

REPRINT

ALABAMA
JOURNAL
OF
MATHEMATICS

VOL. 7 No. 2
Fall 1983

01/9
8653
7420
13591
0286
873
40519
6728
1648
5089
435
789
5217
0849
650

MAXIMIZING AREA WITHOUT CALCULUS

By P. G. Casazza

An important part of any calculus course involves maximizing areas subject to given constraints. Unfortunately, few students have previous experience with trying to maximize area. As a consequence, they find this topic slightly confusing. Moreover, since they have not tried to do similar problems without calculus, they do not realize how much is gained by using the tools of calculus. In this note we will use geometric, intuitive and elementary algebraic arguments to identify certain figures with maximum area and a fixed perimeter. This can easily be done in an elementary geometry course or used as a “lead-in” to maximization in a calculus course. The arguments presented here are not exact “proofs” (for example, showing that the circle is the figure of maximal area with a fixed perimeter requires a sophisticated argument in the calculus of variations and we will “prove” it here with a simple picture), but they appeal to intuition, are visual, are very convincing, and will get the students thinking.

Let us begin with a simple problem:

Since the rabbits ate all my vegetables last year, I have decided to fence an area of a field behind my house and build my garden inside of it. I go to the barn and discover that I have one 42-meter straight and rigid section of fence left over from a previous project and forty-eight 1-meter sections. How should I put up my fence so as to have the largest possible garden?

If you give this problem to the students, most will immediately “see” there are just 3 possibilities for the dimensions of the garden (see figure 1): $3 \times 42 = 126\text{m}^2$, $2 \times 43 = 86\text{m}^2$ and $1 \times 44 = 44\text{m}^2$.

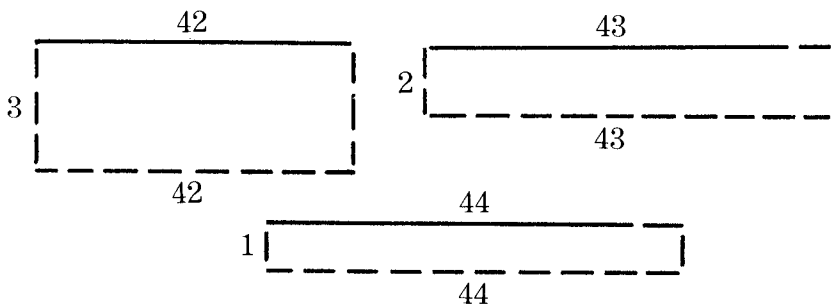


Figure 1

So the maximum area must be 126m^2 ? Of course not! In fact, if I leave the 42-meter section of fence in the barn and put up the forty-eight 1-meter sections in a square, I can enclose $12 \times 12 = 144\text{m}^2$. So what is the maximum? To find out, we need to examine some elementary figures. Since the students may have forgotten how elementary areas are found, we might review this procedure briefly with them. We assume they know that the area of a rectangle of length b and width h is bh . It follows that the area of a right triangle with legs of lengths b and h is $\frac{bh}{2}$. (Since we merely put two such triangles together to form a rectangle with area bh — see figure 2)

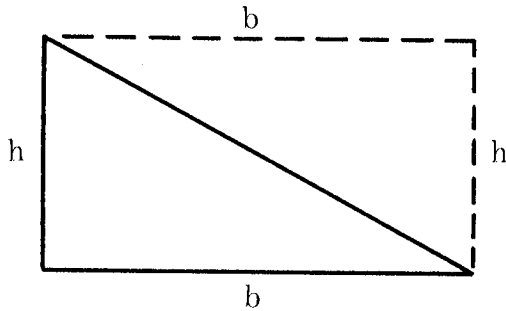


Figure 2

Next consider the parallelogram of length b and altitude h . (see figure 3)

