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Operator algebras and the Fredholm spectrum of advective equations of linear hydrodynamics ☆

R. Shvydkoy [a,*], Y. Latushkin [b]

[a] *Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, USA*

[b] *Department of Mathematics, University of Missouri, Columbia, MO 65211, USA*

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Abstract

In this paper we give a complete description of the Fredholm spectrum of the linearized 3D Euler equation in terms of the dynamical spectrum of the cocycle governing the evolution of shortwave perturbations. The argument is based on the pseudo differential representation of the Euler group and an application of an abstract isomorphism theorem for operator C^*-algebras. We further show that for all but countably many times the Fredholm spectrum of the group is rotationally invariant, and thus may consist of one or two concentric annuli. The results are obtained for a variety of other equations of ideal hydrodynamics.

Keywords: Euler equations; Fredholm spectrum; C^*-algebras; WKB-method

1. Introduction

In this paper we treat pseudo differential operators in terms of topological dynamics using methods of operator C^*-algebras. Specifically, we study the group of linear operators generated by a class of advective partial differential equations that appear as linearizations in many prob-

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* Corresponding author.
E-mail addresses: shvydkoy@math.uic.edu (R. Shvydkoy), yuri@math.missouri.edu (Y. Latushkin).

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lems of hydrodynamics. We describe the Fredholm spectrum of the operators via the exponential dichotomy (uniform hyperbolicity) of the related finite dimensional linear skew-product flow, called the bicharacteristic amplitude system (BAS). Based on this result, we derive new topological conditions under which the Fredholm spectrum is rotationally invariant.

Spectral analysis of the linear operators obtained by linearizing fluid equations about steady states is a classical topic that recently received much attention. That the radius of the Fredholm spectrum is related to the maximal Lyapunov exponent of the bicharacteristic amplitude system is known since the breakthrough work in [33,34] and [20,21]. Further relations of the Fredholm and the dynamical spectrum of BAS as *sets* were studied in [28,31], where one can also find an extensive bibliography on the subject.

In the current paper we introduce a novel key tool into the subject: abstract Isomorphism Theorems for operator C^*-algebras. Theorems of this type can be traced back at least to [5], see also [7,24], and have been extensively studied more recently in [1–4,18] where one can find further references. The Isomorphism Theorem has many important applications, see [2,3]. As one of them, the Isomorphism Theorem allows one to construct symbols for pseudo differential operators with shifts, and thus express the Fredholm property of the operators in terms of the spectra of the symbols. This is exactly what one needs for the above mentioned linearized problems of hydrodynamics.

The class of pseudo differential operators that we can handle is quite general. We postpone the description of the general setup until the next section, and briefly illustrate our approach using the most notable example, the linearized Euler equations. Let $u_0 \in C^2(\mathbb{T}^n)$ be a solution to the steady Euler equations

$$(u_0 \cdot \nabla)u_0 + \nabla p = 0, \quad \nabla \cdot u_0 = 0,$$

on the torus $\mathbb{T}^n$. The corresponding the linearized problem is given by

$$v_t = -(u_0 \cdot \nabla)v - (v \cdot \nabla)u_0 - \nabla q, \quad \nabla \cdot v = 0. \tag{1.1}$$

The evolution of the infinite dimensional system (1.1) is described by a C_0-group $\{G_t\}_{t\in\mathbb{R}}$ of operators acting on the energy space L^2_{div} of divergence-free vector fields. According to the method of geometric optics, one considers solutions to (1.1) in the form of an oscillating localized wave

$$v(x,t) = b(x,t)e^{iS(x,t)/\delta} + O(\delta), \quad \delta > 0. \tag{1.2}$$

A direct calculation reveals the following bicharacteristic amplitude system (BAS) stated in the Lagrangian coordinates of the flow u_0:

$$x_t = u_0(x), \tag{1.3a}$$

$$\xi_t = -\partial u_0(x)^\top \xi, \tag{1.3b}$$

$$b_t = -\partial u_0(x)b + 2\big(\partial u_0(x)b, \xi\big)\xi|\xi|^{-2}. \tag{1.3c}$$

This is a finite dimensional system of ODEs, which describes dynamics of the leading order term in the ansatz (1.2) along a particle trajectory at initial position $x(0) = x_0$, initial frequency

$\xi(0) = \xi_0 \neq 0$ and amplitude $b(0) = b_0$ subject to the condition $b_0 \perp \xi_0$. The latter insures incompressibility of (1.2) at time $t = 0$. One can further notice that $b \cdot \xi = 0$ is a conservation law of (1.3), which validates incompressibility at all times.

