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Perturbations of Strongly Continuous Operator Semigroups, and Matrix Muckenhoupt Weights[*]

G. M. Gubreev and Yu. D. Latushkin

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Abstract. Let A and A_0 be linear continuously invertible operators on a Hilbert space $\mathfrak{H}$ such that $A^{-1} - A_0^{-1}$ has finite rank. Assuming that $\sigma(A_0) = \varnothing$ and that the operator semigroup $V_+(t) = \exp\{iA_0 t\}$, $t \geqslant 0$, is of class C_0, we state criteria under which the semigroups $U_\pm(t) = \exp\{\pm iAt\}$, $t \geqslant 0$, are of class C_0 as well. The analysis in the paper is based on functional models for nonself-adjoint operators and techniques of matrix Muckenhoupt weights.

Key words: nonself-adjoint operator, perturbation of a semigroup, functional model, Muckenhoupt condition.

To the centenary of M. G. Krein

In this paper, we apply functional models of nonself-adjoint operators and the technique of matrix Muckenhoupt weights to the theory of one-parameter operator semigroups in Hilbert spaces.

1. *w*-perturbations of linear operators. Let A_0 and A be linear unbounded densely defined and continuously invertible operators on a Hilbert space $\mathfrak{H}$ such that

$$A^{-1}h = A_0^{-1}h + \sum_{k=1}^{n} (h, f_k) g_k, \qquad h \in \mathfrak{H}, \tag{1}$$

where $f_k, g_k \in \mathfrak{H}$, $1 \leqslant k \leqslant n$, and the parentheses stand for the inner product in $\mathfrak{H}$. We shall assume that the operator A_0 belongs to the class $\Sigma^{(\exp)}$, that is,

(a) A_0 does not have spectrum and generates a C_0-semigroup $V_+(t) = \exp\{iA_0 t\}$, $t \geqslant 0$ [1].

(b) The entire operator-valued function $(A_0 - zI)^{-1}$ has finite exponential type.

It is of significant interest to solve the problem of describing the systems of vectors $\{f_k\}_1^n$ and $\{g_k\}_1^n$ such that the operator A related to $A_0 \in \Sigma^{(\exp)}$ by (1) generates a C_0-semigroup $U_+(t) := \exp\{iAt\}$ or a C_0-semigroup $U_-(t) := \exp\{-iAt\}$, $t \geqslant 0$. In this paper, we indicate a class of finite-dimensional perturbations A satisfying (1) in which this problem admits a solution.

An $n \times n$ matrix weight almost everywhere positive on the real line will be denoted by $w^2(x)$, $x \in \mathbb{R}$. Further, by $\mathscr{M}_n^2$ we denote the class of matrix weights w^2 satisfying Muckenhoupt's condition (A_2) [2]:

$$\sup_{\Delta} \|(M(w^{-2}))^{1/2}(M(w^2))^{1/2}\| < \infty, \tag{A_2}$$

where $M(w^{\pm 2}) := |\Delta|^{-1}\int_\Delta w^{\pm 2}(x)\,dx$, Δ is an arbitrary interval in $\mathbb{R}$, and $|\Delta|$ is its length. Using the operator $A_0 \in \Sigma^{(\exp)}$ and the vectors $\{g_k\}_1^n$, we construct the n-dimensional row

$$\mathscr{A}_0(z,h) := \operatorname{row}\{(A_0(A_0 - zI)^{-1}g_k, h)\}_1^n, \qquad h \in \mathfrak{H},\ z \in \mathbb{C},$$

and introduce the following definition.

Definition 1. Suppose that the operators A and A_0 are related by (1).We say that A is a w-perturbation of rank n of the operator $A_0 \in \Sigma^{(\exp)}$ if there exists a weight $w^2(x)$ of the class $\mathscr{M}_n^2$ such that the following conditions hold:

1. For each $h \in \mathfrak{H}$, the function $\|\mathscr{A}_0(x,h)w^{-1}(x)\|$ belongs to $L_2(\mathbb{R})$.

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2. There exists a constant $m > 0$ such that

$$m\|h\|^2 \leqslant \int_{\mathbb{R}} \|\mathscr{A}_0(x,h)w^{-1}(x)\|^2\,dx, \qquad h \in \mathfrak{H}.$$

