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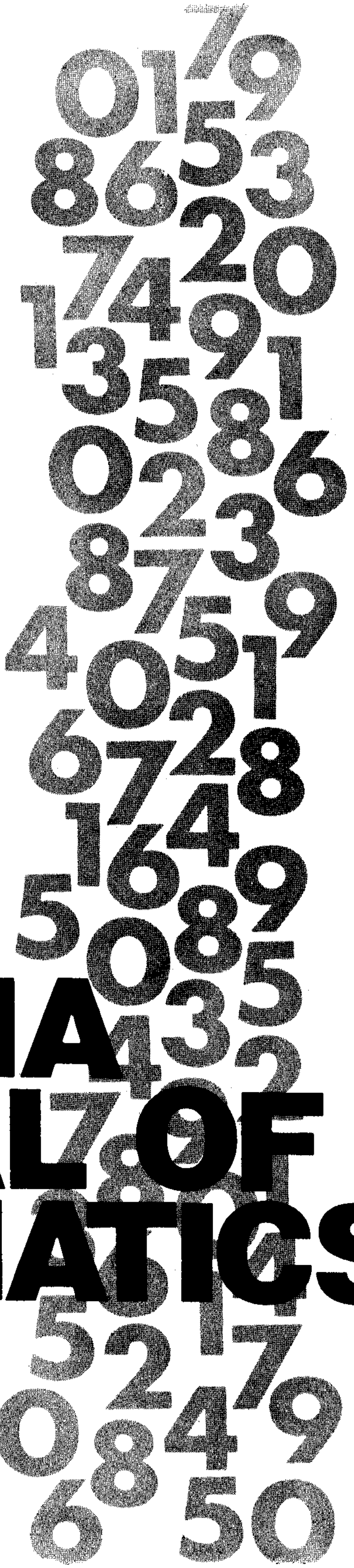

GIVE ME A TABLESPOON OF PAINT AND I WILL PAINT THE WHOLE UNIVERSE

By Cathy Peters

The purpose of this note is to demonstrate how we might use a standard elementary mathematical example to generate interest and enthusiasm in students while also bringing some important ideas to light. I have presented this example to both Juniors and Seniors in high school with very good results, but if it is too long as a classroom example, it could be done in a mathematics club. To construct the example, we start with the graph of the function $y = 1/x$ for $0 < x < 1$ (see figure 1). Now we grasp ther "top" and "bottom" of the y-axis between our fingers and spin the plane around in circles so that the graph traces out "the bell of a trumpet" whose neck extends indefinitely upwards (see figure 2).

The first thing we want to show is that our trumpet has finite volume. To do this, we construct right circular cylinders, each of height 1, around our trumpet and touching the graph of $y = 1/x$ at the points $(1,1)$, $(1/2, 2)$, $(1/3, 3)$, ... (See figure 3). Since each of our cylinders has height 1 (and the formula for the volume of a cylinder is $V = \pi r^2 h$) we see that the volume of the n^{th} cylinder, denoted V_n, is $V_n = \pi r_n^2$. Since the n^{th} cylinder touches the graph at the point $(1/n, n)$, we see that $r_n = 1/n$. Hence the sum V of the volumes of the cylinders is:

$$V = V_1 + V_2 + V_3 + \dots \quad = \pi\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

The sum of the series on the right is $\pi^3/6$ but it is not trivial to show this. However, all we really need to know is that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

is a finite real number. In calculus we could use the integral test to show this; but not to eleventh-graders. However, a simple variation of the comparison test works here. Namely,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$= \left(\frac{1}{1^2}\right) + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right)$$

$$+ \left(\frac{1}{8^2} + \frac{1}{9^2} + \dots \frac{1}{15^2}\right) \quad + \dots$$