Let $\varphi_t : \mathbb{T}^n \to \mathbb{T}^n$ denote the flow of particle trajectories of the stationary solution u_0 induced by (1.3a). Eqs. (1.3a), (1.3b) define another flow, $\{\chi_t\}_{t\in\mathbb{R}}$, acting on the space $\Omega = \mathbb{T}^n \times \mathbb{S}^{n-1}$. Let $B_t(x,\xi)$ denote the fundamental matrix solution of (1.3c). It satisfies the identity $B_{t+\tau}(x,\xi) = B_t(\chi_\tau(x,\xi))B_\tau(x,\xi)$, and thus defines a cocycle, we call it the b-cocycle, over χ_t on the nontrivial bundle $\mathcal{F}$ with fibers $F(x,\xi) = \{b \in \mathbb{R}^n\colon\ b \perp \xi\}$. Let Σ_B denote the dynamical spectrum of the b-cocycle, that is, the set of $\alpha \in \mathbb{R}$ such that the rescaled cocycle $\{e^{-\alpha t}B_t\}_{t\in\mathbb{R}}$ *does not* have exponential dichotomy (uniform exponential hyperbolicity) on the bundle, see [25] and [8,17]. A technical, but important point of our paper is that one can pass to a cocycle with the same spectrum but acting on a trivial bundle (Section 3.1).

The main result of the present paper (Theorem 3.2) relates the Fredholm spectrum of the evolution operator G_t and the dynamical spectrum Σ_B. From the cocycle calculus developed in [28] for general advective equations it follows that the system of ODEs (1.3) determines the action of each G_t uniquely up to a compact operator. Specifically, the action of the group $\{G_t\}_{t\in\mathbb{R}}$ for each fixed t is given by

$$G_t f(x) = \sum_{k\neq 0} B_t\big(\varphi_{-t}(x), k\big)\hat{f}(k)e^{ik\cdot\varphi_{-t}(x)} + (K_t f)(x), \tag{1.4}$$

for $f \in L^2_{\mathrm{div}}$, where $K_t \in \mathfrak{K}$, the set of compact operators. Thus, G_t is a pseudo differential operator with the shift, that is, it belongs to the C^*-algebra $\mathfrak{B}'$ generated by the C^*-algebra $\mathfrak{A}'$ of pseudo differential operators, by $\mathfrak{K}$, and by the shift operators $T'_m f = f \circ \varphi_{-mt}$, $m \in \mathbb{Z}$.

On the other hand, it is well known since the celebrated paper [22] that the dynamical spectrum of the b-cocycle is related to the spectrum of the weighted shift operator

$$E_t u(\omega) = \big|\det \partial\chi_{-t}(\omega)\big|^{\frac{1}{2}} B_t\big(\chi_{-t}(\omega)\big)u\big(\chi_{-t}(\omega)\big),$$

see [8, Chapter 6] for more information regarding this relation. The latter operator belongs to the C^*-algebra $\mathfrak{B}$ generated by the algebra $\mathfrak{A}$ of operators of multiplication by continuous matrix-valued functions, and by the shift operators $T_m u = |\det \partial\chi_{-mt}|^{\frac{1}{2}} u \circ \chi_{-mt}$, $m \in \mathbb{Z}$. The algebra $\mathfrak{A}$ serves as the algebra of symbols for the pseudo differential operators from $\mathfrak{A}'$, and thus there is an isomorphism between $\mathfrak{A}$ and $\mathfrak{A}'/\mathfrak{K}$. Therefore, to be able to relate the Fredholm spectrum of G_t and the spectrum of E_t, one needs to prove that this isomorphism extends to an isomorphism between $\mathfrak{B}$ and $\mathfrak{B}'$. This is exactly when an application of the Isomorphism Theorem comes into play; it is briefly described in Appendix A.