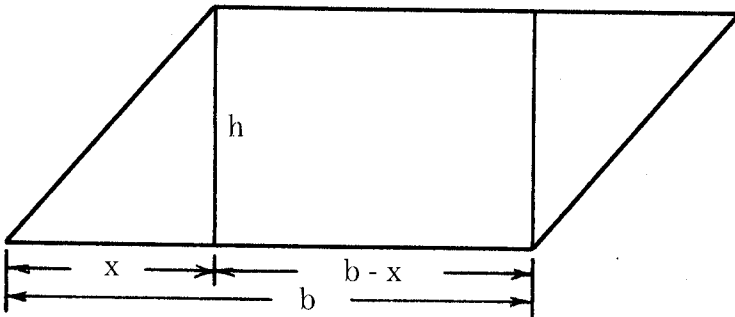


Figure 3

Letting x be the indicated length, the area of the parallelogram is the area of the center square $h(b-x)$ plus twice the area of the end triangle $2(\frac{hx}{2})$. So the area of the parallelogram is $bh - hx + hx = bh$.

Putting two triangles together to form a parallelogram (see figure 4) shows that the area of any triangle of altitude h and base b is $\frac{bh}{2}$.

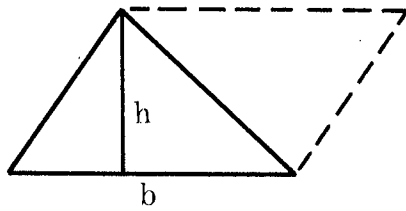


Figure 4

These are, of course, trivial arguments. But, we overlooked an interesting fact in the process. Suppose we fix the lengths of the sides of the parallelogram and let the angle θ increase to θ_1 (see figure 5).

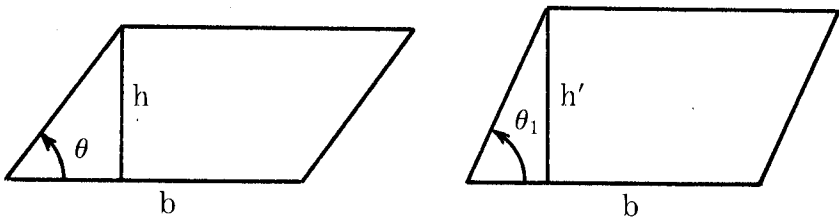


Figure 5

Then clearly our altitude h' is larger than h and so the area of the second parallelogram (bh') is larger than the area of the first (bh). That is, we have discovered,

Theorem 1: The area of a parallelogram increases as its acute angle increases towards 90° (i.e. area is an increasing function of the acute angle) and hence, the parallelogram (with fixed length sides) of maximum area is the rectangle.

Next let us consider a rectangle of fixed perimeter p , length b and width h , where for convenience, we assume that $b > h$. So $2b + 2h = p$ and its area A is,

$$\begin{aligned} A = bh &= b\left(\frac{p-2b}{2}\right) = -\left(b^2 - \frac{bp}{2}\right) \\ &= -\{(b - p/4)^2 - (p/4)^2\} \\ &= (p/4)^2 - (b - p/4)^2. \end{aligned}$$

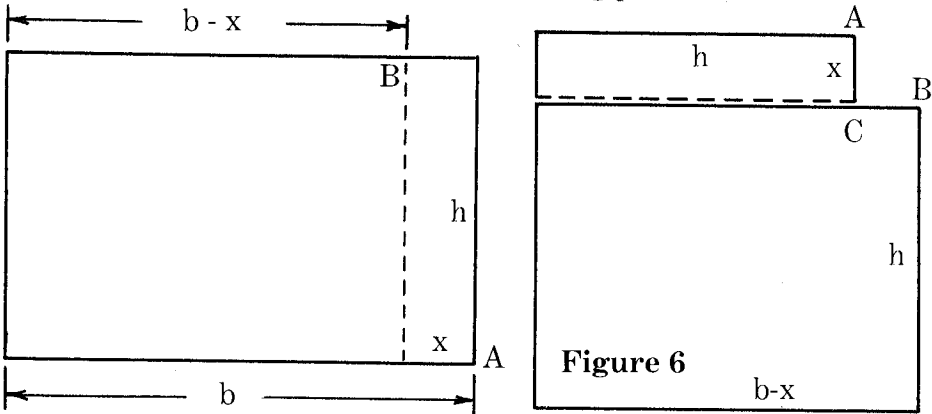
As b decreases to $p/4$, $(b - p/4)^2$ decreases to 0 and so A increases to $(p/4)^2$. That is,

Theorem 2: The area of a rectangle of fixed perimeter increases as the length of its length b decreases towards its width h (that is, when the perimeter is fixed and $b > h$, area is a decreasing function of the length b). So the unique rectangle of maximum area with fixed perimeter p is the square with maximum area $\frac{p^2}{16}$.