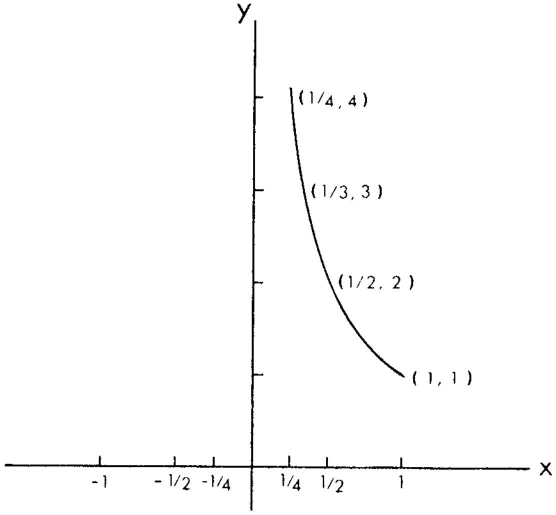

Figure 1

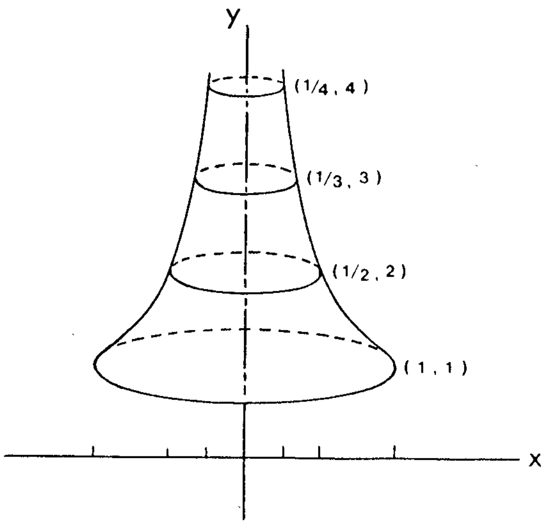

Figure 2

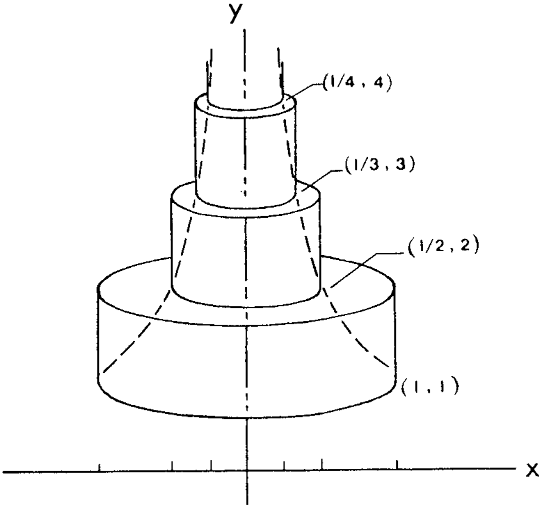

Figure 3

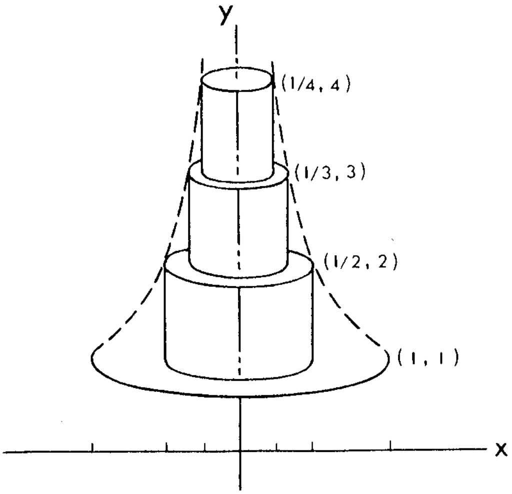

Figure 4

where each parenthesis contains twice as many terms as the preceding one. Decreasing a denominator makes a fraction bigger, thus

$$\frac{1}{2^2}+\frac{1}{3^2}<\frac{1}{2^2}+\frac{1}{2^2}=2\cdot\frac{1}{2^2}=\frac{1}{2}$$

$$\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}<\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2}=4\cdot\frac{1}{4^2}=\frac{1}{4}$$

and so on. The terms grouped in parenthesis are less than, respectively,

$$\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\ldots$$

and therefore

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots<1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$$

The sum on the right hand side is 2 since

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=1$$

(For a simple geometric proof of this last fact, see P. Casazza's article "Paradox Lost" in the Alabama Journal of Mathematics, Volume 1, No. 1, Spring 1977.) Notice that what we've done is to derive a special case of the Cauchy Condensation Test.