The paper is organized as follows. In Section 2 we describe the general setup for a class of advective equations. In Section 3 we formulate and proof the main theorem relating the Fredholm spectrum of the evolution operator for these equations and the dynamical spectrum of the corresponding b-cocycle, and give its extension for Sobolev spaces. In Section 4 we take a look inside of the Fredholm spectrum, and formulate the topological conditions under which it is rotationally invariant. In Section 5 we give examples, and, finally, in Appendix A we briefly review the Isomorphism Theorem.

2. The general setup

Let $u_0 \neq 0$ be a smooth vector field on the n-dimensional torus $\mathbb{T}^n$. Incompressibility of u_0 will be our standing hypothesis, although it is not always necessary.

We study linear partial differential equations of the form

$$f_t = -(u_0 \cdot \nabla) f + Af, \quad t \geqslant 0, \tag{2.1}$$

subject to periodic boundary conditions, and zero mean $\int f = 0$. Solution f assumes values in $\mathbb{C}^d$, and A is a discrete pseudo differential operator (PDO) defined by

$$Af(x) = \mathrm{Op}[a] f(x) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} a(x,k) \hat{f}(k) e^{ik \cdot x}, \quad x \in \mathbb{T}^n, \tag{2.2}$$

where a is a $d \times d$ matrix-valued semiclassical symbol of zero order. Specifically, we define the class $\mathcal{S}^m$, $m \in \mathbb{R}$, of all smooth symbols $a \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n \setminus \{0\})$ such that for any multi-indices α and β there exists a constant $C_{\alpha,\beta}$ for which the following estimate holds:

$$\left| \partial_\xi^\alpha \partial_x^\beta a(x,\xi) \right| \leqslant C_{\alpha,\beta} |\xi|^{m-|\alpha|}, \quad x \in \mathbb{T}^n, \ |\xi| \geqslant 1.$$

We assume that $a \in \mathcal{S}^0$ is semiclassical, i.e.,

$$a = a_0 + a_1, \tag{2.3}$$

where $a_0 \in \mathcal{S}^0$ is homogeneous of degree 0 in ξ, and $a_1 \in \mathcal{S}^{-1}$. We remark that PDO's of the form (2.2) with symbols smooth in ξ obey the same classical principles as in the case of $\mathbb{R}^n$ (see [11]). In particular A is bounded on $L^2 = L^2(\mathbb{T}^n)$, and $\mathrm{Op}[a_1]$ is compact. So, the right-hand side of (2.1) is a bounded perturbation of the advective derivative $-u_0 \cdot \nabla$, and hence generates a C_0-group of operators on L^2. Let us denote the group by $\mathcal{G} = \{\mathcal{G}_t\}_{t \in \mathbb{R}}$, and its generator by $\mathcal{L} = -u_0 \cdot \nabla + A$.

Typically, solutions of (2.1) are required to satisfy additional constraints such as incompressibility, which translate into a geometric condition in the frequency space. In general, we can define this condition via a smooth linear bundle $\mathcal{F}$ over the space of non-zero frequencies $\mathbb{R}^n \setminus \{0\}$. Let us denote the fibers of $\mathcal{F}$ by $F(\xi) \subset \mathbb{C}^d$, and assume that $F(\xi)$ is 0-homogeneous. We separately consider the fiber at zero, $F(0) = \{0\}$, which reflects the zero mean condition. For example, incompressibility corresponds to $F(\xi) = \{b\colon\ b \cdot \xi = 0\}$. Let us assume that (2.1) respects the frequency constraints defined by the bundle $\mathcal{F}$, i.e., that the group $\mathcal{G} = \{\mathcal{G}_t\}_{t \in \mathbb{R}}$ leaves invariant the subspace

$$L^2_{\mathcal{F}} = \left\{ f \in L^2\left(\mathbb{T}^n\right)\colon\ \hat{f}(k) \in F(k) \right\}. \tag{2.4}$$