Theorems 1 and 2 give the corollary,

Corollary 3: The parallelogram of largest area with a fixed perimeter is a square.

Theorem 2 can be justified by an intuitive geometric argument. Assume we have a rectangle with length b and width h and $b > h$. We cut a portion off the end of the rectangle (see figure 6) so that $h + x = b - x$. So, we've cut a section with dimensions h and $\frac{b-h}{2}$ off the end and put this piece on top of the remaining portion.



Note that the new figure has the same area and the same perimeter (the latter is not as obvious as the former) as the original rectangle but is not now a rectangle. But we can construct a square with this same perimeter by replacing line AC by line DB and replacing line CB by line AD (see figure 7). Clearly this gives a figure with larger area and the same perimeter as before. So as long as $b > h$, we do not have the rectangle of maximum area.

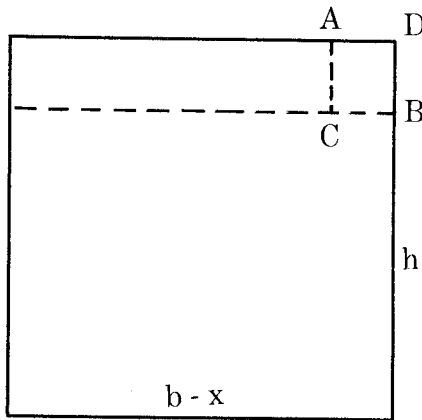


Figure 7

A variation of this problem occurs when we have a “free-side” on our rectangle. These are also standard problems in calculus.

Suppose a farmer has p meters of fence and he wishes to fence a rectangular plot along the bank of a river (see figure 8). What are the dimensions of the maximum area he can enclose? (Note that he does not fence along the river banks).

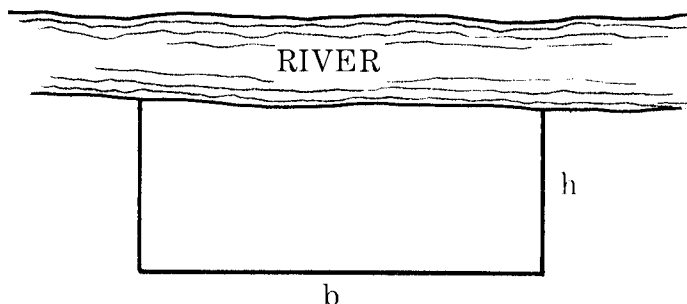


Figure 8

We could solve this problem algebraically just as we did the rectangle problem only now our equations become $b + 2h = p$ and $bh = A$. But, this result is actually clear from what we have already done. Just consider the rectangle of base b and height $2h$ obtained by “reflecting” the fence in the river (see figure 9). The area of this rectangle is exactly twice the area of the fenced region and its perimeter is exactly twice the fixed length of the fence. When the area of the rectangle is maximum, the area of its lower half — our fenced region — will also be maximum. (Note that this conclusion follows only because the “reflection” of a rectangle is also a rectangle. As we will see later, this same “technique” applied to a triangle could lead to a false conclusion.) Since the rectangle of maximum area is a square, it follows that h must be $\frac{b}{2}$. We also get uniqueness here by the uniqueness of the maximum area for a rectangle being a square.

