MATH PROJECTS

This section contains projects a teacher can give to a bright high school student who wants to learn more about a topic that is being studied in school. Manuscripts submitted for this section should have as a central theme some topic normally found in the high school curriculum; the project should be self-contained, including necessary hints and references. A complete set of answers should be submitted with the project. The answers will not normally be published, but should be available so that the teachers can write to the Journal for a copy. Manuscripts for this section should be sent to the managing editor of the Journal.

RIGHT TRIANGLES AND THE PYTHAGOREAN THEOREM

by P. G. Casazza

Pythagorean Theorem: *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the legs. (i.e.* $c^2 = a^2 + b^2$ *in Figure 1.)*

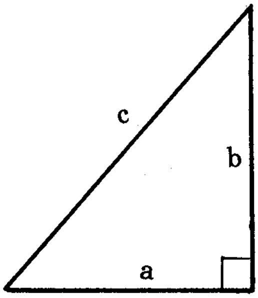


Figure 1

In this project you will study some variations of the Pythagorean Theorem as well as some consequences of it. It is assumed that you have already studied the Theorem and have done some of the standard problems related to it.

Part I: This part of the project is designed to review some of the standard properties of right triangles which will be used in the rest of the project. If you have already done these in class, you may skip them.

Problem 1

Prove that if one angle of a triangle equals the sum of the other two angles, then the triangle is a right triangle.
Hint: Use the fact that the sum of the angles of a triangle is 180°.

Problem 2

Prove that the altitude to the hypotenuse of a right triangle divides it into two triangles, each of which is similar to the whole triangle (and hence similar to each other).
Hint: Check that all these triangles have the same 3 angles.

Problem 3

Prove that if one acute angle of a right triangle is double the other, then the hypotenuse is double the shorter side.
Hint: Prove that the triangle is a 30°-60°-90° right triangle.

Problem 4

Prove that the following inequalities hold in any right triangle labeled as in Figure 1.

$$c < a + b \leq \sqrt{2}\, c.$$

Also, show that $a + b = \sqrt{2}\, c$ if and only if $a = b$.
Hint: For the right hand inequality, expand $0 \leq (a - b)^2$ and deduce that $2ab \leq a^2 + b^2$, where the equality holds if and only if $a = b$. Now expand $(a + b)^2$ and use the Pythagorean Theorem together with the above inequality to get the final result.

Part II: In this part of the project you will discover and use some variations of the Pythagorean Theorem.

Problem 5

(The converse of the Pythagorean Theorem)

Prove that if the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.
Hint: Draw a right triangle with legs a and b and hypotenuse of length x. By the assumption and the Pythagorean Theorem, $x = c$. Now isn't this triangle congruent to an arbitrary triangle with sides of length a, b, c?

Although this Theorem is generally not used as often as the Pythagorean Theorem, there are people who use it almost daily in their work and who probably do not know the statement of the Theorem. Carpenters often check to see if a large structure has a "square" corner by measuring from a corner 3 feet down one side, 4 feet down another and checking to see if the "hypotenuse" of the triangle they form is 5 feet. If it is, they know they have a true right angle. You might talk to a carpenter about this technique and others they have for guaranteeing "their angles are true".

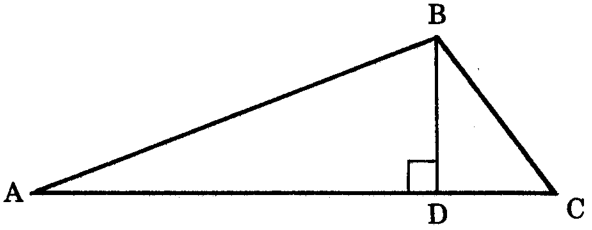
Problem 6

(Generalized Pythagorean Theorem for Arbitrary Triangles)

Given an arbitrary triangle ABC with altitude BD (see figure), prove that

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 - 2\overline{AC} \cdot \overline{AD}$$

Hint: Apply the Pythagorean Theorem to triangles ABD and BDC and set like terms equal. Then square each side of the equation $\overline{DC} = \overline{AC} - \overline{AD}$.. Combine your results.