Let us denote by $\Pi : L^2 \to L^2_{\mathcal{F}}$ the orthogonal projection, and $p(\xi) : \mathbb{C}^d \to F(\xi)$ the projection onto the fiber $F(\xi)$. Clearly, p is the symbol of Π. We will deal with the restrictions $G_t = \mathcal{G}_t|_{L^2_{\mathcal{F}}}$. The generator of the group $G = \{G_t\}_{t \in \mathbb{R}}$ is given by $Lf = \mathcal{L} f$ with the domain $D(L) = L^2_{\mathcal{F}} \cap D(\mathcal{L})$ (see [12, Section II.2.3]).

3. The Fredholm spectrum of G_t

Our main result concerns the description of the Fredholm spectrum of G_t over $L^2_{\mathcal{F}}$, which we denote $\sigma_\Phi(G_t; L^2_{\mathcal{F}}) = \{\lambda \in \mathbb{C}\colon\ G_t - \lambda \text{ is not Fredholm in } L^2_{\mathcal{F}}\}$. The Fredholm spectrum of any bounded operator is contained in the Browder essential spectrum (the latter consists of all spectral points except isolated eigenvalues of finite algebraic multiplicity); the essential spectral radius coincides with the radius of the Fredholm spectrum [23].

Let us consider the Hamiltonian system on the cotangent bundle $T^*(\mathbb{T}^n)$,

$$x_t = u_0(x), \tag{3.1}$$

$$\xi_t = -\partial u_0(x)^\top \xi, \tag{3.2}$$

so that the Hamiltonian is given by $H(x,\xi) = u_0(x)\cdot\xi$. If φ_t denotes the integral flow of the field u_0, then the solution to (3.1)–(3.2) is given by

$$(x_0,\xi_0) \to \bigl(\varphi_t(x_0), \partial\varphi_t^{-\top}(x_0)\xi_0\bigr), \tag{3.3}$$

where $-\top$ denote the inverse of the transpose. Assuming $\xi_0 \neq 0$, we define the projectivized flow on $\Omega = \mathbb{T}^n \times \mathbb{S}^{n-1}$ given by

$$\chi_t(x_0,\xi_0) = \left(\varphi_t(x_0), \frac{\partial\varphi_t^{-\top}(x_0)\xi_0}{|\partial\varphi_t^{-\top}(x_0)\xi_0|}\right).$$

Let $\mathcal{B} = \{\mathcal{B}_t(\omega)\}_{t\in\mathbb{R},\omega\in\Omega}$ be the cocycle generated by the dynamical system

$$b_t = a_0\bigl(\chi_t(\omega)\bigr)b \tag{3.4}$$

on the trivial bundle $\Omega \times \mathbb{C}^d$.

Lemma 3.1. *The cocycle $\mathcal{B}$ leaves the frequency bundle $\mathcal{F}$ invariant.*

Proof. We recall that $p(\xi): \mathbb{C}^d \to F(\xi)$ is the orthogonal projection. Clearly, $p \in \mathcal{S}^0$. The orthogonal projection $\Pi: L^2 \to L^2_{\mathcal{F}}$ is the Fourier multiplier with symbol $p(\xi)$. We have $\Pi\mathcal{L} = \mathcal{L}$ on $D(L)$ by invariance of (2.4). So, by the composition formula (see, e.g., [27, Section I.3.6]) the symbol of $\Pi\mathcal{L}$ is given by

$$-iu_0(x)\cdot\xi - \partial u_0^\top(x)\xi\cdot\nabla p(\xi) + p(\xi)a_0(x,\xi) + \widetilde{a}_1(x,\xi),$$

where $\widetilde{a}_1 \in \mathcal{S}^{-1}$ (see the calculation in [28, Section 3.2]). Notice that the second symbol is simply the directional derivative $p_t(\xi)$ along the characteristics of (3.3). Comparing symbols of zero order for the operators $\Pi\mathcal{L}$ and $\mathcal{L}$, and using the fact that the actions of these operators are equal on $D(L)$, we obtain:

$$pa_0p + p_tp = a_0p.$$

Denoting $q = \mathrm{id} - p$ we have $qa_0 p = p_t p$. Now let us compute

$$(qb)_t = -p_t b + qb_t = -p_t b + qa_0 pb + qa_0 qb = -p_t qb + qa_0 qb = (-p_t + qa_0)(qb).$$