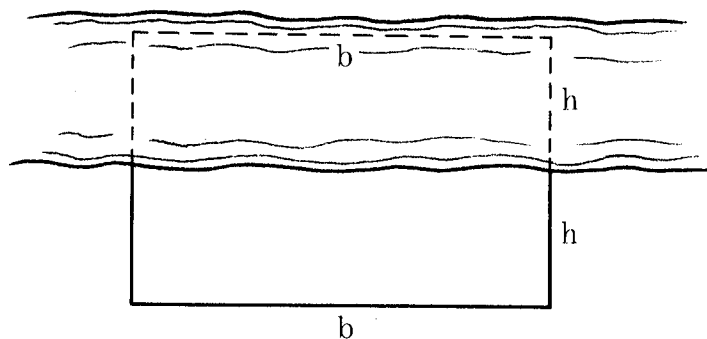


Figure 9

Theorem 4: The largest rectangular area enclosed by a fixed length p for three sides of a rectangle is the rectangle whose length is twice its width.

Now we can return to our original problem. If we use part of the 42 meter section of fence as one boundary of a rectangle, then, just as in the river problem, the most efficient use of the 48 one meter sections is illustrated in figure 10.

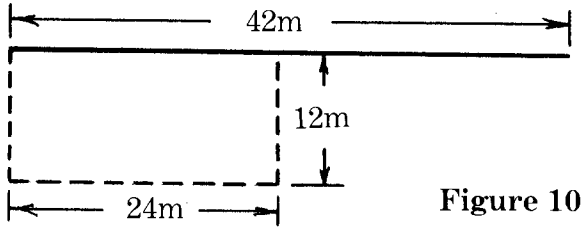


Figure 10

The area enclosed here is $12 \times 24 = 288\text{m}^2$. But, is this really the maximum? Let us continue. Perhaps we should not restrict ourselves to rectangles. Can we do any better by using triangles?

We found that the rectangle of maximum area is a square. By analogy, we would expect that the triangle of maximum area would be an equilateral triangle. An equilateral triangle of perimeter p has height $\frac{p}{3} \frac{\sqrt{3}}{2}$ and hence area $\frac{1}{2} \left(\frac{p}{3} \frac{\sqrt{3}}{2} \right) \frac{p}{3} = \frac{p^2 \sqrt{3}}{18 \cdot 2} < \frac{p^2}{18} < \frac{p^2}{16}$. Since $\frac{p^2}{16}$ is the largest rectangular area enclosed by a perimeter p , we would expect to enclose less area with a triangle than a rectangle for a fixed perimeter. Since the area of a right triangle with fixed legs of lengths a and b is $\frac{ab}{2}$, the argument of Theorem 1 (see figure 11) gives:

Corollary 5: The area of a triangle, with fixed sides of lengths a and b and angle θ between them, is an increasing function of θ . Hence, the triangle with maximum area and two fixed sides is a right triangle.

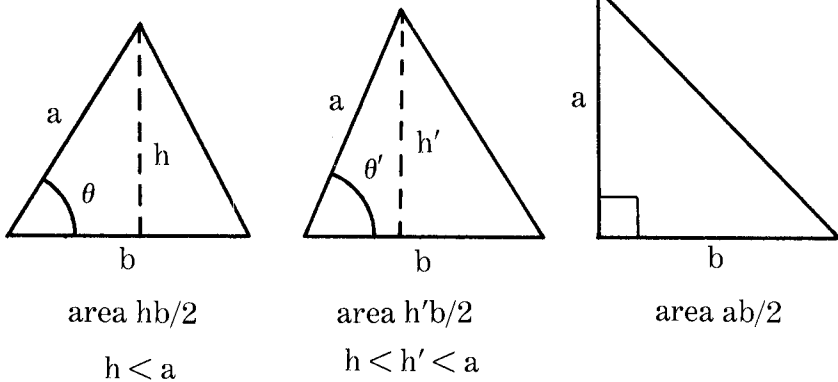


Figure 11

Now suppose we have a right triangle with legs of lengths h and b and hypotenuse c . (see figure 12)

