CLASSIFYING TIGHT WEYL-HEISENBERG FRAMES

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Abstract. A Weyl-Heisenberg frame for $L^2(R)$ is a frame consisting of the set of translates and modulates of a fixed function in $L^2(R)$. i.e. $(E_{mb}T_{na}g)_{m,n\in Z}$, with $a, b > 0$, and $g \in L^2(R)$. WH-frames were introduced in 1946 by D. Gabor when he formulated a fundamental approach for signal decomposition in terms of elementary signals. Since then, a central question in this area has been to give necessary and sufficient conditions on g, a, b so that $(E_{mb}T_{na}g)_{m,n\in Z}$ forms a frame. In this paper we will give necessary and sufficient conditions for this family to form a tight WH-frame. Since every WH-frame is equivalent to a tight WH-frame, we hope that this will eventually lead to the solution to the general problem. We also give a corresponding classification for alternate dual frames to Weyl-Heisenberg frames.

1. Introduction

In 1946 D. Gabor [5] introduced a technique for signal processing which soon became a paradigm for the spectral analysis associated with time-frequency methods. His introduction of the short-time (windowed) Fourier transform led eventually to Wavelet theory. Gabor received the Nobel prize in 1971 for his fundamental work in signal analysis. In 1952, Duffin and Schaeffer [4] introduced frame theory which put Gabor's technique into the framework of a larger model - that of a frame for a Hilbert space.

A sequence $(f_i)_{i\in I}$ of elements of a Hilbert space H is called a **frame** for H if there are constants $A, B > 0$ so that

$$A\|f\|^2 \leq \sum_{i\in I} |<f, f_i>|^2 \leq B\|f\|^2, \quad \text{for all} \quad f \in H. \tag{1.1}$$

The numbers A, B are called the **lower** (resp. **upper**) frame bounds. The largest A and the smallest B satisfying (1.1) are called the **optimal frame bounds** and

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are denoted A_{op}, B_{op}. The frame is a **tight frame** if $A = B$ and a **normalized tight frame** if $A = B = 1$. For the basic results on frames we refer to [2,4,7].

Gabor's work gave rise to Gabor frames, also called Weyl-Heisenberg frames. Fix $a, b \in R$ and a function $g \in L^2(R)$. If the family $(E_{mb}T_{na}g)_{m,n\in Z}$ is a frame for $L^2(R)$ (see section 2 for notation) we call this frame a Weyl-Heisenberg frame or a WH-frame. A central question here is to classify the WH-frames. That is, to find necessary and sufficient conditions on g, a, b so that $(E_{mb}T_{na}g)_{m,n\in Z}$ is a frame for $L^2(R)$. In this paper we will give a complete classification of the normalized tight WH-frames. Since it is known that every WH-frame is equivalent to a normalized tight WH-frame, we hope that this will lead eventually to a solution to the general problem. The general conditions required for g, a, b to give a normalized tight WH-frame are given in Theorem 3.2. In section 4, we will give a concrete representation of the functions satisfying these conditions for a few cases. In section 5 we will give similar conditions which characterize all alternate dual WH-frames for a given WH-frame.

2. Preliminaries

If we replace f in (1.1) by f_j, we see that

$$\|f_j\|^4 + \sum_{i\in I-\{j\}} |< f_j, f_i >|^2 \leq B\|f_j\|^2.$$

This yields immediately the following remark:

Remark 2.1. *For all $j \in Z$, $\|f_j\|^2 \leq B$ and $\|f_j\|^2 = B$ implies $f_j \perp span_{i\neq j} f_i$. In particular, if $(f_i)_{i\in I}$ is a normalized tight frame, then $\|f_j\| \leq 1$ and $\|f_j\| = 1$ if and only if $f_j \perp span_{i\neq j} f_i$.*

If $(e_i)_{i\in I}$ is an orthonormal basis for H and $(f_i)_{i\in I}$ is a sequence of vectors in H, we will call the operator $T : \ell_2 \to H$ given by: $T(e_i) = f_i$ the **pre-frame operator** associated with $(f_i)_{i\in I}$. A direct calculation shows that

$$T^*f = \sum_{i\in I} < f, f_i > e_i, \quad \text{for all} \quad f \in H.$$

It follows that the pre-frame operator is bounded if and only if there is a constant $B > 0$ satisfying:

$$\sum_{i\in I} |< f, f_i >|^2 \leq B\|f\|^2, \quad \text{for all} \quad f \in H.$$

