

REPRINT

**ALABAMA
JOURNAL
OF
MATHEMATICS**

**VOL. 6 NO. 1
Spring 1982**

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THIS ARTICLE NEEDS A TITLE

by P. G. Casazza

Recently I was asked to write a section of a book, *Mathematical Projects*, to be produced by NCTM. The book will consist of a set of projects to be completed by students. The guidelines were simple enough: (1) Each project must be complete and self-contained, including all necessary references and hints; (2) students should be able to complete it in 10-30 hours; (3) it should help students develop problem solving skills; (4) it should encourage the students to use resource materials; (5) it should be on a topic that students would not normally encounter in the traditional mathematics curriculum. Since I had been collecting paradoxes in mathematics for several years, and this topic seemed to fit the requirements, I knew I already had a good topic and the required material "at my fingertips". The project looked easier the more I thought about it (even though the deadline was just three weeks away).

Having written articles in mathematics, I knew that the first thing to do was to list a set of general guidelines for the project (and stick to them, lest the project "get out of hand".) To satisfy my ego it was clear what the requirements had to be:

- (1) The article had to be clever (perhaps even ingenious);
- (2) It had to be interesting (perhaps even fascinating);
- (3) It had to be instructive (perhaps even authoritative).

(Note that this modest set of general guidelines was chosen to keep the project from getting out of hand. Yet, despite this care, the project was soon almost out of sight.)

The next step was to set down some "specific guidelines" which would guarantee that the "general guidelines" were accomplished. These were: (1) Steps in the project should lead students through some of the thought processes used by mathematicians in solving their problems; (2) One of the problems for students to do should be paradoxical in some sense; (3) Students should have an opportunity to be original and make up their own paradoxes; (4) Students should be introduced to most (maybe all) of the great authors and books on the subject. With these guidelines in hand, I was ready to start.

Obviously the first requirement for students should be the first thing done by mathematicians when confronted with a problem: try to understand it. So the students were asked to begin by learning the definition of the word paradox. This seemed straightforward enough — until I got to the library and consulted the Oxford English Dictionary. It lists six different uses of the word paradox, including: Paradox: "A statement or tenent contrary to received opinion or belief. (This isn't the one I wanted.)

Paradox: "A statement which on the face of it seems self-contradictory but may actually be well-founded." (Is this the right one?)

Paradox: "Often applied to a statement which is actually self-contradictory." (How could I get them to pick out the right one?) This problem had an obvious solution: I'd ask them to study *all* the variants of the definition.

The second assignment for the students was motivated by experience in trying to solve problems: they should study a large number of examples to get a feeling for the nature of the subject. So I begun by stating Bertrand Russell's famous barber's paradox:

A barber in a certain town shaves all those people and only those people who do not shave themselves. Who shaves the barber?

(You are to assume that the barber is an adult, potentially bearded, person.) Then I gave them the standard "box paradox":

THE STATEMENT IN THIS BOX IS FALSE.

Next came some common sayings:

Every generality is false.

I've told you a million times —
never exaggerate.

I planned to wind up with Bertrand Russell's story about Tristram Shandy.

Tristram Shandy decided to write his autobiography. Not wanting to leave out any detail, he spent two years chronicling the first two days of his life and lamented that, at this rate, material would accumulate much faster than he could deal with it. As the years went by he would be further and further from the end of his task. What he did not realize was if he had only lived forever he could have written his entire autobiography. For he could have finished the first year of his life in 365 years, the tenth year in 3650 years and, for all n , the n^{th} year in $365n$ years (modulo leap years).

One day's research in the library produced the appropriate reference to B. Russell's *The Principles of Mathematics*, Lawrence Stern's *Tristram Shandy*, and all of the others. Now I was "really rolling". Only three days had passed and I was (nearly) half-way through the project.

The third assignment for the students was a little more complicated. I wanted them to have more examples, read some of the world's great literature and use the library. Also, it was time for one of the problems itself to be a paradox. I decided to make the paradoxical problem depend on them looking up a group of paradoxes in the library; the instructions would have to be written to put them in a paradoxical situation at some point in the process. The first (great literature?) paradox for them to look up would be the one in *Don Quixote*:

Don Quixote tells of a land with a curious custom. Every visitor there is asked: "Why did you come here?" If he answers truthfully, all is well. If he lies, he is hanged. Everything is fine until a visitor answers the question with: "I came to be hanged."

Next the students would go to the *Bible* and look up the original "Liars Paradox".

One of themselves, even a prophet of his own, said the Cretans are always liars . . .

