# Real Analysis II

## Homework 1

(1) Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Let $X \subset C(\mathbb{T})$ be the set of all functions $f \in C(\mathbb{T})$ such that $f(-1) = 0$.
  (a) Prove that $X$ is a closed linear subspace of $C(\mathbb{T})$.
  (b) Let $\mathcal{A}$ be an algebra in $C(\mathbb{T})$, which separates points and vanishes only at $-1$. Assume that for any $f \in \mathcal{A}$, $\bar{f} \in \mathcal{A}$. Prove that $\mathcal{A}$ is dense in $X$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0.$$

Prove that for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and a polynomial $P$ of degree less than $2n$ such that

$$\sup_{x \in \mathbb{R}} \left| f(x) - \frac{P(x)}{(1+x^2)^n} \right| < \varepsilon.$$

(Hint: Prove that the functions $\frac{P(x)}{(1+x^2)^n}$ form an algebra in $C(\mathbb{R})$. Let $\varphi : \mathbb{R} \to \mathbb{T}$ be a function defined by $\varphi(x) = \exp(2i \tan^{-1} x)$. Consider the image of this algebra under the mapping $\varphi$.)

(3) Prove that $L_p(0, +\infty) \not\subset L_p(0, +\infty)$, whenever $p \neq q$.

(4) Let $1 \leq p < \infty$ and let $\ell_p$ be the space of all sequences $\{x_n\}_{n=1}^{\infty}$, such that

$$\|\{x_n\}_{n=1}^{\infty}\|_p := \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < +\infty.$$

Prove that if $1 \leq p < q < +\infty$, then $\ell_p \subset \ell_q$ and $\ell_p$ is dense in $\ell_q$.

(5) Let $X$ be the set of all continuous functions $f : [0, 1] \to \mathbb{C}$ satisfying $f(0) = f(1)$. Prove that $X$ is dense in $L_p([0, 1])$ for any $p \in [1, +\infty)$.