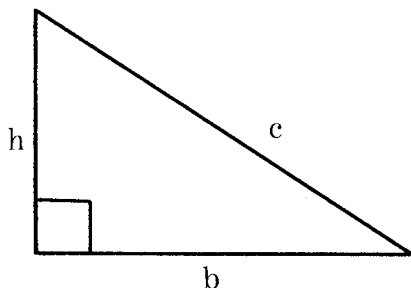


Figure 12

perimeter $p = h + b + c$

Then $c < h + b$. Also, since $(h - b)^2 = h^2 - 2hb + b^2 \geq 0$, we have $2hb \leq h^2 + b^2$. Thus, $(h + b)^2 = h^2 + 2hb + b^2 \leq 2(h^2 + b^2) = 2c^2$ (by the Pythagorean Theorem) and it follows that $h + b \leq \sqrt{2}c$. Since $h + b + c = p$, $(h + b)^2 = (p - c)^2$ and so $h^2 + 2hb + b^2 = p^2 - 2pc + c^2$. Since $h^2 + b^2 = c^2$, $2hb = p^2 - 2pc$ and so the area A of this triangle is

$$A = \frac{hb}{2} = \frac{(p^2 - 2pc)}{4}$$

It follows that as c decreases, $p^2 - 2pc$ increases and therefore the area of the triangle increases as the length of its hypotenuse decreases.

Theorem 6: The area of a right triangle with fixed perimeter p is maximized when the length of the hypotenuse is a minimum.

It follows that the maximum area occurs when $h + b = \sqrt{2}c$ and from the calculations above, this occurs when $h = b$.

Theorem 7: The largest area enclosed by a right triangle of perimeter p is the isosceles right triangle (with legs of length $\frac{\sqrt{2}}{2}c$ where c is the hypotenuse).

Corollary 5 and Theorem 7 seem to imply that the triangle of largest area with a fixed perimeter is a right triangle. This is false, in fact, as we will now show. First, let us consider triangles with fixed base b and the sum of the lengths of the other two sides s . (see figure 13).

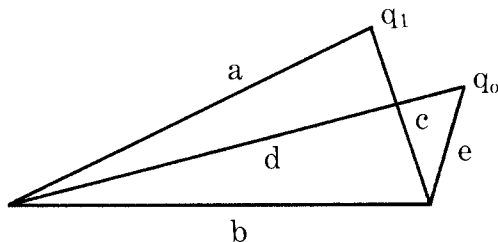


Figure 13

$$a + c = d + e = s$$

Note that the set of points q_0, q_1 , etc. forms the locus of points for which the sum of their distances from two fixed points is a constant s . That is, they sweep-out an ellipse (see figure 14).

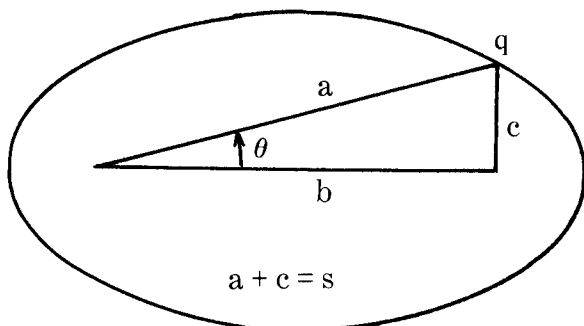


Figure 14

The area of any such triangle is its altitude times its base (which is b for all of them). It is also clear that the altitudes of these triangles are an increasing function of θ as θ increases to 90° , so we have,

Theorem 8: The unique triangle of largest area with a fixed base and the sum of the other two sides a constant is an isosceles triangle with that base.

Now suppose triangle ABC encloses the maximum area for its perimeter $a + b + c = p$. Then, in particular, it must enclose the maximum area for a triangle of fixed base b and lengths of the other two sides $a + c = p - b$. Hence, it is the unique isosceles triangle with $a = c$. Repeating this argument with base a gives $a = c$ and we have

Theorem 9: The unique triangle of maximum area and fixed perimeter is the equilateral triangle (with sides of length $p/3$).