Moreover, $(f_i)_{i\in I}$ is a frame if and only if the pre-frame operator is a bounded, linear, onto map. The dimension of the kernel of T is called the **excess** of the frame. It follows that $S = TT^*$ is a self-adjoint, positive isomorphism of H onto H, called the **frame operator**, which has the form:

$$S(f) = \sum_{i\in I} < f, f_i > f_i, \quad \text{for all} \quad f \in H. \tag{2.2}$$

Now, for all $f \in H$ we have,

$$< Sf, f >= \sum_{i\in I} | < f, f_i > |^2. \tag{2.3}$$

One can check then that $(f_i)_{i\in I}$ is a normalized tight frame if and only if $S = I$. It is a straightforward calculation that for all $f \in H$,

$$f = \sum_{i\in I} < S^{-1}f, f_i > f_i = \sum_{i\in I} < f, S^{-1/2}f_i > S^{-1/2}f_i. \tag{2.4}$$

One interpretation of (2.4) is that $(S^{-1/2}f_i)_{i\in I}$ is a normalized tight frame for H which is equivalent to the frame $(f_i)_{i\in I}$.

The particular frames of interest to us will be the Weyl-Heisenberg frames. To define these frames, let $a, b \in R$ and define the operators of **modulation** E_b, and **translation** T_a for functions $f \in L^2(R)$ by:

$$E_b f(x) = e^{2\pi ibx} f(x),$$

and

$$T_a f(x) = f(x - a).$$

Given $g \in L^2(R)$, and $a, b > 0$, we say that (g, a, b) **generates a WH-frame for** $L^2(R)$ if $(E_{mb}T_{na}g)_{m,n\in Z}$ is a frame for $L^2(R)$. The function g is referred to as the **mother wavelet**. The numbers a, b are the **frame parameters**, with a being the **shift parameter** and b the **modulation parameter**.

Notation 2.2. *The class of functions we will be working with are those $g \in L^2(R)$ for which $(E_{mb}T_{na}g)_{m,n\in Z}$ has a finite upper frame bound. We call these functions the* **pre-frame functions** *and denote this class of functions by* **PF**.

It is easily seen that if $g \in$PF, then g is bounded. The class PF has not been identified yet but our earlier discussion on frame properties yields:

Proposition 2.3. *The following are equivalent:*

(1) $g \in$PF.

(2) The pre-frame operator $T : \ell_2 \to L^2(R)$ given by: $T(e_{m,n}) = E_{mb}T_{na}g$; is a bounded linear operator, where $(e_{m,n})_{m,n\in Z}$ is any orthonormal basis for ℓ_2.

(3) There is a constant $B > 0$ satisfying for all $f \in L^2(R)$:

$$\sum_{m,n} |< f, E_{mb}T_{na}g >|^2 \leq B\|f\|^2.$$

A minor modification of the proof of Theorem 2.1 of Casazza and Christensen [1] yields a sufficient condition for $g \in$PF:

Proposition (Casazza/Christensen [1]). *If*

$$\sum_{k\in Z} |\sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)}| \leq B \quad a.e.,$$

then $g \in$PF.

It can be shown that a variation of this produces a necessary condition for $g \in$PF. That is, if $g \in$PF, then,

$$\sum_{k\in Z} |\sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)}|^2 < \infty.$$

We will not show that this condition is necessary since it is clearly not strong enough to classify PF. Actually, we conjecture that the sufficient condition of Casazza and Christensen is also necessary for $g \in$PF.

In the above proposition and in the rest of the paper, we will be working with infinite sums of the form:

$$\sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)}.$$

So let us discuss the question of convergence here, and ignore it for the rest of the paper.

Proposition 2.4. *If $g \in L^2(R)$ then the series*

$$\sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)}$$

converges unconditionally a.e. $x \in R$.

Proof. We just compute,

$$\|g\|^2 = \int_R |g(x)|^2 \, dx = \int_0^a \sum_{n\in Z} |g(x-na)|^2 \, dx < \infty.$$

It follows that

$$G(x) = \sum_{n\in Z} |g(x-na)|^2 < \infty \ \text{ a.e..}$$

Applying Holder's Inequality, we get that

$$\sum_{n\in Z} |g(x-na)\overline{g(x-na-k/b)}| < \infty \ \text{ a.e.}$$

We need a result that appeared the first time in [7]:

WH-Frame Identity. *If* $g \in L^2(R)$ *and* $f \in L^2(R)$ *is bounded and compactly supported, then*

$$\sum_{n\in Z}\sum_{m\in Z} | < f, E_{mb}T_{na}g > |^2 = F_1(f) + F_2(f) \tag{2.5}$$

where

$$F_1(f) = b^{-1}\int_R |f(x)|^2 \sum_n |g(x-na)|^2 \, dx, \tag{2.6}$$

$$F_2(f) = b^{-1}\sum_{k\neq 0}\int_R \overline{f(x)}f(x-k/b)\sum_n g(x-na)\overline{g(x-na-k/b)} \, dx = \tag{2.7}$$

$$b^{-1}\sum_{k\geq 1} 2Re\int_R \overline{f(x)}f(x-k/b)\sum_n g(x-na)\overline{g(x-na-k/b)} \, dx.$$

To simplify the notation a little we introduce the following auxilliary functions:

$$G(x) = \sum_{n\in Z} |g(x-na)|^2, \tag{2.8}$$

and for all $k \in Z$,

$$G_k(x) = \sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)}, \tag{2.9}$$

It follows that $G_0 = G$, and G_k are periodic functions on R of period a.

