**Real Analysis II**

**Homework 2**

(1) Let $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ be sequences of vectors in a Hilbert space $H$. Prove that if $\|x_n\|, \|y_n\| \leq 1$ for all $n \in \mathbb{N}$ and $\langle x_n, y_n \rangle \to 1$, then $\|x_n - y_n\| \to 0$.

(2) Let $B_\infty = \{f \in L_2(\mu) |\, |f(t)| \leq 1 \text{ a.e.}\}$.
(a) Show that $B_\infty$ is closed in $L_2(\mu)$.
(b) For $f \in L_2(\mu)$ find $P_{B_\infty} f$.

(3) Let $H$ be a Hilbert space, and let $P : H \to H$ be a bounded linear operator. Prove that $P$ is a projection onto a closed subspace of $H$ if and only if $P^2 = P$ and $\|P\| \leq 1$.

(4) Let $P$ and $Q$ be projections onto closed subspaces of $H$. Prove that $PQ$ is a projection if and only if $PQ = QP$.

(5) Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal sequence in $L_2[0,1]$.
(a) Prove that for any $x \in [0,1]$

$$x \geq \sum_{n=1}^{\infty} \left| \int_0^x u_n(t)dt \right|^2 .$$

(b) Prove that $\{u_n\}_{n=1}^{\infty}$ is a basis if and only if for any $x \in [0,1]$

$$x = \sum_{n=1}^{\infty} \left| \int_0^x u_n(t)dt \right|^2 .$$

(c) Prove that $\{u_n\}_{n=1}^{\infty}$ is a basis if and only if

$$\int_0^1 \sum_{n=1}^{\infty} \left| \int_0^x u_n(t)dt \right|^2 dx = \frac{1}{2}.$$