Recall that we maximized area when fencing against a river by “reflecting” the region and maximizing the area of the new rectangle. Let us try this argument on a triangle. That is, suppose we want to maximize the area enclosed in a triangular region by a fence of length s along a river bank (see figure 15).

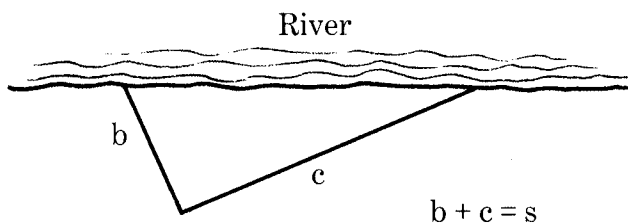


Figure 15

- One way to use the argument of Theorem 4 is to “reflect” this figure in the river and try to maximize the area of the enclosed region. However, there is an immediate problem in that the new region is not a parallelogram (see figure 16).

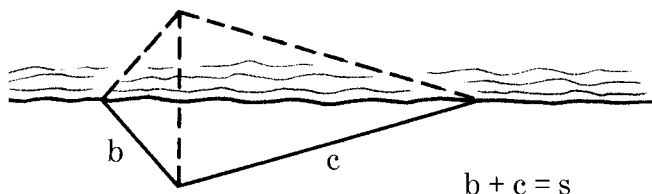


Figure 16

Although the maximum area enclosed by a quadrilateral of fixed perimeter is a square, we have not shown this.

Another approach might be to assert that the triangle of maximum area with fixed perimeter is an equilateral triangle. Maximizing this area for perimeter $2s$ should maximize half its area with the sum of the lengths of 2 of its sides equal to s (see figure 17).

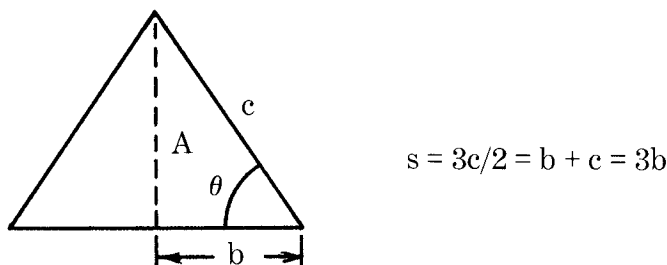


Figure 17

But angle θ here is 60° and we know that the maximum occurs when $\theta = 90^\circ$. So what happened? The argument was correct but our interpretation of the conclusion was wrong. We do not have the maximum triangular area fenced against a river, but instead we have the maximum triangular area fenced against a river with the additional constraint that one section of fence must be perpendicular to the river (see figure 18, which illustrates the maximum triangular area fenced against a river with one side perpendicular to the river).

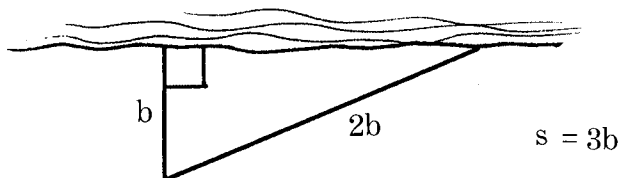


Figure 18

Note how this result compares to Theorem 4.

By Theorem 9 and the discussion preceding Corollary 5 we can state

Theorem 10: For a fixed perimeter, the square with this perimeter encloses a larger area than any triangle with this perimeter.

Now suppose we have n straight line segments of variable lengths but the sum of their lengths is a constant p . Assume we have connected them together to enclose the maximum area possible. Let us look at any 3 consecutive points where the lines join together. It is clear that the figure “bows-out” between these points (in mathematical language, the figure is said to be convex). For otherwise (see figure 19) replacing the line segments AB and BC by the equal length segments AD and DC will produce a figure of larger area (but uses the original set of line segments) and so has the same perimeter.