3. Classifying Tight WH-Frames

Now we will classify the tight WH-frames both abstractly in terms of the behavior of related families of vectors and concretely in terms of the behavior of the functions $G_k(x)$. In the next section we will write down explicitly the functions which satisfy the conditions of our theorem for some cases. We start with a basic fact which will simplify parts of the theorem and will be used in section 5 to classify the alternate dual frames for a WH-frame.

Proposition 3.1. *Let $g, h \in L^2(R)$ and $a, b \in R$.*

(1) $h \perp E_{mb}g$, for all $m \neq 0$ if and only if there is a constant C so that

$$\sum_{n \in Z} h(x - n/b)\overline{g(x - n/b)} = C, \quad a.e..$$

(2) If $n \neq 0$, then $h \perp E_{mb}T_{na}g$, for all $m \in Z$ if and only if,

$$\sum_{k} h(x - k/b)\overline{g(x - k/b - na)} = 0, \quad a.e..$$

Proof. (1): We have that $h \perp E_{mb}g$, for all $m \neq 0$ if and only if

$$0 =< h, E_{mb}g >= \int_R h(x)\overline{E_{mb}g(x)}\ dx = \int_R h(x)\overline{g(x)}E_{-mb}\ dx =$$

$$\int_0^{1/b} \sum_{n \in Z} h(x - n/b)\overline{g(x - n/b)}E_{-mb}\ dx, \quad \text{for all} \quad m \in Z.$$

(1) now follows.

(2): Similar to (1) we have that $h \perp E_{mb}T_{na}g$, for all $m \in Z$ and $n \neq 0$ if and only if

(3.1)

$$0 =< h, E_{mb}T_{na}g >= \int_R h(x)\overline{E_{mb}g(x - na)}\ dx = \int_R h(x)\overline{g(x - na)}E_{-mb}\ dx =$$

$$\int_0^{1/b} \sum_{k \in Z} h(x - k/b)\overline{g(x - k/b - na)}E_{-mb}\ dx, \quad \text{for all} \quad m \in Z.$$

Now, (3.1) is equivalent to

$$\sum_{k \in Z} h(x - k/b)\overline{g(x - k/b - na)} = 0, \quad \text{a.e..}$$

We are now ready to prove the classification theorem for tight WH-frames.

Theorem 3.2. *Let $g \in L^2(R)$ and $a, b \in R$. The following are equivalent:*

(1) $(E_{mb}T_{na}g)_{n,m\in Z}$ is a normalized tight Weyl Heisenberg frame for $L^2(R)$.

(2) We have:

(a) $G(x) = \sum_{n\in Z} |g(x-na)|^2 = b$ a.e.,

(b) $G_k(x) = \sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)} = 0$ a.e. for all $k \neq 0$.

(3) We have, $g \perp E_{n/a}T_{m/b}g$, for all $(n,m) \neq (0,0)$, and $\|g\|^2 = ab$.

(4) $(E_{n/a}T_{m/b}g)_{n,m\in Z}$ is an orthogonal sequence in $L^2(R)$ and $\|g\|^2 = ab$.

Moreover, in case the conditions are satisfied, $(E_{mb}T_{na}g)_{m,n\in Z}$ is an orthonormal basis for $L^2(R)$ if and only if $\|g\| = 1$.