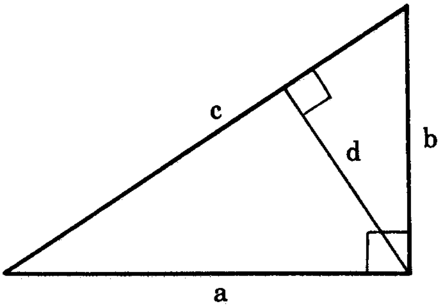


Problem 7

(Inverse Pythagorean Theorem)

Prove: Given a right triangle ABC with legs a, b and altitude drawn to the hypotenuse of length d (see figure), show that:

$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2}$$



Hint: Divide each side of the Pythagorean Theorem by a^2b^2. Then use the result of problem 2 to show that $\frac{a}{c} = \frac{d}{b}$

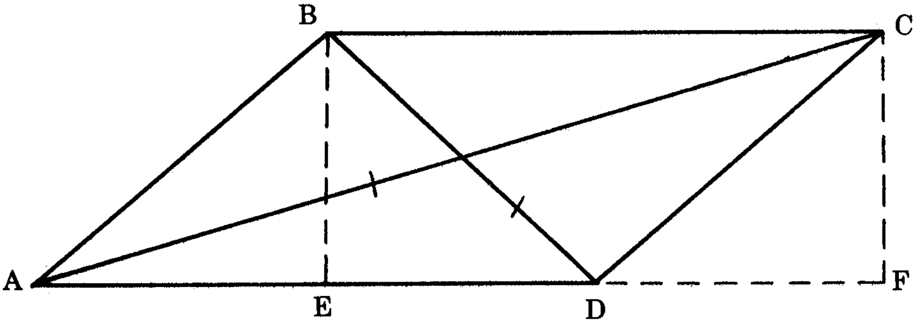
Problem 8

(Hypotenuse-Leg Theorem)

Prove that if the hypotenuse and one leg of one right triangle are congruent to the hypotenuse and one leg of another, then the triangles are congruent.

Problem 9

Prove that if the diagonals of a parallelogram are equal, then the parallelogram is a rectangle.
Hint: This is immediate from the hypotenuse-leg theorem (applied to triangles BDE and ACF below) and the following diagram for the parallelogram ABCD:



This Theorem also is a standard tool used by carpenters in their work. As long as the sides are parallel, an object is rectangular if and only if the diagonals are of equal length.

Problem 10

Prove that for a fixed perimeter, the right triangle which encloses the greatest area is an isosceles right triangle.

Hint: Fix the perimeter d for the triangle in Figure 1. Then $a + b + c = d$ and so $a + b = d - c$. Squaring both sides and using $c^2 = a^2 + b^2$ produces (where A is the area of the triangle): $A = \frac{1}{2}ab = \frac{1}{4}(d^2 - 2dc)$. It follows that the area is maximized when the hypotenuse c is minimized. But when c is minimized, we must have $\sqrt{2}\,c = a + b$ (for otherwise, by problem 4, $\sqrt{2}\,c > a + b$ and we could decrease c slightly and adjust a and b to get a right triangle with perimeter d and a smaller c). But now, by problem 4, $a = b$.

Problem 11

Take the triangle in Figure 1 and draw squares out from the 3 sides as in Figure 2. Note that the Pythagorean Theorem states that the area of square C equals the sum of the area of squares A and B. (It is thought that this observation about areas is what led the Pythagorean School to discover the Pythagorean Theorem 2500 years ago.) Show that replacing the squares with equalateral triangles (see Figure 3) with areas A_1, B_1, C_1 produces the equality $C_1 = A_1 + B_1$. Similar formulas hold if we replace the squares with any regular polygons or even half circles (see Figure 4). In fact, the same formula holds if we draw any similar figures out from the triangle as long as the "similar" sides of these figures are the sides of the triangle (see Figure 5). Try to prove this.