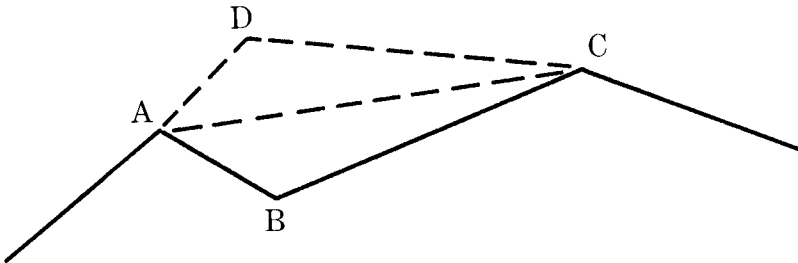


Figure 19

Our curve must look like figure 20 and clearly the triangle ABC is of maximum area for its base AC and the sum of the lengths of the other two sides equal to $AB + BC$ (see figure 20).

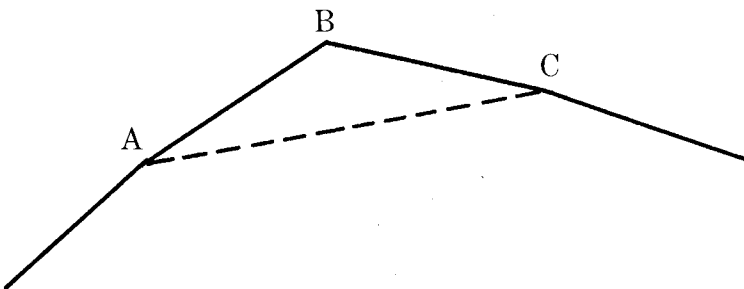


Figure 20

For if not, we could move the point B until the triangle ABC is isosceles, and has the sum of the lengths $\overline{AB} + \overline{BC}$ unchanged and get a figure with the same perimeter as our original but with larger area. It follows from Theorem 8 that $\overline{AB} = \overline{BC}$. But then, since every pair of adjacent sides must have equal length, we see that all sides are equal.

Theorem 11: The n-sided figure with fixed perimeter which encloses the largest area is the regular n-gon.

Finally, let us assume we have enclosed the largest area which can be enclosed with a fixed perimeter p with no restrictions on the shape of the region. Pick two points a and b on our figure so that the perimeter is cut in half (see figure 21).

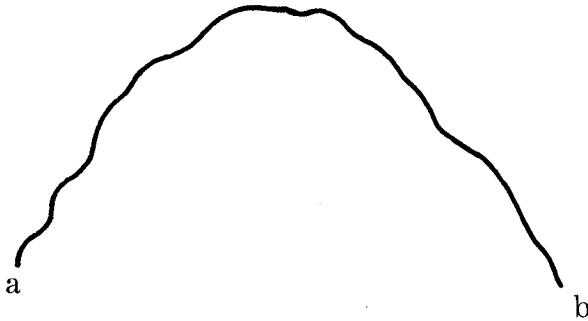


Figure 21

Note that we may assume that the figure is symmetric around the line through a and b . For if not, by merely reflecting the half with larger area around this line we have a symmetric figure with the same perimeter as the original and larger area. Pick a point d on the locus and think of this figure as having a hinge at point d (see figure 22).

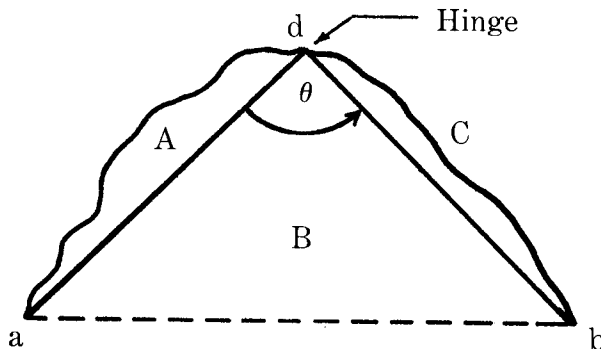


Figure 22

By opening and closing the “hinge” (see figure 23) we are varying the area B but leaving areas A and C fixed and also leaving the outer perimeter fixed. (The other half of the figure, symmetric to Figures 22 and 23 and thus not illustrated, has a hinge at the point d' symmetric to d .) Therefore, for the half of the figure illustrated, the total area $A + B + C$ will be maximized when the area B is maximized and this occurs (see Corollary 5) when the angle θ is a right angle. Hence θ is a right angle for all points d if our figure encloses the maximum area.

