admits a unique extension to a (pure) state on $\mathcal{L}(l_2)$, the C^*-algebra of all bounded linear operators on l_2. The problem of Kadison and Singer reduces to (and is equivalent to) the Paving Conjecture (see also [D-Sz]).

Proposition 4.1. *The Paving Conjecture implies Conjecture 2.2.*

Proof. Let $\{f_i\}_{i=1}^n$ be a unit norm Bessel sequence with Bessel constant B. Define the linear operator T on l_2^n by setting $Te_i = f_i$ for all i. Then $\|T\| \leq B$. Consider the operator $S = T^*T - I$. Then the (i,j)-entry of the matrix of S is

$$\langle Se_i, e_j \rangle = \begin{cases} \langle f_i, f_j \rangle, & i \neq j, \\ 0, & i = j. \end{cases}$$

By the Paving Conjecture, there exists a number $M = M(\varepsilon)$ and a partition $(\sigma_k)_{k \leq M}$ of the set $\{1, \cdots, n\}$ such that

$$\|P_{\sigma_k} S P_{\sigma_k}\| \leq \varepsilon \|S\| \quad \text{for all } k.$$

Applying this with $\varepsilon = \frac{1}{2\|S\|} \geq \frac{1}{2B^2}$, we obtain:

$$\|P_{\sigma_k} S P_{\sigma_k} x\| \leq \frac{1}{2}\|x\| \quad \text{for all } k.$$

Now,

$$\begin{aligned} \langle P_{\sigma_k} S P_{\sigma_k} x, x \rangle &= \langle P_{\sigma_k}(T^*T - I) P_{\sigma_k} x, x \rangle \\ &= \langle (T^*T - I) P_{\sigma_k} x, P_{\sigma_k} x \rangle \\ &= \langle T^*T P_{\sigma_k} x, P_{\sigma_k} x \rangle - \langle P_{\sigma_k} x, P_{\sigma_k} x \rangle \\ &= \langle T P_{\sigma_k} x, T P_{\sigma_k} x \rangle - \|P_{\sigma_k}\|^2 \\ &= \|T P_{\sigma_k} x\|^2 - \|P_{\sigma_k} x\|^2. \end{aligned}$$

Hence

$$\frac{\left| \|T P_{\sigma_k} x\|^2 - \|P_{\sigma_k} x\|^2 \right|}{\|x\|} = \frac{|\langle P_{\sigma_k} S P_{\sigma_k} x, x \rangle|}{\|x\|} \leq \|P_{\sigma_k} S P_{\sigma_k} x\| \leq \frac{1}{2}\|x\|.$$

In particular,

$$\left| \|T P_{\sigma_k} x\|^2 - \|P_{\sigma_k} x\|^2 \right| \leq \frac{1}{2}\|P_{\sigma_k} x\|^2 \quad \text{for all } x.$$

Thus,

$$\frac{1}{2}\|P_{\sigma_k}(x)\|^2 \leq \|T P_{\sigma_k}(x)\|^2 \leq \frac{3}{2}\|P_{\sigma_k}(x)\|^2, \quad \text{for all } x.$$

By the definition of T, this implies that $\{f_i\}_{i \in \sigma_k}$ is a Riesz basic sequence with Riesz constant $\sqrt{2}$. The proposition is proved. ■

The Paving Conjecture is known to be true for various classes of operators T on l_2^n; see [B-Tz] for references. In particular, the Paving Conjecture is proved for the operators whose matrices have small entries, $O(1/\log^{1+\gamma} n)$ for some $\gamma > 0$.