Proof. (1) $\Rightarrow$ (2): Assume $(E_{mb}T_{na}g)_{n,m\in Z}$ is a normalized tight frame for $L^2(R)$. For any function $f \in L^2(R)$ which is bounded and supported on an interval of length $< 1/b$ we have for all $x \in R$ and all $0 \neq k \in Z$ that $\overline{f(x)}f(x-k/b) = 0$. That is, $F_2(f) = 0$. Now, by the WH-frame Identity:

$$\|f\|^2 = \int_R |f(x)|^2 \, dx = \sum_n \sum_m |< f, E_{mb}T_{na}g >|^2 = F_1(f) + F_2(f)$$

$$= b^{-1} \int_R |f(x)|^2 \sum_n |g(x-na)|^2 \, dx = b^{-1} \int_R |f(x)|^2 G(x) \, dx.$$

Since this equality holds for all bounded $f \in L^2(I)$, for any interval I of length $< 1/b$, it follows easily that $G(x) = b$, a.e. Hence, for all $f \in L^2(R)$, $F_1(f) = \|f\|^2$. But now, again by the WH-frame Identity, we have for all bounded, compactly supported $f \in L^2(R)$,

$$\|f\|^2 = F_1(f) + F_2(f) = \|f\|^2 + F_2(f).$$

That is, $F_2(f) = 0$, for all bounded compactly supported $f \in L^2(R)$. Now fix $k_0 \geq 1$ and let I be any interval in R of length $\leq 1/b$. Define a function $f \in L^2(R)$ by:

$$f(x) = e^{i \arg H_{k_0}(x)}, \quad \text{for all} \quad x \in I,$$

and $f(x - k_0/b) = 1$ for all $x \in I$ and $f(x) = 0$, otherwise. Then, by the WH-frame Identity,

$$0 = F_2(f) = b^{-1} \sum_{k\geq 1} 2\text{Re} \int_R \overline{f(x)} f(x-k/b) \sum_n g(x-na)\overline{g(x-na-k/b)} \, dx =$$

$$b^{-1}2\text{Re}\int_R \overline{f(x)}f(x-k_0/b)H_{k_0}(x)\ dx = b^{-1}2\int_I |H_{k_0}(x)|\ dx$$

It follows that $H_{k_0}(x)=0$, a.e. on I. Since k_0 and I were arbitrary, we have $(2b)$.

(2) $\Rightarrow$ (1): By assumption $(2b)$ and the WH-frame Identity, we have $F_2(f)=0$ for all bounded, compactly supported $f\in L^2(R)$. Hence, applying assumption $(2a)$ and the WH-frame Identity:

$$\sum_n\sum_m |< f, E_{mb}T_{na}g >|^2 = F_1(f) = b^{-1}\int_R |f(x)|^2\sum_n |g(x-na)|^2\ dx =$$

$$\int_R |f(x)|^2\ dx = \|f\|^2.$$

Since this equality holds on a dense subset of $L^2(R)$, it holds for all $f\in L^2(R)$. So $(E_{mb}T_{na}g)_{n,m\in Z}$ is a normalized tight frame for $L^2(R)$.

(2) $\Leftrightarrow$ (3): By (2) of Proposition 3.1, (2b) is equivalent to $g\perp E_{n/a}T_{m/b}g$, for all $m\neq 0$. Also, we have,

$$\|g\|^2 = \int_R |g(x)|^2\ dx = \int_0^a \sum_{n\in Z}|g(x-na)|^2\ dx = \int_0^a G(x)\ dx.$$

So the rest of the equivalence follows from (1) of Proposition 3.1.

(3) $\Leftrightarrow$ (4): This is immediate from the observation that for all $m,n,\ell,k\in Z$ we have:

$$< E_{n/a}T_{m/b}g, E_{k/a}T_{\ell/b}g >= e^{2\pi i\frac{n-k}{a}\frac{m}{b}} < g, E_{\frac{k-n}{a}}T_{\frac{\ell-m}{b}}g > .$$

For the moreover part of the theorem, we just observe that for all $m,n\in Z$, $\|g\| = \|E_{mb}T_{na}g\|$. Hence, if $\|g\|=1$, then our tight frame consists of norm 1 elements which now form an orthonormal basis by Remark 2.1.

Now we can give one simple class of functions which produce normalized tight WH-frames.

Corollary 3.3. *Let $a,b\in R$, and assume that $g\in L^2(R)$ is supported on an interval I with $|I|\leq\frac{1}{a}$. If*

$$\sum_{m\in Z}|g(x-mb)|^2 = a\quad a.e.$$

then $(E_{mb}T_{na}\hat{g})$ is a normalized tight WH-frame.

Proof. By Theorem 3.2, the conditions stated in Corollary 3.3 guarantee that $(E_{na}T_{mb}g)$ is a normalized tight WH-frame for $L^2(R)$. Taking the Fourier Transform of this yields that $(E_{mb}T_{na}\hat{g})$ is a normalized tight WH-frame.

Since $ab \leq 1$ for a WH-frame (see below), it follows that $b \leq 1/a$. So it is easy to produce many functions satisfying the conditions in Corollary 3.3.