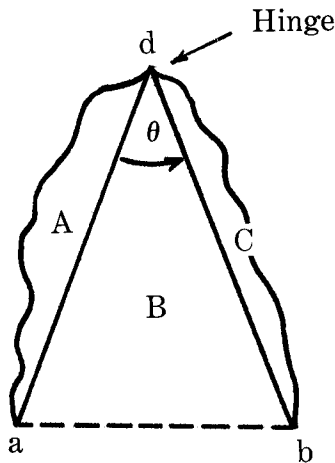


Figure 23

Now we merely need to recall a result from geometry: Any locus of points with the property that there exist two fixed points a and b such that for any point d on the locus the line ad is perpendicular to the line bd , must be a circle. If this result is not obvious, we can prove it analytically by putting in an axis system as in figure 24 where our two points are placed at $(-c,0)$ and $(c,0)$.

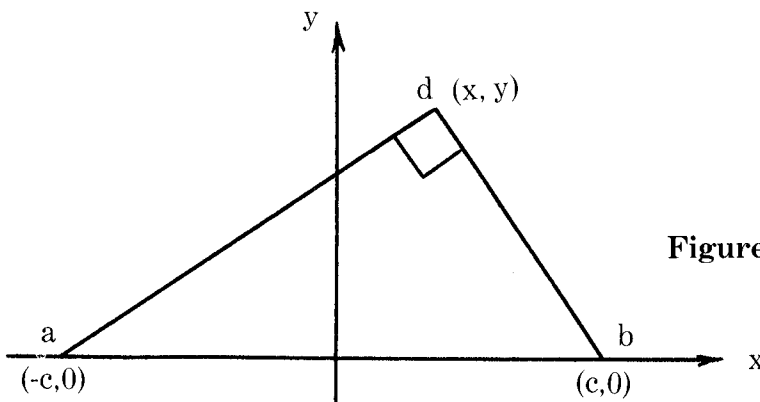


Figure 24

By the Pythagorean Theorem,

$$(x + c)^2 + y^2 + (x - c)^2 + y^2 = (2c)^2.$$

Hence, $2x^2 + 2y^2 = 2c^2$ and so $x^2 + y^2 = c^2$.

Theorem 12: The largest area enclosed by a fixed perimeter is a circle.

By the same type of symmetry argument used to prove Theorem 4, we have

Theorem 13: The largest area enclosed against a flat wall with a fixed length of fence is a semicircle.

Returning once again to our original problem, we see that the maximum area should be enclosed as in figure 25 if we could make a perfect semicircle out of our 48 one meter sections of fence.

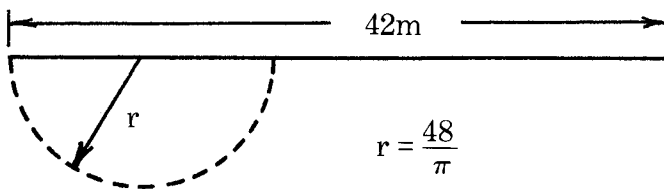


Figure 25

Hence, we could enclose an area of $\frac{\pi}{2} \left(\frac{48}{\pi} \right)^2 \approx 367\text{m}^2$ by this method.

So the student's original estimate of 126m^2 as the maximal possible area was considerably off. Since we treated our fence as if it formed a semi-circle when it really forms itself of a regular 96-gon, our estimate is a little optimistic but not too much so.

One of the main uses of calculus is in finding maximums and minimums, for engineering and science problems, for problems in economics and business, and for many other problems. As we have seen, we can find certain maximums and minimums by constructing various ad hoc geometric arguments. But calculus will give us a much more systematic approach to maximum and minimum problems. Having worked through the results of this paper, students should be ready to appreciate and enjoy the power which the calculus brings to bear on maximum and minimum problems.

Department of Mathematics
University of Alabama
University, AL 35486