Next we want to show that our trumpet has infinite surface area. To see this we will use a technique similar to the above. This time we construct disjoint right circular cylinders, each of height 1, inside our trumpet and touching the graph of $y = 1/x$ at the points (1/2, 2), (1/3, 3), (1/4, 4), ... (See figure 4) Since our cylinders are disjoint and contained entirely within our trumpet, it follows that the surface area of our trumpet is greater than or equal to the sum of the surface areas (without the circular ends) of our cylinders. Since each cylinder has height 1 (and the formula for the surface area of a cylinder is $S = 2\pi rh$) we have that the surface area of the n^{th} cylinder, denoted S_n, is $S_n = 2\pi r_n$. As before, we see that $r_n = \frac{1}{n+1}$ for each n and therefore the sum of the surface areas

S_n will be $2\pi S$ where:

$$S=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots$$

We shall show that the sum S is infinite; that is, we shall show that the partial sums of S are unbounded. Once we know that S is infinite, it will follow that the area of the trumpet is infinite since it is even larger than S.

Using a similar technique to the above we see:

$$1+\frac{1}{2}+\frac{1}{3}+\ldots=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\ldots+\frac{1}{8}\right)+\ldots$$

$$\geqslant\frac{1}{2}+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\ldots+\frac{1}{8}\right)+\ldots$$

$$=\frac{1}{2}+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\ldots=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots=+\infty.$$

Therefore, our trumpet has infinite surface area but finite volume. This fact will not startle your students at first since they don't understand its implications. I find, at this point, that it is best to be dramatic and tell them the following story. This trumpet is sitting on my front lawn exposed to the wind and rain when one day I discover that it is beginning to rust. Not wanting such a beautiful object to be ruined, I go down to my local paint store to get some paint to protect it from the elements, but the first question the salesman asks me is: "How large is the area you need to paint?" I respond: "Oh, the area is infinite." "But it will take an infinite amount of paint to paint an infinite surface area," he responds, "And I certainly don't carry that much paint." "But wait," I say, "it has finite volume so I'll just fill it with paint."

By now the students will be saying: "Hey, wait a minute. What's going on here? How can you fill something with paint, but still not paint it?" Now you've got them. They are intrigued. At this point I usually tell them to think about the problem tonight and we'll discuss it tomorrow. Hopefully, some of them will lose a night's sleep over it. They now have been exposed to the wonderful feeling which comes when they are so interested in learning something that they can't even sleep. The next day I let them argue about the example for a while, making sure they don't dismiss it as a mere trick I've pulled on them or a mistake in the mathematics. Then I lead them into a discussion about solids in general which have infinite area and finite volume. They should soon realize that such things just don't exist in the real world. Now you point them in the right direction by asking: "Is it perfectly clear that you can't paint an infinite area with a finite amount of paint?" Let them discuss it for a time before you drop the blockbuster on them by saying: "Actually I claim that I can paint the entire trumpet with one tablespoon of paint." When things quiet down, remind them that they just concluded that such things don't actually exist in the "real world." So if we are going to paint it, we should paint it with "mathematical paint." This paint has the nice property that if you have a finite area to paint and any given amount of paint, you can spread the paint thin enough to cover the area. So, to paint our trumpet with one tablespoon of paint, we merely have to paint the finite part of it between one and two units above the x-axis with 1/2 a tablespoon of paint, the part between 2 and 3 units above the x-axis with 1/4 of a tablespoon of paint, and so on. Then we've painted the whole trumpet with

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$$

tablespoon of paint. Now tell them to go out and paint the universe.

Department of Mechanical Engineering
The University of Alabama
University, Alabama 35486