With the characterization in Theorem 3.2, we can now recover easily and directly from the definition the basic properties of WH-frames which formerly required some heavy machinery. Recently, Janssen [8,9] has derived some of these results also in an easier fashion. For any WH-frame $(E_{mb}T_{na}g)_{m,n\in Z}$ with frame operator S, a direct computation shows that

$$S(E_{mb}T_{na}g) = E_{mb}T_{na}Sg, \quad \text{for all} \;\; m,n \in Z.$$

It follows that $S^{-1/2}$ also commutes with $E_{mb}T_{na}$ and so $(E_{mb}T_{na}S^{-1/2}g)_{m,n\in Z}$ is a normalized tight WH-frame which is equivalent to $(E_{mb}T_{na}g)_{m,n\in Z}$ and hence must satisfy the conditions of Theorem 3.2. In particular, for all $(m,n) \neq (0,0)$ we have:

$$< S^{-1}g, E_{n/a}T_{m/b}g > = < S^{-1/2}g, E_{n/a}T_{m/b}S^{-1/2}g > = 0.$$

Corollary 3.4. *Let $(E_{mb}T_{na}g)_{m,n\in Z}$ be a WH-frame for $L^2(R)$.*

(1) $S^{-1}g \perp E_{n/a}T_{m/b}g$, for all $(m,n) \neq (0,0)$ and

$$< S^{-1}g, g > = < S^{-1/2}g, S^{-1/2}g > = \|S^{-1/2}g\|^2 = ab \leq 1.$$

(2) If $ab < 1$ then the WH-frame is overcomplete.

(3) If $ab = 1$ then the WH-frame is a Riesz basis for $L^2(R)$.

Proof. (1): All of this was observed before we stated the theorem except the last inequality $ab \leq 1$ which follows immediately from Remark 2.1.

(2): If our WH-frame is exact then $(E_{mb}T_{na}S^{-1/2}g)$ is an orthonormal basis. Hence,

$$1 = \|S^{-1/2}g\|^2 = ab.$$

(3): If $ab = 1$ then $(E_{mb}T_{na}S^{-1/2}g)$ is a tight WH-frame and hence $(E_{n/a}T_{m/b}S^{-1/2}g)$ is an orthogonal sequence by Theorem 3.2 (4). But, $ab = 1$ implies that $n/a = nb$ and $m/b = na$, so $(E_{nb}T_{ma}S^{-1/2}g)$ is an orthogonal basis for $L^2(R)$. Since $S^{-1/2}$ is an isomorphism, it follows that $(E_{mb}T_{na}g)$ is a Riesz basis for $L^2(R)$.

Rieffel [10] proves a stronger statement than (1) of Corollary 3.4. Using Von Neumann algebra theory he shows that if $ab > 1$ then for any $g \in L^2(R)$ there is an $f \in L^2(R)$ so that $f \perp \text{span}_{m,n\in Z}(E_{mb}T_{na}g)$.

Finally, we point out what appears to be a surprising consequence of Theorem 3.2 which seems to indicate the existence of a relationship between the values of a function and its Fourier Transform which we have not seen before.

Corollary 3.5. *If $g \in L^2(R)$ and $ab \leq 1$, the following are equivalent:*

(1) The function g satisfies:

$$\sum_n g(x-na)\overline{g(x-na-k/b)} = 0 \quad a.e. \text{ for all } \quad k \neq 0.$$

and

$$\sum_n |g(x-na)|^2 = b \quad a.e.$$

(2) The Fourier transform of the function g satisfies:

$$\sum_n \hat{g}(x-nb)\overline{\hat{g}(x-nb-k/a)} = 0 \quad a.e. \text{ for all } \quad k \neq 0.$$

and

$$\sum_n |\hat{g}(x-nb)|^2 = a \quad a.e.$$

Proof. Since the Fourier transform of$E_{mb}T_{na}g$ is $E_{ma}T_{nb}\hat{g}$, Corollary 3.5 comes from Theorem 3.2 applied to the families $(E_{mb}T_{na}g)$ and $(E_{ma}T_{nb}\hat{g})$ which are normalized tight WH-frames together.

4. The Functions g Giving Tight WH-Frames

In this section we will give an explicit representation for the functions giving tight WH-frames given in Theorem 3.2 (2) for two specific cases. If $f(x,y)$ is any function of two variables, we denote by $f_y(x)$ the function:

$$f_y(x) = f(x,y).$$

We start with a simple proposition which contains the basic notions which will be used in our characterization.

Proposition 4.1. *The following are equivalent:*

(1) The sequence $z = (c_n)_{n\in Z} \in \ell_2(Z)$ is orthogonal to all of its proper shifts and $\|z\|^2 = b$.

(2) The unique function $h : [0,1] \to C$ with $\hat{h}(n) = c_n$, for all $n \in Z$ has $|h(x)|^2 = b$ a.e.