Let us construct examples of w-perturbations of operators A_0 of the class $\Sigma^{(\exp)}$. Let θ be an entire $n \times n$ matrix function inner in the half-plane $\mathbb{C}_+$ and normalized by the condition $\theta(0) = E_n$. (See [3] and [5]; a multiplicative representation of θ is given in [5].) We denote the set of such matrix functions by J_n. Also, let $H^2_+(\mathbb{C}^n)$ denote the Hardy class of $\mathbb{C}^n$-valued vector functions on the half-plane $\mathbb{C}_+$. In the model space

$$\mathscr{K}_\theta := H^2_+(\mathbb{C}^n) \ominus \theta H^2_+(\mathbb{C}^n), \tag{2}$$

we consider the operator B,

$$Bh := z^{-1}(h(z) - \theta(z)h(0)), \qquad h \in \mathscr{K}_\theta, \tag{3}$$

which has trivial kernel and satisfies $\sigma(B) = \{0\}$ and $(B - B^*)/i \leqslant 0$ (antidissipativity). Since the elements of the space $\mathscr{K}_\theta$ are entire vector functions, it follows that the operator B is well defined by formula (3). It is worth mentioning ([3, p. 310]; [6, p. 91]) that for each antidissipative Volterra operator B with trivial kernel and n-dimensional imaginary part there exists a matrix function $\theta \in J_n$ (the characteristic matrix function of B^*) such that B is unitarily equivalent to an operator of the form (2), (3). Further, for each weight $V^2 \in \mathscr{M}^2_n$ there exists a matrix function V_+ (respectively, V_-) that is outer [3] in $\mathbb{C}_+$ (respectively, $\mathbb{C}_-$) and satisfies [2]

$$V^2(x) \overset{\text{a.e.}}{=} V^*_+(x+i0)V_+(x+i0) \overset{\text{a.e.}}{=} V^*_-(x-i0)V_-(x-i0), \qquad x \in \mathbb{R}, \tag{4}$$

where by $V_+(x+i0)$ (respectively, $V_-(x-i0)$) we denote the nontangential limit values of $V_+(z)$, $z \in \mathbb{C}_+$ (respectively, $V_-(z)$, $z \in \mathbb{C}_-$) on the real line. Both matrix functions are determined up to a left constant unitary factor, and the entries of the matrix $V_+(z)(z+i)^{-1}$ (respectively, $V_-(z)(z-i)^{-1}$) belong to H^2_+ (respectively, H^2_-).

In $\mathscr{K}_\theta$, consider the system of vectors

$$G_k := (I + iB)^{-1}\mathbb{P}_\theta \frac{V^k_+(x+i0)}{x+i}, \qquad 1 \leqslant k \leqslant n, \tag{5}$$

where $V^k_+(x+i0)$ is the kth column of the matrix $V_+(x+i0)$, $\mathbb{P}_\theta$ is the orthogonal projection of $H^2_+(\mathbb{C}^n)$ onto $\mathscr{K}_\theta$, and B is given by (3). Note that the operator $\widetilde{A}_0 := B^{-1}$ belongs to the class $\Sigma^{(\exp)}$. Let $\widetilde{A}$ be the densely defined operator in $\mathscr{K}_\theta$ whose inverse is defined by the formula

$$(\widetilde{A})^{-1}h = (\widetilde{A}_0)^{-1}h + \sum_{k=1}^{n}\langle h, F_k\rangle G_k, \qquad h \in \mathscr{K}_\theta, \tag{6}$$

where $F_k \in \mathscr{K}_\theta$, $1 \leqslant k \leqslant n$, and the angle brackets stand for the inner product in $\mathscr{K}_\theta$. One can show that $\widetilde{A}$ is a w-perturbation of rank n of the operator $\widetilde{A}_0$ and that conditions 1 and 2 in Definition 1 are satisfied for the weight $w^2 = V^2$. *Therefore, each matrix function $\theta \in J_n$ and each weight $V^2 \in \mathscr{M}^2_n$ generate a w-perturbation* (5), (6).

