Real Analysis II

Homework 3

(1) Let $f, g \in L_1([0, 2\pi])$. Assume that these functions are extended periodically on $\mathbb{R}$. Define the convolution of f and g by

$$f * g(x) = \int_0^{2\pi} f(x-y)g(y)\,dy.$$

(a) Prove that $f * g \in L_1([0, 2\pi])$.

(b) Prove that if $f, g \in L_\infty([0, 2\pi])$, then $f * g$ is continuous.

(c) Prove that $\widehat{f * g}(n) = \hat{f}(n) \cdot \hat{g}(n)$.

(2) (a) Let $u \in \mathbb{C} \setminus \mathbb{R}$. Find the Fourier transform of the function $f \in L_2(\mathbb{R})$, where

$$f(x) = \frac{1}{x+u}.$$

(b) Let $f \in L_1(\mathbb{R})$ be a rational function. Prove that there exist constants $C, c > 0$ such that

$$|f^{(}t)| \le Ce^{-c|t|}$$

for all $t \in \mathbb{R}$.

(3) (a) Show that there exists a non-zero function $f \in S(\mathbb{R})$ such that

$$\int_{\mathbb{R}} x^n f(x)\,dx = 0 \quad \text{for all } n \in \mathbb{N}.$$

(b) Let $f \in S(\mathbb{R})$ be a function with compact support such that

$$\int_{\mathbb{R}} x^n f(x)\,dx = 0 \quad \text{for all } n \in \mathbb{N}.$$

Prove that $f = 0$.