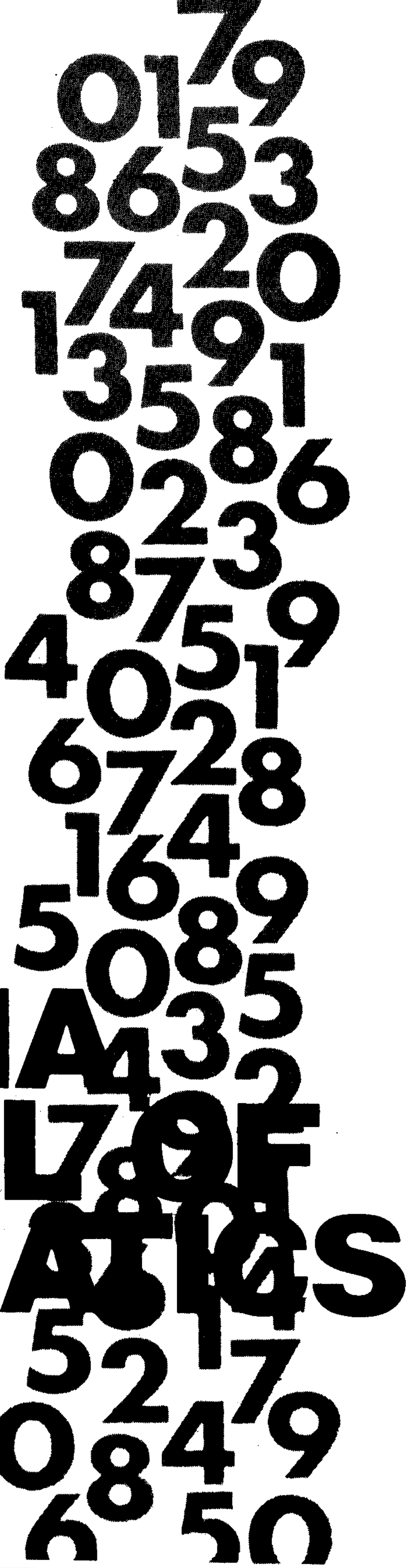
REPRINT

FROM

# ALABAMA JOURNAL OF MATHEMATICS

Vol. 5, No. 1
Spring 1981



# The Irrationality of $\pi$ and e

### BY GINA PHILLIPS

In 1761, J. H. Lambert used continued fractions to give the first proof of the irrationality of $\pi$. Since that time, there have been dozens of books written about $\pi$; high school students have formed clubs to study it; and in 1967 two French mathematicians, Jean Gilloud and Michele Dichamp, had a computer crank out the first 500,000 decimals of $\pi$ (which by the way took 444 hours and 45 minutes to compute at a cost of about two cents per decimal — not including programming costs). It is therefore very disappointing that almost no students have ever seen a proof of the irrationality of $\pi$, despite the fact that there are short proofs in existence which require nothing more than a knowledge of Calculus. In this note, we will review how the irrationality of $\pi$ and e can easily be done by alloting only a few minutes of your time in a calculus course. The style of both these proofs is quite similar and both are elementary.

(1) *The Irrationality of e.* This proof can be done immediately after you have derived the power series expansion for $e^x$ as being $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Therefore, $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$. The only formula required is $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $0 < |x| < 1$.

For each n, if we let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$ be the (n + 1) –st partial sum of the series for e then,

$$\begin{aligned}
0 < e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\
&\leqslant \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots\right) \\
&= \frac{1}{(n+1)!}\;\frac{1}{1 - (\frac{1}{n+1})} \\
&= \frac{1}{(n+1)!} \cdot \frac{n+1}{n} = \frac{1}{n!n} \qquad (1)
\end{aligned}$$

Now let us assume that e is rational and get a contradiction.

If e is rational, then $e = \frac{m}{n}$ for some natural numbers m,n with $n > 1$. Then by (1) above,

$$0 < n!\,(e - s_n) < n!\frac{1}{n!n} = \frac{1}{n} < 1. \qquad (2)$$

But,

$$\begin{aligned} n!\,(e-s_n) &= n!\frac{m}{n} - n!\,s_n \\ &= (n-1)!m - n!\,(1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}) \\ &= (n-1)!m - (n!+n!+\frac{n!}{2!}+\cdots+\frac{n!}{n!}) \qquad (3) \end{aligned}$$

Since every number in the sum on the right side of (3) is an integer, it follows that $n!\,(e-s_n)$ is an integer. But by (2), $0 < n!\,(e-s_n) < 1$, and so we have produced an integer between 0 and 1. This contradiction completes the proof of the irrationality of e.

You might mention that there is a more general result which says that $e^r$ is irrational for all non-zero rational numbers r.

(2) *The Irrationality of* $\pi$. This proof is due to I. Niven and can be done as soon as the students know how to integrate and differentiate the trigonometric functions and know that $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0$ for all real numbers a. If you want to wait and do e and $\pi$ together, then this last fact can be established easily by using the ratio test to show that the series $\sum\limits_{n=0}^{\infty} \frac{a^n}{n!}$ is absolutely convergent and consequently that the limit of the $n^{th}$ term is zero. It is clear that if r is a rational number then so is $r^2$. Hence, if $r^2$ is irrational then so is r. Thus it suffices to prove the stronger result that $\pi^2$ is irrational. We will proceed as above by assuming that $\pi^2$ is rational and produce an integer between 0 and 1 for a contradiction. So assume $\pi^2 = \frac{p}{q}$ where p, q are natural numbers.