Conversely, suppose that formula (1) gives a w-perturbation of the operator A_0, and let w^2 be the weight corresponding to this perturbation by Definition 1. Let a be the exponential type of the resolvent $(A_0 - zI)^{-1}$. The set of all left divisors [3] $\tilde{\theta} \in J_n$ of the matrix function $e^{iaz}E_n$ contains divisors satisfying the following property:

$$\mathscr{A}_0(z,h)w^{-1}_+(z)\tilde{\theta}(z) \in H^2_+(\mathbb{C}^n) \tag{7}$$

for each $h \in \mathfrak{H}$, where w_+ is taken from the factorization (4) of the weight w^2 and $\mathscr{A}_0(z,h)$ has the same meaning as in Definition 1. *In the set of the left divisors $\tilde{\theta}$ for which* (7) *holds, there exists a $\theta \in J_n$ that is the "least" in the following sense*:

1. *Each matrix function $\tilde{\theta}$ has θ as a left divisor.*
2. *The inclusion* (7) *holds for $\tilde{\theta} = \theta$.*

3. $\theta(0) = E_n$.

Once the weight $w^2 \in \mathscr{M}_n^2$ has been chosen, the inner matrix function θ is determined by these conditions uniquely. Thus, the converse assertion holds; that is, *to each w-perturbation there correspond a weight $w^2 \in \mathscr{M}_n^2$ and a matrix function $\theta \in J_n$*. In the following statement, the operators A and A_0 are related by (1), the operator B is given by formulas (2)–(3), and the vectors G_k are determined by formulas (5).

Theorem 1. *Let A be a w-perturbation of rank n of an operator $A_0 \in \Sigma^{(\exp)}$ with the corresponding pair $w^2 \in \mathscr{M}_n^2$, $\theta \in J_n$. Then there exists an isomorphism S of the space $\mathfrak{H}$ onto the model space $\mathscr{K}_\theta$ such that*

$$B = SA_0^{-1}S^{-1}, \qquad Sg_k = G_k, \quad 1 \leqslant k \leqslant n, \qquad \widetilde{A} = SAS^{-1},$$

where $\widetilde{A}$ is defined by formula (6) with $F_k := (S^*)^{-1} f_k$, $1 \leqslant k \leqslant n$.

This theorem has the following corollary.

Corollary 1. *Let A be an operator defined by formula (1) in a Hilbert space $\mathfrak{H}$, and suppose that A_0^{-1} is an antidissipative operator with nonself-adjointness subspace $\mathscr{L} := (A_0^{-1} - (A_0^{-1})^*)\mathfrak{H}$ of dimension n. If the vectors g_k, $1 \leqslant k \leqslant n$, in formula (1) form a basis in $\mathscr{L}$, then A is a w-perturbation of A_0 with constant weight $w^2(x) = E_n$, $x \in \mathbb{R}$.*

2. Description of the generators of C_0-semigroups. To each operator A defined by (1), we assign the entire matrix function $\Phi(z)$ with entries

$$\Phi_{kj}(z) = \delta_{kj} - z(A_0(A_0 - zI)^{-1} g_j, f_k), \qquad 1 \leqslant k,\ j \leqslant n,$$

and consider the determinant $\Delta(z) := \det \Phi(z)$. Note that $\sigma(A)$ coincides with the set $\Lambda := \{\lambda_k\}$ of roots of the function Δ.

Let A be an w-perturbation of the operator A_0. By the considerations in Sec. 1, to the operator A there corresponds an A_2-weight w^2 and a left divisor $\theta \in J_n$ of the matrix function $e^{iaz}E_n$, where a is the exponential type of the resolvent $(A_0 - zI)^{-1}$. Note that the inequalities $0 < -i\,\mathrm{Sp}(\frac{d}{dz}\theta\big|_{z=0}) \leqslant na$ hold. We denote by $w_\pm$ the outer matrix functions in the factorization (4) of the weight w^2.