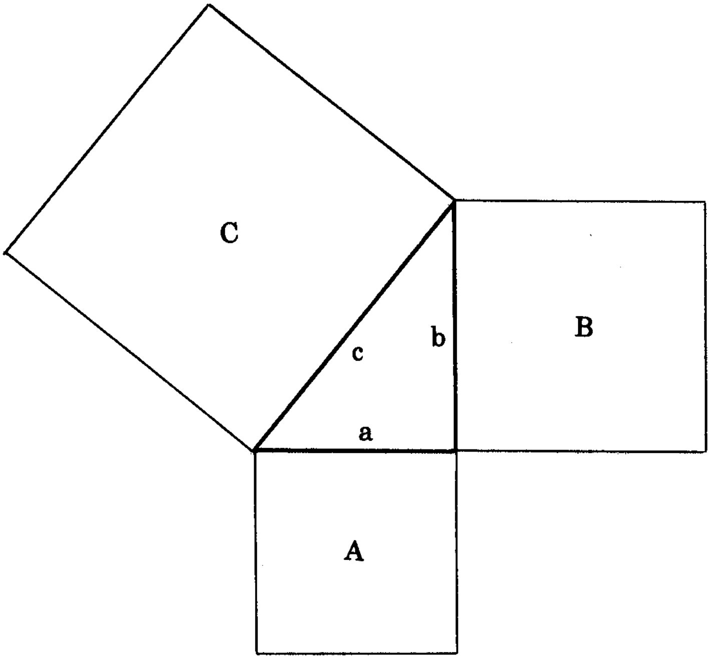


Figure 2

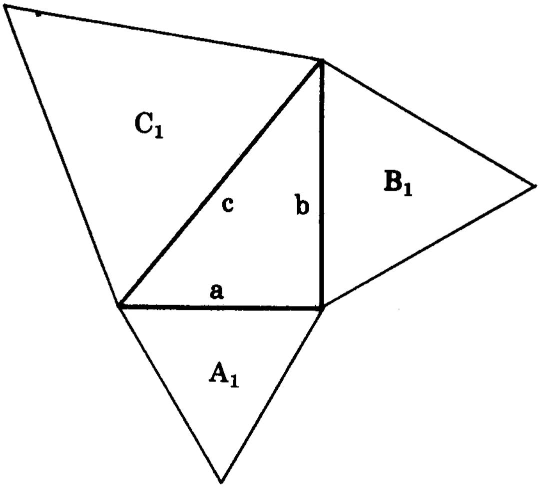


Figure 3

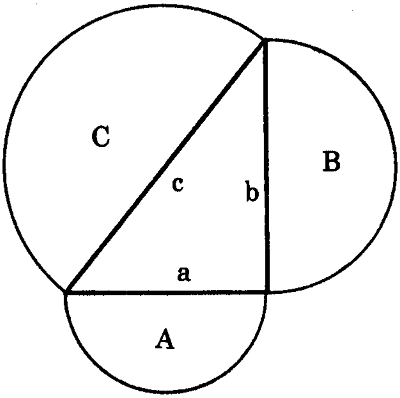


Figure 4

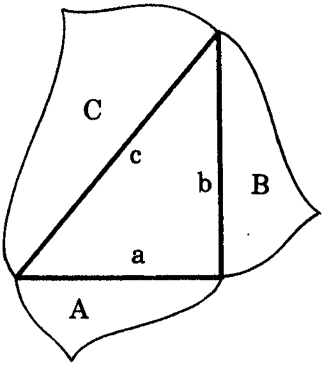


Figure 5

Hint: For the specific figures listed, you can actually compute the respective areas and use the Pythagorean Theorem to show equality. In general, it is true that for similar figures such as those in Figure 4, the quotient of the respective areas equals the square of the quotient of the respective sides. Hence,

$$\frac{B}{C} = \left(\frac{b}{c}\right)^2 , \quad \frac{A}{C} = \left(\frac{a}{c}\right)^2$$

Dividing through the Pythagorean Theorem by c^2 and using these equalities gives $1 = \frac{A}{C} + \frac{B}{C}$ and hence $C = A + B$.

EXTRA STUDY: You might like to read about the life of Pythagorous or about the Pythagorean school in books on the history of mathematics in your school library.

PROBLEMS

This section is intended to provide material for classroom use for a teacher who wants interesting or unusual problems of varying degrees of difficulty. Whenever possible, it will include problems of historical interest or with a particular emphasis. These problems should help the teacher stimulate curiosity and activity by students. Even the teacher may enjoy solving these problems.

Readers of this section are encouraged to share their favorite problems with the readers of this Journal by sending them to Peter G. Casazza, Department of Mathematics, University of Alabama, University, Alabama 35486.

1. Three boxes are labeled "apples", "oranges" and "apples and oranges." Each label is incorrect. How can you take one fruit from only one box and then be able to label all 3 boxes correctly? (You may assume that one of the boxes actually contains both apples and oranges.)
2. A carpenter wants to cut a cube of wood into 27 equal cubes. What is the least number of straight cuts necessary to accomplish this? (He may stack up any number of pieces he has and cut them all at once.)
3. What is the maximum number of parts into which a circle can be divided by 4 straight lines? Now try 5, 6, 7 . . . straight lines.
4. The sum of a certain number of consecutive integers is 1000. Find the integers. There are exactly 4 solutions to this problem.
5. Verify the rule: To multiply a number by 12.5, move the decimal point two units to the right and divide this result by 8.
6. Pete gave half his records plus half a record more to Chuck. He then gave half of what was left plus half a record more to Tavan. That left Pete with one record. How many records did Pete have originally.
7. Prove: If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.
8. Prove: If $a, b > 0$, then