Theorem 4.2 (Bourgain-Tzafriri). *Let S be a linear operator on l_2^n whose matrix has zero diagonal and all entries are bounded by $1/\log^{1+\gamma} n$ for some $\gamma > 0$. Then S satisfies the conclusion of the Paving Conjecture: there exists a partition $(\sigma_k)_{k \le M}$ of the set $\{1, \cdots, n\}$, where $M = M(\gamma, \varepsilon)$, and such that*

$$\|P_{\sigma_k} S P_{\sigma_k}\| \le \frac{1}{2}\|S\| \quad \text{for all } k.$$

Actually, the partition (σ_k) constructed by Bourgain and Tzafriri is random, i.e. σ_k is the image of the interval $\{1, \cdots, n/M\}$ under a random permutation π of the interval $\{1, \cdots, n\}$; for such a partition, the conclusion holds with probability close to one.

Theorem 4.2 implies the positive answer to Conjecture 2.2 for well separated sequences.

Corollary 4.3. *Let $\{f_i\}_{i=1}^n$ be a Bessel sequence with Bessel constant $B > 0$ and with $\|f_i\| = 1$ for all i. Assume that*

$$|\langle f_i, f_j \rangle| \le \frac{1}{\log^{1+\gamma} n} \quad \text{for all } i \ne j.$$

Then the sequence $\{f_i\}_{i=1}^n$ can be written as a union of $M = M(B, \gamma)$ Riesz basic sequences each with Riesz constant $K = K(B, \gamma)$.

Proof. This follows from Theorem 4.2 with an argument analogous to that of Proposition 4.1. ∎

5. Weyl-Heisenberg Frames

In this section we will show that Weyl-Heisenberg frames over rational lattices can always be written as finite unions of Riesz basic sequences. Gröchenig [G1] obtained this result under extra assumptions on the generator g.

Theorem 5.1. *Let $g \in L^2(\mathbb{R})$ and $0 < ab < 1$ with ab rational. If $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a Weyl-Heisenberg frame for $L^2(\mathbb{R})$ then it can be written as a finite union of Riesz basic sequences.*

Proof: After a change of variables we may assume that $b = 1$ and $a = \frac{p}{q}$ with $p, q \in \mathbb{N}$. We first reduce the problem to the case of integer oversampling. Notice that

$$\begin{aligned} \{E_m T_{\frac{1}{q}n} g\}_{m,n \in \mathcal{Z}} &= \cup_{k=0}^{p-1} \{E_m T_{\frac{1}{q}(np+k)} g\}_{m,n \in \mathcal{Z}} \\ &= \cup_{k=0}^{p-1} \{E_m T_{\frac{p}{q}n + \frac{k}{q}} g\}_{m,n \in \mathcal{Z}} \end{aligned}$$

Since each of the families $\{E_m T_{\frac{p}{q}n+\frac{k}{q}} g\}_{m,n \in \mathcal{Z}}$ is a Bessel sequence (and a frame for $k = 0$) we conclude that $\{E_m T_{\frac{1}{q}n} g\}_{m,n \in \mathcal{Z}}$ is a frame. In the rest of the proof we show that $\{E_m T_{\frac{1}{q}n} g\}_{m,n \in \mathcal{Z}}$ is a finite union of Riesz basic sequences. Since

$\{E_m T_{\frac{p}{q}n} g\}_{m,n\in\mathcal{Z}} \subset \{E_m T_{\frac{1}{q}n} g\}_{m,n\in\mathcal{Z}}$ the conclusion of the theorem follows from here.

We use a result by Ron and Shen [R-S], stating that $\{E_{qm}T_n g\}_{m,n\in\mathcal{Z}}$ is a Riesz basic sequence. Now,

$$\begin{aligned}\{E_m T_{\frac{n}{q}} g\}_{m,n\in\mathcal{Z}} &= \cup_{k=0}^{q-1}\{E_m T_{\frac{nq+k}{q}} g\}_{m,n\in\mathcal{Z}} \\ &= \cup_{k=0}^{q-1}\{E_m T_{n+\frac{k}{q}} g\}_{m,n\in\mathcal{Z}} \\ &= \cup_{k=0}^{q-1}\cup_{j=0}^{q}\{E_{qm+j} T_{n+\frac{k}{q}} g\}_{m,n\in\mathcal{Z}}\end{aligned}$$