Then the students could look up Zeno's paradoxes, the paradox of the "unexpected hanging" (from Martin Gardner's book — *The Unexpected Hanging and Other Mathematical Diversions*), and Lewis Carroll's paradoxical discussion between Achilles and the Tortoise. Now, I still had to figure out how to make this requirement "paradoxical." After several hours without success, I decided to "think about it" while I went to the library to look up exact references for them to use. It seemed clear that this was the beginning of the end (or the end of the beginning) of this project. A (minor) problem arose when I could find no hint in the chapter headings of *Don Quixote* (and its two bulky volumes didn't leave too many options) as to where the paradox was. The next best thing would be to find it in James R. Newman's *The World of Mathematics* (This is 4 volumes!) where I was sure I had seen it. No success here nor in the other standard mathematical works. Having wasted a whole day, I decided to switch to the Bible (only 1 volume! — but 66 books). After half a day of reading I found the reference (Titus I, 12-13 in the King James version of the Bible). Next I examined the works of Lewis Carroll and Charles Lutwidge Dodson (both great authors). These "authors" had written a considerable amount of literature over the years (besides "Alice in Wonderland"). After many hours, I found the discussion between Achilles and the Tortoise in *The Rectory Umbrella* and a reproduction of it in Newman's book. Finding Zeno's paradoxes would be easy (I thought) since they are discussed in countless books and articles. But it was difficult to find a source giving exact statements of all four of them. Each reference seemed to give only part of them — frequently only one. After some time, I found them in Adolf Grunbaum's *Modern Science and Zeno's Paradoxes*. Although I still had not found the reference for *Don Quixote* (I wanted a page number since I could not ask students to read the whole book — could I?), some other progress had been made. I had figured out what the paradoxical question would be. This was a very delicate problem since it would be necessary for the students not to realize too soon that the question is a paradox; for otherwise they might merely find it confusing and not know what to do at all. They had to "discover" it was a paradox, but only after having done it. The problem read as follows:

I can never lie to you. This problem contains references to 5 paradoxes and one verse (which is not a paradox) about paradoxes. If you look these up in the exact order given, I guarantee that the one which is not a paradox will be a complete surprise to you.

So where is the paradox? That's easy: we have the same structure that appears in the "surprise hanging". The "verse" is number six. So after doing parts one through five and discovering they are all paradoxes, the student is in a "paradoxical" situation. For if one of these six items is not a paradox, the only remaining choice is number 6, so number 6 must be the verse. But then it can't be a "surprise" that number six is not a paradox. Realizing I must have lied to them, but that there is no way of their knowing exactly which of my statements was the lie, they should decide they can draw no conclusion whatsoever about number 6 (and hence be "surprised" that it is a verse). Oh, how clever we can be when we put our minds to it (and have read the "surprise hanging" recently). But wait — I had not written step 6 yet. What was it going to be? Then I remembered a cute verse of W. S. Gilbert (This is the famous Gilbert from "Gilbert and Sullivan"):

How quaint the ways of paradox,
At common sense she clearly mocks.

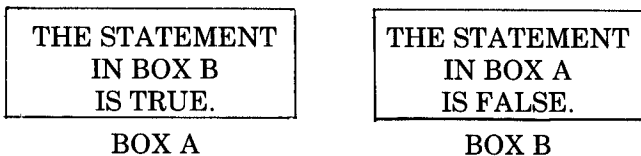
This was perfect (famous writer, etc.). But I had to find it (while still looking for the Don Quixote quote).

Do you know how many operas Gilbert and/or Sullivan wrote? (I do *now*!) The best part of a day in the library brought me to Act II of "The Pirates of Penzance". Now, finally, assignment three was done (as soon as I got the reference for *Don Quixote*). And, I still had a little more than a week left before the article was due.

I had planned assignment four in advance. They were to make up their own paradoxes. It is easy to make up variations of Russell's "barber's paradox":

How about the robot which fixes all those robots and only those robots which do not fix themselves; or the encyclopedia which lists all those encyclopedias and only those which do not list themselves?

Also, they could generalize the "box paradox" to:



Next, they could think about:

- (1) A button which says "ban buttons",
- (2) A sign which says "Don't read this",
- (3) Writing on a wall which says "down with graffiti".

With the end of this project in sight, I decided to play it safe and checked the two volumes of *Don Quixote* out of the library (I could scan it during lunch hours, meetings, etc.).

Assignment five would teach the students an important idea: mathematicians draw pictures to "see" what is going on in a problem, but they must be careful not to rely too heavily on the "exact form" of their

picture. To show the danger of pictures, I asked them to find the flaw in the standard "proof" (see, for example, "False Proofs — Right or Wrong" by Al Mather, this journal, vol. 1, No. 2, Fall, 1977) that all angles are right angles.

Finally, for assignment six, I would ask them to study "super-machine" paradoxes. Super-machines can accomplish any finite task in any specified finite amount of time. This leads to trouble if you divide a finite amount of time, say one second, into an infinite number of finite segments and give the machine something interesting to do in each segment. It is very likely that you won't know what the machine is doing at the end of a second. For example, if I attach my machine to a light bulb and push a ball off a one foot high table, I can create a paradox as follows. During the time it takes the ball to travel half the distance to the floor, my machine is to turn the light bulb on. While the machine travels half the remaining way to the floor (to within $\frac{1}{2^2}$ of a foot of the floor) my

machine turns the light off. It turns the bulb on again while the ball travels half its distance to the floor again (to within $\frac{1}{2^3}$ of a foot of the

floor). Repeating these steps, my machine turns the light on and off an infinite number of times in one second (the time it takes the ball to reach the floor). What is the final state: Is the light on or off at the end of the second? Students are asked to design five paradoxical tasks for the super-machine to perform: perhaps it could paint your house red while the light is on and blue while it is off; or open a door while the light is on and close it while the light is off; or start at 0 and add one each time the light is turned on; . . .