$$\frac{a}{\sqrt{b}} + \frac{b}{\sqrt{a}} \geqslant \sqrt{a} + \sqrt{b}$$

9. Prove For $a = 1, 2, 3, \ldots,$

$$\left(1 + \frac{1}{6a}\right)^{-a} > \frac{5}{6}$$

10. Prove: For all x, $\dfrac{x^2}{1 + x^4} \leqslant \dfrac{1}{2}$
11. Given that there is an x such that

$$\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}} = 2,$$

determine x.

12. Convince yourself that if $0 < \alpha < \frac{\pi}{2}$, then $\sin \alpha > \alpha > \tan \alpha$.

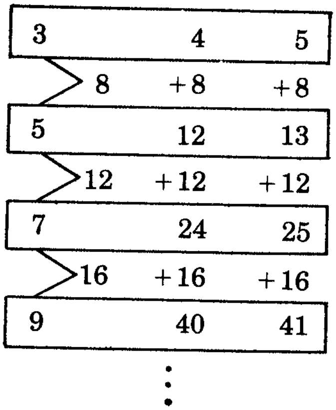
13. Explain why the assumptions $n > 1, 0 < n\alpha < \frac{\pi}{2}$ lead to the conclusion

$$\frac{\sin \alpha}{\alpha} > \frac{\sin n\alpha}{n\alpha}.$$

14. Show that if $0 < \alpha < \beta < \frac{\pi}{2}$, then $\alpha - \sin \alpha < \beta - \sin \beta$

15. Consider the following: $1 \cdot 1 = 1$, $11 \cdot 11 = 121$, $111 \cdot 111 = 12321$, $1111 \cdot 1111 = 1234321$ . . . , $111{,}111{,}111 \cdot 111{,}111{,}111 = 12345678987654321$. Can you figure out why this pattern occurs?

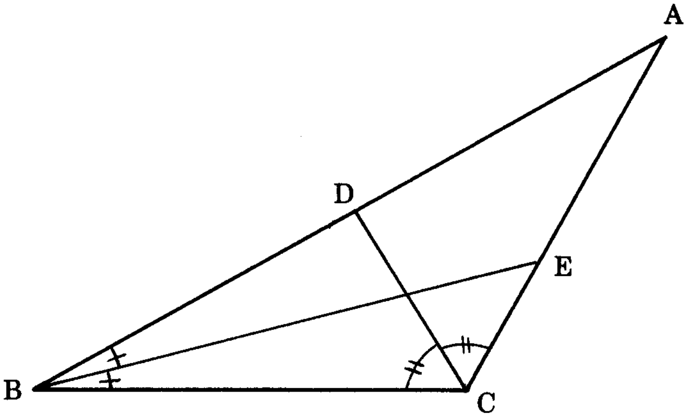
16. Submitted by: Paul Saunders from Jemison High School, Jemison, Al. Show that the following procedure produces all Pythagorean triples of the form $(a, n, n + 1)$:



That is, if a is odd and the triple $(a, n, n + 1)$ is given, then the next triple is $(a + 2, n + 2a + 2, n + 1 + 2a + 2)$. Moreover, if a is even, then there is no triple of the form $(a, n, n + 1)$.

Can you find a way to generate all Pythagorean triples of the form $(a, n, n + k)$ where $k \geqslant 1$ is fixed?

17. In the triangle ABC below, BE = CD and the angles B and C are bisected. Prove that ABC is an isoceles triangle.



(This problem is known as the Steiner-Lehmer problem).

Hint: We may suppose that angle CBA = 2a and angle ACB = 2b where, say, b > a. Construct CF such that angle DCF is equal to EBA. Construct EG parallel to CF. Argue that CDF is similar to BEG and use the fact that $\overline{BE} = \overline{CD}$ to conclude that the triangles are congruent. Then, since $\overline{BG} = \overline{FC}$ and $\overline{BG} > \overline{BF}$, we have $\overline{BF} < \overline{FC}$. Note that the triangle BCF has base angles 2a and a + b where a < b, yet the sides opposite these angles don't fit: the bigger side is opposite the smaller angle. (The figure shows point G between F and A. It could perhaps fall outside the triangle, somewhere beyond A on the line BA. That would not affect the argument since we only use the fact that F is between B and G.)