(3) There is a measurable function $f : [0,1] \to R$ so that the function h in (2) is of the form

$$h(x) = \sqrt{b}\ e^{2\pi i f(x)}.$$

Proof. (1) $\Rightarrow$ (2) If $(\hat{h}(n)) = (c_n)$ is orthogonal to all its shifts then $h \perp e^{2\pi i m}h$, for all $m \neq 0$. That is, for all $m \neq 0$ we have

$$0 =< h, e^{2\pi i m}h >=< |h|^2, e^{2\pi i m} >$$

Hence, $|h|^2 = C$ a.e. Also, $\|h\|^2 = \|(c_n)\|^2 = b$.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (1): Given h as in (3), for any $m \in Z$ we have:

$$< h, E_m h >= \int_0^1 h(x)\overline{h(x)E_m}\, dx = \int_0^1 |h(x)|^2 e^{-2\pi i m x} dx =$$

$$\int_0^1 b\; e^{-2\pi i m x} dx = b\delta_m.$$

So h is orthogonal to $e^{2\pi i m}h$ for all $m \neq 0$. Hence, $(\hat{h}(n))$ is orthogonal to all its proper shifts and $\|(\hat{h})(n))\|^2 = b$.

Now we are ready to give an explicit representation for the functions g giving tight WH-frames.

Theorem 4.2. *Let $ab = 1$ and $g \in L^2(R)$.. The following are equivalent:*

(1) $(E_{mb}T_{na}g)$ is a normalized tight WH-frame for $L^2(R)$.

(2) There is a measurable function $f : [0,1] \times [0,a) \to R$ and

$$h(x,y) = \sqrt{b}\; e^{2\pi i f(x,y)}$$

so that

$$g(y + na) = \hat{h_y}(n), \quad \textit{for all} \;\; y \in [0,a).$$

Proof. This is essentially immediate from our assumptions and Theorems 3.2 and 4.1. Since $ab = 1$, we have $a = 1/b$ so Theorem 3.2 (2) becomes:

(a) $G(x) = \sum_{n \in Z} |g(x - na)|^2 = b$, a.e.,

(b) $G_k(x) = \sum_{n \in Z} g(x - na)\overline{g(x - (n-k)a)} = 0$, a.e. for all $k \neq 0$.

But condition (b) is equivalent to: $z_y = (g(y - na))_{n \in Z}$ is orthogonal to all of its proper shifts and (a) is equivalent to $\|z_y\|^2 = b$. By Theorem 4.1 these conditions are equivalent to: For each $y \in [0,a)$ there is a function $f_y : [0,1] \to R$ and $h_y : [0,a] \to C$ with

$$h_y(x) = e^{2\pi i f_y(x)},$$

and

$$\hat{h_y}(n) = g(y - na).$$

So defining $f(x, y) : [0, 1] \times [0, a) \to R$ and $h(x, y)$ by:

$$f(x, y) = f_y(x), \quad h(x, y) = h_y(x),$$

yields the theorem modulo the measurability conditions which are obvious.

The condition in Theorem 4.2 is new, so we will now show that it quickly and easily gives the standard examples from the literature on tight WH-frames as well as new examples.

Example 4.3. *Letting $f(x, y) = C$, for all $y \in [0, a]$ gives the function*

$$g(x) = \sqrt{b}\ \chi_{[0,a)}$$

which yields a normalized tight frame (in fact, an orthonormal basis) for $L^2(R)$. Letting

$$f(x, y) = C_y, \quad \text{for all} \quad x \in [o, 1],$$

gives a function g of the form:

$$g(x) = \sqrt{b} e^{ih(x)} \chi_{[0,a)}$$

where $h : [0, 1] \to R$.

Example 4.4. *Letting $f(x, y) = f(x, y')$, for all $y, y' \in [0, a]$ gives the function*

$$g(x) = \sum_{n \in Z} c_n \chi_{[na,(n+1)a)}$$

where (c_n) is an ℓ_2 sequence which is orthogonal to all its proper shifts. This now yields a normalized tight WH-frame.

Example 4.5. *If we partition $[0, a)$ into disjoint measurable sets $(A_n)_{n \in Z}$, let $B_n = A_n + \{n\}$ and define*

$$f(x, y) = d_n + nx, \quad \text{for all} \quad y \in A_n.$$

Then

$$g(x) = \sum_{n \in Z} c_n \chi_{B_n},$$

yields a normalized tight frame.

We leave it to the reader to check other variations for the functions $f(x, y)$ to yield old and new examples of tight WH-frames.

We proceed to the next level of the representations of the functions g giving WH-frames.