At this stage, the project could have been called complete (well, almost — there was still *Don Quixote*). But — it did not satisfy the general requirements: It wasn't clever enough for me. The paradoxical assignment was a step in the right direction. But would this be enough? Shouldn't the project contain something more to surprise the careful reader? After some thought I came up with a truly clever (ingenious?) idea. It would take a lot of work, but there was still a week left before the deadline. The first step was to rewrite the paradox of *Don Quixote* (This is the "still unreferenced" *Don Quixote*) into an "acrostic" using the word "paradox". Not being clever at this sort of thing, I used up two days designing it:

Perhaps you have heard of Don Quixote:

And the strange land of which he loves to tell;

Recent visitors are asked: "Why did you come here?"

Answer truthfully and all is well;

Don't lie (lest they hang you immediately);

Or say: "I came to be hanged today."

Xenophobia just cannot be cured that way.

The hard part was yet to come, however. For my idea was to make the entire project into an acrostic. That is, the first letters of all the sentences would spell out a new requirement (unfortunately it had to be chosen more for the number of letters involved than for its new ideas).

REQUIREMENT VII: EXPLAIN THE FOLLOWING PARADOX: THE NEXT SENTENCE YOU READ WILL BE FALSE. THE SENTENCE YOU JUST READ IS TRUE.

This, of course, would involve rewriting the entire article and restructuring every sentence to start with the appropriate letter.

The time left was short (5 days) but I could do it somehow (working around my teaching duties, etc., and ignoring a case of the flu). Two days of university duties, writing (rewriting) by day and reading *Don Quixote* by night produced slow progress (How many sentences do you know which start with the letter x needed for "next"?). Also, once and for all, I had to find the *Don Quixote* reference. So, on the third day, I started reading it in earnest. By 5:00 a.m. the following morning I had finished Volume I but had found no paradox. (Could I have missed it?) And I only had 2 days left and a lot of rewriting still to do. I asked for help from the English Department — left that problem to them — and concentrated on rewriting. Two days later the article was done (I gave up on the word "next" and changed it to "following") except for writing some not too obvious "hint" to the students to look for the acrostic (and *Don Quixote*?) The typist's deadline of tomorrow morning at 9:00 a.m. didn't look so ominous anymore. I talked to a friend who I knew was working on a similar project to announce my victory. He asked, "Do you know that there must be a complete answer sheet for this project to be submitted with it?" The look on my sweaty face certainly answered that question. Now I was really in trouble and three nights with little sleep merely compounded the problem. By 10:00 p.m. that night I had the last "subtle hint" for the students completed. The project would end with (the acrostic):

For further study look in the references.

At this point you think you are finished.

Let me remind you, I may never lie.

So believe me — there is one requirement yet to do.

Every effort should be made to find and do it.

Feverishly (a pun?) working all night I had the answer sheet done by 7:00 a.m. At 8:00 I was in my office (just in time to find out that the English Department hadn't found the paradox in *Don Quixote* — Does it really exist? Should I have asked the Spanish Department?) Now I was really in trouble. The article was like a jigsaw puzzle. To remove a piece would make the acrostic meaningless. Then I remembered (how lucid we become when desperate) that I had put this paradox in an earlier paper of my own. I could just reference it there. In an hour the appropriate lines were rewritten and I proudly submitted my 44 handwritten pages to the typist. She had previously agreed to type it in one day so it would meet the deadline. This, of course, required her to be at work that day. However, she had caught the flu also (That's regretable), but would try to type it at home (That's commendable). I collapsed (That's understandable) and slept all day (That's wonderful) and awoke to find the typed manuscript on my desk (That's incredible) with a note attached: "Did you know that this project was to be no more than three pages long?" (That's . . .). Fragments of ideas were passing through my mind:

"If you are really clever," . . . "do it in three pages," . . . "a whole new much shorter acrostic," . . . "The final deadline is not until tomorrow morning," . . . "It can't be done!" I folded up the manuscript, put it in an envelope, and mailed it to the NCTM.

Now it is a week later, I am over the flu and can look back on the project with the wisdom of experience (hindsight?). I still haven't found the paradox in *Don Quixote*. Perhaps I should read Volume II? But I don't have the time. This article is due to the *Alabama Journal of Mathematics* tomorrow, and I don't even have a title for it yet. And I am still not sure if it is ingenious (or even clever); if it is fascinating (or even interesting); if it is authoritative (or even instructive). Does it fulfill the "general guidelines" I set for it? Maybe I should rewrite it into an acrostic (or maybe a poem, this time). I can't: I just don't have the time. You see, I have been asked to write another article after this, the deadline is rapidly approaching, and I haven't even set down my general (or specific) guidelines for it yet. It seems that I am always behind. Can't I ever catch up (maybe if I could live forever . . .)? At least this time I won't be struggling at the end trying to find a title for it. That part, at least, is clear. Of course, you can guess what it must be: "How I wrote the Article 'This Article Needs a Title' " . . .

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