Theorem 2. *Let A be a w-perturbation of an operator $A_0 \in \Sigma^{(\exp)}$ with the corresponding pair w^2, θ. Suppose that the following conditions hold for some $\varepsilon \geqslant 0$:*

1. $\sup_{\lambda_k \in \Lambda} \operatorname{Im} \lambda_k < \varepsilon$.
2. $\limsup_{y \to +\infty} y^{-1} \log |\Delta(iy)| = -i\,\mathrm{Sp}(\theta'(0))$.
3. *The matrix weight $W_\varepsilon^2(x) := \Phi(x+i\varepsilon)(w_+^*(x+i\varepsilon)w_+(x+i\varepsilon))^{-1}\Phi^*(x+i\varepsilon)$ satisfies condition (A_2) on $\mathbb{R}$.*

Then the semigroup $U_-(t) := \exp\{-iAt\}$, $t \geqslant 0$, belongs to the class C_0, and its exponential type does not exceed ε.

Conversely, if $U_-(t) \in C_0$ and the type of $U_-(t)$ is equal to ω, then conditions 1–2 hold for each $\varepsilon > \omega$. In addition, if the matrix function $e^{-i\delta z}\theta(z)$ is bounded in $\mathbb{C}_+$ for some $\delta > 0$, then condition 3 holds for each $\varepsilon > \max\{0, \omega\}$.

Theorem 3. *Let A be a w-perturbation of an operator $A_0 \in \Sigma^{(\exp)}$ with the corresponding pair w^2, θ. Suppose that the following conditions hold for some $\varepsilon \geqslant 0$:*

1. $\inf_{\lambda_k \in \Lambda} \operatorname{Im} \lambda_k > -\varepsilon$.
2. $\limsup_{y \to -\infty} |y|^{-1} \log |\Delta(iy)| = 0$.
3. *The matrix weight $W_\varepsilon^2(x) := \Phi(x-i\varepsilon)(w_-^*(x-i\varepsilon)w_-(x-i\varepsilon))^{-1}\Phi^*(x-i\varepsilon)$ satisfies condition (A_2) on $\mathbb{R}$.*

Then the semigroup $U_+(t) := \exp\{iAt\}$, $t \geqslant 0$, belongs to the class C_0, and its exponential type does not exceed ε.

Conversely, if $U_+(t) \in C_0$ and the type of $U_+(t)$ is equal to ω, then conditions 1–2 hold for each $\varepsilon > \omega$. In addition, if the matrix function $e^{-i\delta z}\theta(z)$ is bounded in $\mathbb{C}_+$ for some $\delta > 0$, then condition 3 holds for each $\varepsilon > \max\{0, \omega\}$.

Corollary 2. *For the class of w-perturbations in Corollary* 1, *one can set $w_\pm(z) = E_n$ in the statements of Theorems* 2 *and* 3.

Let us discuss an example of application of Theorems 2 and 3. Let A be a w-perturbation of the operator A_0 in Corollary 1. Since the operator $L := A_0^{-1}$ is antidissipative, it follows that there exists an orthogonal basis $\{u_k\}_1^n$ of the subspace $\mathscr{L}$ such that $(L^* - L)h = i\sum_{k=1}^n (h, u_k)u_k$ for all $h \in \mathfrak{H}$. The characteristic matrix function of the operator L^* is defined by the formula

$$\theta_{kj}(z) = \delta_{kj} + iz((I - zL^*)^{-1}u_k, u_j), \qquad L = A_0^{-1}, \tag{8}$$

and belongs to the class J_n [5]. Suppose also that the vectors $\{f_k\}_1^n$ and $\{g_k\}_1^n$ (see formula (1)) form bases of the subspace $\mathscr{L}$; that is, $f_k = \sum_{p=1}^n d_{kp}u_p$, $g_k = \sum_{p=1}^n c_{kp}u_p$, and the matrices $D := \|d_{kp}\|$ and $C := \|c_{kp}\|$ are invertible.

Theorem 4. *Suppose that A is a w-perturbation of the operator A_0 in Corollary* 1, *both systems $\{f_k\}_1^n$ and $\{g_k\}_1^n$ are bases of the subspace $\mathscr{L}$, and the matrix function θ is defined by formula* (8).

If $\lim_{y\to+\infty}\theta(x + iy) = 0$ uniformly with respect to $x \in \mathbb{R}$, then the semigroup $U_-(t)$ belongs to the class C_0. If, in addition, we assume that the matrix $E_n + iCD^*$ is invertible, then the semigroup $U_+(t)$ belongs to the class C_0 as well.