Since $\lim\limits_{n\to\infty} \frac{\pi p^n}{n!} = 0$, choose a natural number n so that

$$\frac{\pi p^n}{n!} < 1. \qquad (4)$$

Since for $0 \leqslant k \leqslant n$, $\frac{n(n-1)\cdots(n-k+1)}{k!}$ is the $(k+1)$ –st coefficient in the binomial expansion of $(a+b)^n$, this is an integer. Hence,

$$\frac{(n+k)!}{n!} \cdot \frac{n(n-1)\cdots(n-k+1)}{k!}$$

is a product of two integers and so it also is an integer. If we let $g(x) = c\,x^{n+k}$ where $c = \frac{n(n-1)\cdots(n-k+1)}{n!\,k!}$, then for any natural number j, the $j^{th}$ derivative of g(x) evaluated at 0 is,

$$g^{(j)}(0) = \begin{cases} 0 & \text{if } j \neq n+k \\ c\,(n+k)! & \text{if } j = n+k. \end{cases} \qquad (5)$$

That is, $g^{(j)}(0)$ is an integer for every j. Now define

$$f(x) = \frac{x^n (1-x)^n}{n!}$$

Then for $0 < x < 1$, it follows that $0 < x^n < 1$ and $0 < (1-x)^n < 1$, therefore

$$0 < f(x) < \frac{1}{n!}. \tag{6}$$

Since $0 \leqslant \sin x$ for all $0 \leqslant x \leqslant 1$, (4) and (6) above imply,

$$0 < \pi p^n \int_0^1 f(x)\sin\pi x \, dx \leqslant \pi p^n \int_0^1 \frac{1}{n!} \, dx = \frac{\pi p^n}{n!} < 1. \tag{7}$$

Also, by the binomial therem,

$$\begin{aligned} f(x) &= \frac{x^n}{n!}\Big(1 - nx + \frac{n(n-1)}{2!}x^2 + \cdots + (-1)^k \frac{n(n-1)\cdots(n-k+1)}{k!}x^k + \\ &\qquad \cdots + (-1)^n x^n\Big) \\ &= \frac{1}{n!}x^n - \frac{n}{n!}x^{n+1} + \frac{n(n-1)}{n!\,2!}x^{n+2} + \cdots + \\ &\qquad (-1)^k \frac{n(n-1)\cdots(n-k+1)}{n!\,k!}x^{n+k} + \cdots + \frac{(-1)^n}{n!}x^n \end{aligned}$$

By (5), $f^{(j)}(0)$ is a sum of $n + 1$ integers for every j and hence is an integer. Since $f(1-x) = f(x)$ we have that $f^{(j)}(1)$ is also an integer for every j. Now define

$$\begin{aligned} F(x) = q^n \{&\pi^{2n} f(x) - \pi^{2(n-1)} f^{(2)}(x) + \pi^{2(n-2)} f^{(4)}(x) \quad - \\ &\cdots + (-1)^n f^{(2n)}(x)\}. \end{aligned}$$

Since $\pi^{2(n-k)} = \dfrac{p^{n-k}}{q^{n-k}}$ for every k, $q^n \pi^{2(n-k)}$ is an integer for every k and hence from what we know about the behavior of $f^{(j)}(x)$ for $x = 0,1$, it follows that F(0) and F(1) are integers. Therefore,

$$F(0) + F(1) \text{ is an integer.} \tag{8}$$

We now finish the proof in the following manner. We will use the function F(x) above to give explicitly an antiderivative for the function $\pi p^n f(x)\sin\pi x$. Using this antiderivative, we will show that $\pi p^n \int_0^1 f(x)\sin\pi x\, dx = F(0) + F(1)$. But the left side of this equality is in (0,1) by (7) while the right side is in an integer by (8). This contradiction will complete the proof of the irrationality of $\pi^2$, and hence $\pi$. To compute the required antiderivative we note that

$$F''(x) = q^n \{\pi^{2n} f^{(2)}(k) - \pi^{2(n-1)} f^{(4)}(x) + \cdots + (-1)^n f^{(2n+2)}(x)\}$$

and

$$\pi^2 F(x) = q^n \{\pi^{2n+2} f(x) - \pi^{2n} f^{(2)}(x) + \cdots + (-1)^n \pi^2 f^{(2n)}(x)\}$$

imply

$$F''(x) + \pi^2 F(x) = q^n \pi^{2n+2} f(x).$$

Therefore, by the product rule for derivatives,

$$\begin{aligned}\frac{d}{dx}\left\{\frac{F'(x)\sin \pi x - \pi F(x)\cos \pi x}{\pi}\right\} &= \frac{F''(x) + \pi^2 F(x)}{\pi} \sin\pi x \\ &= q^n \pi^{2n+1} f(x)\sin\pi x \\ &= \pi q^n \frac{p^n}{q^n} f(x)\sin\pi x \\ &= \pi p^n f(x)\sin\pi x. \qquad (9)\end{aligned}$$

Therefore, $\dfrac{F'(x)\sin \pi x - \pi F(x)\cos \pi x}{\pi}$ is an antiderivative of $\pi p^n f(x)\sin\pi x$ and so

$$\begin{aligned}\pi p^n \int_0^1 f(x)\sin\pi x\,dx &= \left[\frac{F'(x)\sin\pi x}{\pi} - F(x)\cos\pi x\right]_0^1 \\ &= F(0) + F(1).\end{aligned}$$

That is,

$$\pi p^n \int_0^1 f(x)\sin\pi x\,dx$$

is an integer (by (8)) which lies between 0 and 1 (by (7)). This contradiction completes the proof of the irrationality of $\pi$.

There is actually a much more general result than the one given here. That is, it can be shown that cos r is irrational for any non-zero rational number r. By letting $r = \pi$, we see that $\cos \pi = -1$ is rational and hence $\pi$ must be irrational.

Department of Mathematics
University of North Alabama
Florence, AL 36630