Proposition 4.6. *Let $ab = 1/q$, $q \in Z$ and $g \in L^2(R)$. The following are equivalent:*

(1) $(E_{mb}T_{na}g)$ is a normalized tight WH-frame for $L^2(R)$.

(2) There are functions $f_j(x,y) : [0,1] \times [0,a) \to C$, $1 \le j \le q$ satisfying:

(a)
$$\sum_{j=1}^{q} |f_{j,y}(x)|^2 = b, \quad \textit{for all} \;\; x \in [0,1].$$

(b)
$$g(y + ja + n/b) = \hat{f_{j,y}}(n), \quad \textit{for all} \;\; 1 \le j \le q, \quad 0 \le y < \frac{1}{qb} = a.$$

Proof. We leave it to the reader to check that g is a well-defined function on all of R. For $0 \le y < \frac{1}{qb} = a$ we have:

(4.1)
$$\sum_{n\in Z} g(y-na)\overline{g(x-na-k/b)} = \sum_{j=1}^{q}\sum_{n\in Z} g(y+ja+nqa)\overline{g(y+ja+nqa+k/b)} =$$

$$\sum_{j=1}^{q}\sum_{n\in Z} g(y+ja+n/b)\overline{g(y+ja+(n+k)/b)} = \sum_{j=1}^{q}\sum_{n\in Z} \hat{f_{j,y}}(n)\overline{\hat{f_{j,y}}(n+k)}.$$

Now, we have the equality in (4.1) equal to 0 for all $k \neq 0$ if and only if:

$$0 = \sum_{j=1}^{q} < f_{j,y}, E_k f_{j,y} >= \int_0^1 \sum_{j=1}^{q} |f_{j,y}(x)|^2 E_k \; dx, \quad \text{for all} \;\; k \neq 0.$$

But, this is equivalent to:

$$\sum_{j=1}^{q} |f_{j,y}(x)|^2 = C_y, \quad \text{for all} \;\; x \in [0,1]$$

for some constant C_y. But, for $k = 0$ and all $0 \le y < 1/kb$ we have:

(4.2)
$$\sum_{n\in Z} |g(y-na)|^2 = \sum_{j=1}^{q}\sum_{n\in Z} |\hat{f_{j,y}}(n)|^2 = \sum_{j=1}^{q} \|f_{j,y}\|^2 =$$

$$\int_0^1 \sum_{j=1}^{q} |f_{j,y}(x)|^2 \; dx = C_y.$$

But, we have a normalized tight WH-frame if and only if the sums in (4.2) equal b a.e. i.e. if and only if $C_y = b$ for all $y \in [0,a)$.

We leave it to the reader to examine the special cases of Proposition 4.6. Common examples are when all the functions f_j are equal or when they are each a constant function.

One could probably find a representation similar to that of Proposition 4.6 for the case where ab = p/q is a rational number. In this case one would need pq-functions $(f_i)_{i=1}^{pq}$ with various equations being satisfied by subsets of them. It is not clear whether such a representation might be too complicated to use in practice. We hope, however, that someone finds a good representation for this case. The case where ab is irrational seems to be a very difficult case and we have not found any good general examples what-so-ever for this case. Yet, we know that every WH-frame is equivalent to a normalized tight WH-frame and so there must be a huge number of such functions even for irrational ab. It would be very interesting to have a characterization of this class of functions.

5. Alternate Dual Frames

If $(f_i)_{i\in I}$ is a frame for a Hilbert space H, a frame $(h_i)_{i\in I}$ for H is called an **alternate dual frame** or a **pseudo-dual** for $(f_i)_{i\in I}$ if

$$f = \sum_{i\in I} < f, h_i > f_i, \quad \text{for all} \quad f \in H. \tag{5.1}$$

We already know one sequence $(h_i)_{i\in I}$ satisfying (5.1). Namely, the sequence $(S^{-1}f_i)_{i\in I}$. That is, for all $f \in H$,

$$f = S(S^{-1}f) = \sum_{i\in I} < S^{-1}f, f_i > f_i = \sum_{i\in I} < f, S^{-1}f_i > f_i.$$