By the commutator relations between the translation and modulation operators, $\{E_{qm+j}T_{n+\frac{k}{q}} g\}_{m,n\in\mathcal{Z}}$ is a Riesz sequence (with the same Riesz basis constants) as $\{E_{qm}T_n g\}_{m,n\in\mathcal{Z}})$ for all j, k, from which the result follows. □

Remark 5.2. *The proof of Theorem 5.1 shows that a frame* $\{E_m T_{\frac{p}{q}n} g\}_{m,n\in\mathcal{Z}}$ *is a union of* pq^2 *Riesz basic sequences.*

6. Concluding Remarks

It is possible that all these conjectures have negative answers. In this case it will be of interest to know the strongest results available. We will look at these now.

First we will show that bounded Bessel sequences can be decomposed into a finit union of linearly independent sets. For this, we need a result of Christensen and Lindner [CL].

Proposition 6.1. *Let* $K \in \mathcal{N}$, I *a finite subset of* $\mathcal{N}$ *and let* $\{f_i\}_{i\in I}$ *be a sequence of nonzero elements in a Hilbert space. The following are equivalent:*

(1) I *can be partitioned into* K *disjoint sets* $I_1, I_2, \cdots, I_K$ *so that each family* $\{f_i\}_{i\in I_j}$ $(j = 1, 2, \cdots, K)$ *is linearly independent.*

(2) For any nonempty subset $J \subset I$ *we have*

$$\frac{|J|}{dim\ span\{f_j : j \in J\}} \le K.$$

Proposition 6.2. *Every Bessel sequence* $\{f_i\}_{i\in I}$ *with Bessel bound* B *and* $\|f_i\| \ge c > 0$, *for every* $i \in I$ *can be decomposed into at most* $\lfloor B/c^2\rfloor + 1$ *linearly independent sets.*

Proof: We proceed by way of contradiction. Let $\{f_i\}_{i\in I}$ be a sequence with Bessel bound B and $\|f_i\| \ge c > 0$ which cannot be decomposed into $\lfloor B/c^2\rfloor + 1$ linearly independent sets. By Propositions 2.3 and 6.1, there is a finite subset $J \subset I$ so that

$$\frac{|J|}{\text{dim span}\{f_j : j \in J\}} \ge \lfloor\frac{B}{c^2}\rfloor + 1.$$

Now, $\{f_j/\|f_j\| : j \in J\}$ is a frame for its span with upper frame bound $B_J \leq \lfloor \frac{B}{c^2} \rfloor + 1$. Denote the corresponding frame operator by S. Now, the sum of the eigenvalues of S equals $|J|$ and since S has exactly dim span$\{f_j : j \in J\}$ eigenvalues (counted with multiplicity), it follows that the largest eigenvalue λ_{max} must satisfy:

$$\lambda_{max} \geq \frac{|J|}{\text{dim span}\{f_j : j \in J\}} > \lfloor \frac{B}{c^2} \rfloor + 1.$$

But, $\lambda_{max} \leq B_J$ and so $B_J > \lfloor \frac{B}{c^2} \rfloor + 1$, which is a contradiction. □

Next, we will observe that, up to a logarithmic factor, the generalized Bourgain-Tzafriri invertability theorem (and hence the (finite) Feichtinger conjecture) is true. We first note that we can iterate Theorem 1.2 to obtain:

Corollary 6.3. *There is a universal constant $c > 0$ and a function $d = d(\|T\|)$ so that whenever $T : \ell_2^n \to \ell_2^n$ is a linear operator for which $\|Te_i\| = 1$, for $1 \leq i \leq n$, where $\{e_i\}_{i=1}^n$ is an orthonormal basis for ℓ_2^n, then there is a partition $\{I_j\}_{j=1}^{d\ell n\ n}$ of $\{1, 2, \cdots, n\}$ so that for each $1 \leq j \leq d\ell n\ n$ and all choices of scalars $\{a_i\}_{i \in I_j}$ we have:*

$$\|\sum_{i \in I_j} a_i T e_i\|^2 \geq c \sum_{i \in I_j} |a_i|^2.$$

Proof: Let c be as in Theorem 1.2, $b = \frac{c}{\|T\|^2} < 1$ and let $d = \frac{-1}{\ell n(1-b)}$. By Theorem 1.2 we can find a set $I_1 \subset \{1, 2, \cdots, n\}$ with $|I_1| \geq bn$ and satisfying inequality (1.1). Let $J_1 = \{1, 2, \cdots, n\} - I_1$. Choose an (possibly into) isometry U_1 : Rng $T \to$ span $\{e_i\}_{i \in J_1}$. Then $UT : \ell_2^{J_1} \to \ell_2^{J_1}$ satisfies Theorem 1.2 so there is a set $I_2 \subset J_1$ with $|I_2| \geq b(1-b)n$ and satisfying inequality (1.1). Continuing, there are disjoint sets $\{I_j\}_{j=1}^{d\ell n\ n}$ with $|I_j| \geq b(1-b)^{j-1} n$ and each I_j satisfies inequality (1.1). It follows that $(1-b)^{d\ell n\ n} < \frac{1}{n}$ and

$$\sum_{j=1}^{d\ell n\ n} |I_j| \geq nd \sum_{j=1}^{d\ell n\ n} (1-b)^j = nb \frac{1 - (1-b)^{d\ell n\ n}}{d} > n[1 - \frac{1}{n}] \geq n - 1.$$

This completes the proof of the Corollary. □

Using Theorem 1.2, Casazza [C] showed:

Theorem 6.4. *There is a function $g(v, w, x) : \mathbb{R}^3 \to \mathbb{R}^+$ with the following property: Let $\{f_i\}_{i=1}^k$ be a frame for an n-dimensional Hilbert space H_n with frame bounds A, B, $\|f_i\| = 1$, for all $1 \leq i \leq k$, and let $0 < \epsilon < 1$. Then there is a subset $\sigma \subset \{1, 2, \cdots, n\}$ with $|\sigma| \geq (1-\epsilon)n$ so that $\{f_i\}_{i \in \sigma}$ is a Riesz basis for its span with Riesz basis constant $g(\epsilon, A, B)$.*

Vershynin [V 1], [V 2] removed the assumption in Theorem 6.4 that the frame elements be bounded above and below and got the conclusion that there is a "large" subset of the frame which is an *unconditional basis* for its span.

Precisely, he proved that for every frame $\{f_i\}$ in H_n and any $\varepsilon > 0$ there exists a set σ of cardinality $|\sigma| > (1-\varepsilon)n$, such that $\{f_i/\|f_i\|\}_{i\in\sigma}$ is a Riesz basic sequence whose Riesz basis constant depends on ε only.

Again we can iterate the proof of Theorem 6.4 to obtain:

Theorem 6.5. *There is a universal constant $K > 0$ and a $d = d(A,B)$ so that whenever $\{f_i\}_{i=1}^k$ is a frame for an n-dimensional Hilbert space $\mathcal{H}$ with $\|f_i\| = 1$ for all $1 \le i \le k$ and frame bounds A, B then there is a partition $\{I_j\}_{j=1}^{d\ell n\ n}$ of $\{1,2,\cdots,n\}$ so that for each $1 \le j \le d\ell n\ n$, $\{f_i\}_{i\in I_j}$ is a Riesz basic sequence with Riesz basis constant K.*

Theorem 6.5 says that, up to a logarithmic factor, the generalized Bourgain-Tzafriri invertability theorem (and hence the finite Feichtinger conjecture) is true.

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