Note that in Theorem 4 the numerical range of the operator A_0^{-1} lies in the closed lower half-plane. It follows from formula (1) that if $g_k = \sum_{p=1}^n Q_{kp}f_p$, $1 \leqslant k \leqslant n$, and $\operatorname{Im} Q \leqslant 0$, where $Q := \|Q_{kp}\|$, then the perturbation preserves this property; that is, $\operatorname{Im}(Af, f) \geqslant 0$, $f \in \mathscr{D}_A$, and hence the semigroup $U_+(t)$ belongs to the class C_0. Theorem 4 leads to the same conclusion, since in this case $C = QD$ and the matrix $E_n + iQDD^*$ is always invertible; however, the first condition (regarding the behavior of $\theta(x + iy)$) becomes redundant.

Although the statements of the two theorems look similar, they solve essentially different problems. Theorem 3 discusses the semigroup $U_+(t)$ assuming that the semigroup $V_+(t)$, $t \geqslant 0$, exists, while Theorem 2 gives a criterion for the existence of $U_-(t)$, although the semigroup $V_-(t) := \exp\{-iA_0t\}$ does not exist.

Examples show that condition 3 may hold even without the assumption that the matrix function $e^{-i\delta z}\theta(z)$ is bounded in $\mathbb{C}_+$. Nevertheless, we conjecture that this condition cannot be dropped in the class of all w-perturbations.

3. The semigroup of shifts along mean-periodic vector functions. Consider a system of linear bounded functionals φ_k, $1 \leqslant k \leqslant n$, on the Sobolev space $W_2^1([0, a], \mathbb{C}^n)$ such that no nonzero linear combination of these functionals can be continued to a bounded functional on $L_2([0, a], \mathbb{C}^n)$. We also assume that the conditions $\varphi_k(e_j) = \delta_{kj}$ hold, where the $\{e_k\}_1^n$ are the standard unit vectors in $\mathbb{C}^n$ viewed as elements of the Sobolev space. By way of example, consider the operators

$$A = i\frac{d}{dx}, \quad \mathscr{D}_A = \bigcap_{k=1}^n \ker\varphi_k, \quad A_0 = i\frac{d}{dx}$$

in $L_2([0, a], \mathbb{C}^n)$ such that the domain of A_0 consists of the functions $h \in W_2^1([0, a], \mathbb{C}^n)$ satisfying $h(0) = 0$. The operator A is a w-perturbation of rank n of $A_0 \in \Sigma^{(\exp)}$ with the corresponding weight $w^2(x) = E$, matrix function $\theta(z) = e^{iaz}E_n$, and entire function Φ with entries $\Phi_{kj}(z) := \varphi_k(e^{-iza}e_j)$. Conditions 1–3 in Theorem 2, where one should set $-i\operatorname{Sp}(\theta'(0)) = na$ and $w_+(z) = E_n$, are necessary and sufficient for $U_-(t)$ to be of class C_0. The semigroup acts on functions $h \in \mathscr{D}_A$ by the formula $(U_-(t)h)(x) = \widetilde{h}(t + x)$, where $\widetilde{h}$ is the unique extension of h to the right such that $\widetilde{h}(t + x) \in \mathscr{D}_A$ for each $t > 0$. In other words, for each $h \in \mathscr{D}_A$ the function $\widetilde{h}$ is the unique solution of the system

$$\varphi_k(\tilde{h}(t + x)) = 0, \qquad \tilde{h}(x) = h(x),\ x \in [0, a],\ 1 \leqslant k \leqslant n,\ \|\tilde{h}(a + x)\|_{L_2} \leqslant M\|h\|_{L_2}, \tag{9}$$

of integro-differential equations with the shift of the argument satisfying the given estimate for the norm in $L_2([0, a], \mathbb{C}^n)$. These extensions $\widetilde{h}$ (by analogy with the case $n = 1$ [4]) can naturally be called mean-periodic with respect to the system of functionals φ_k. Thus, Theorem 2 gives a criterion for the solvability of problem (9) in terms of the matrix function Φ.

We dedicate this paper to the memory of Mark Grigor′evich Krein, whose unforgettable presence we shall always treasure.

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Poltava National Technical University
e-mail: gubreev@yandex.ru

Department of Mathematics, University of Missouri
e-mail: yuri@math.missouri.edu

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