We call $(S^{-1}f_i)_{i\in I}$ the **canonical dual** of (f_i). If $(f_i)_{i\in I}$ is a normalized tight frame, then $S = I$, so the frame equals its canonical dual frame. The converse of this clearly holds also. But in general, there may be many alternate dual frames for a given frame. A simple example would be to let our frame be $\{e_1, e_1, e_2, e_2, \dots\}$ where (e_i) is an orthonormal basis for H, and and observe that $\{e_1, 0, e_2, 0, \dots\}$ is an alternate dual frame while $\{e_1/2, e_1/2, e_2/2, e_2/2, \dots\}$ is the canonical dual frame. In the definition we assumed that our sequence $(h_i)_{i\in I}$ is a frame for H. This assumption is necessary. For example,

$$\{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \dots\}$$

is a normalized tight frame for H and the sequence $(h_i)_{i\in I}$ given by,

$$\{e_1, \sqrt{2}e_2, 0, \sqrt{3}e_3, 0, 0, \dots\}$$

is not a frame for H but satisfies (5.1). However, this is essentially all that can go wrong here. That is, if $(f_i)_{i\in I}$ and $(h_i)_{i\in I}$ are sequences of vectors in H each of which has a bounded preframe operator (i.e. $Te_i = f_i$ and $T_0 e_i = h_i$ are bounded operators) and they satisfy (5.1), then $(h_i)_{i\in I}$ and $(f_i)_{i\in I}$ are alternate dual frames. For the basic properties of alternate dual frames we refer to [6,8,11]. It is known, for example, that a frame has a unique alternate dual frame if and only if it is a Riesz basis. Also, no two distinct alternate dual frames are equivalent to one another. So the canonical dual frame is the only alternate dual frame which is equivalent to the original frame. Finally, if $(h_i)_{i\in I}$ is an alternate dual frame to $(f_i)_{i\in I}$, then $(f_i)_{i\in I}$ is an alternate dual frame to $(h_i)_{i\in I}$. Now we will use the techniques developed in section 3 to characterize the Weyl-Heisenberg alternate dual frames for a given Weyl-Heisenberg frame. We need a beautiful result of Wexler-Raz [11] (see also Janssen [8,9]):

Theorem(Wexler-Raz [11]). *Let $g, h \in PF$. Then $(E_{mb}T_{na}h)$ and $(E_{n/a}T_{m/b}g)$ are alternate dual frames if and only if $h \perp E_{n/a}T_{m/b}g$, for all $(m,n) \neq (0,0)$, and $< h, g >= ab$.*

We proceed with the corresponding result to Theorem 3.2 for alternate dual frames.

Theorem 5.1. *For $g, h \in L^2(R)$ the following are equivalent:*

(1) $(E_{mb}T_{na}h)_{m,n\in Z}$ is an alternate dual frame for $(E_{mb}T_{na}g)_{m,n\in Z}$.

(2) We have:

(a) $\sum_{n\in Z} h(x-na)\overline{g(x-na-k/b)} = 0$ a.e. for all $k \neq 0$.

(b) $\sum_{n\in Z} h(x-na)\overline{g(x-na)} = b$, a.e.

(3) $h = S^{-1}g + f$, where $f \in L^2(R)$ and $f \perp span_{n,m\in Z} E_{n/a}T_{m/b}g$.

Proof. (1) $\Leftrightarrow$ (2): This is Proposition 3.1 combined with the theorem of Wexler-Raz.

(1) $\Rightarrow$ (3): By the Wexler-Raz Theorem, $h \perp E_{n/a}T_{m/b}g$, for all $(m,n) \neq (0,0)$. By Corollary 3.3, we also have that $S^{-1}g \perp E_{n/a}T_{m/b}g$. Hence, $f = h - S^{-1}g \perp$

$E_{n/a}T_{m/b}g$. Again by the Wexler-Raz Theorem, $< h, g >= ab$ and applying Corollary 3.3 again,

$$< h - S^{-1}g, g >=< h, g > + < S^{-1}g, g >= ab - ab = 0.$$

It follows that $h = S^{-1}g + (h - S^{-1}g) = S^{-1}g + f$ and $f \perp E_{n/a}T_{m/b}g$, for all $n, m \in Z$.

(3) $\Rightarrow$ (1): Fix $(m, n) \neq (0, 0)$. We compute using Corollary 3.3:

$$< h, E_{n/a}T_{m/b}g >=< S^{-1}g + f, E_{n/a}T_{m/b}g >=$$

$$< S^{-1}g, E_{n/a}T_{m/b}g > + < f, E_{n/a}T_{m/b}g >= 0 + 0 = 0.$$

Also, using Corollary 3.3,

$$< h, g >=< S^{-1}g + f, g >=< S^{-1}g, g > + < f, g >= ab + 0 = ab.$$

So this implication follows from the Wexler-Raz Theorem.

Note that $(E_{mb}T_{na}g)_{m,n \in Z}$ is a normalized tight frame if and only if we can replace h in Theorem 5.2 by the function g and in this case Theorem 5.1 reduces to Theorem 3.2. Also in this case, $S = I$ so part (3) of the theorem becomes: $h = g + f$ where $f \perp \text{span}_{n,m \in Z} E_{n/a}T_{m/b}g$.

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