Thus, by Grönwall's inequality $|(qb)(t)| \leqslant |(qb)(0)| \exp\{\| - p_t + qa_0\|_\infty t\}$. So, if $b(0) \in F(\xi_0)$, then $b(t) \in F(\xi(t))$. □

We denote by B_t the restriction of the cocycle $\mathcal{B}_t$ onto the vector bundle $(\Omega, \mathcal{F})$. We call it the b-cocycle. Let Σ_B denote the dynamical (Sacker–Sell) spectrum of the b-cocycle. According to the structure theorem of Sacker and Sell [25], see also [17] and [8] and the bibliography therein, the dynamical spectrum of a finite dimensional cocycle consists of finitely many disjoint closed intervals $I_1, \dots, I_k$, where k does not exceed the dimension of the bundle.

Our main theorem is given next.

Theorem 3.2. *The following identity holds*

$$\left|\sigma_\Phi\left(G_t; L^2_{\mathcal{F}}\right)\right| = e^{t\Sigma_B}, \tag{3.5}$$

for all $t \in \mathbb{R}$.

Let $\lambda_{\max}$ denote the maximal exponential growth rate of the b-cocycle:

$$\lambda_{\max} = \lim_{t\to\infty} \frac{1}{t} \log \sup_{(x,\xi)\in\Omega, b\in F(\xi), |b|=1} \left|B_t(x,\xi)b\right|; \tag{3.6}$$

in other words, $\lambda_{\max}$ is the maximal element of Σ_B. According to the celebrated formula of Vishik [33], the spectral radius for the Browder essential spectrum of the operator G_t is given by

$$r_{\mathrm{ess}}(G_t) = e^{t\lambda_{\max}}. \tag{3.7}$$

Using that the radii of the Fredholm and essential spectra for any bounded operator are equal [23], we thus recover (3.7) from our (3.5). Also, we recall from [17] or [8, Theorem 8.15] the formula $\lambda_{\max} = \sup_{\nu\in\mathrm{Erg}} \lambda^1_\nu$, where λ^1_ν is the largest Lyapunov–Oseledets exponent from the Oseledets Multiplicative Ergodic Theorem for the b-cocycle corresponding to the χ_t-ergodic measure ν.

3.1. Trivialization

The following representation of G_t was proved in [28]:

$$G_t f = \Pi\left[\left(\mathrm{Op}[B_t]f\right) \circ \varphi_{-t}\right] + K_t f, \tag{3.8}$$

where K_t is a compact operator on $L^2_{\mathcal{F}}$. Respectively, on the entire L^2 we have

$$\mathcal{G}_t f = \left(\mathrm{Op}[\mathcal{B}_t]f\right) \circ \varphi_{-t} + \mathcal{K}_t f. \tag{3.9}$$

In order to enable the use of the Isomorphism Theorem, we would like to extend G_t onto the whole L^2 in such a way that the extension is produced by an extended cocycle as in (3.9) and at the same time the extended operator has a block diagonal form with respect to the decomposition

$L^2 = L^2_{\mathcal{F}} \oplus L^2_{\mathcal{F}^\perp}$ up to a compact term. Unfortunately, the original pair $\mathcal{B}$ and $\mathcal{G}$ is not good for this purpose since $\mathcal{G}$ may not leave $L^2_{\mathcal{F}^\perp}$ invariant even up to a compact perturbation. We therefore introduce a different extension. We start with a general lemma.

Lemma 3.3. *Let $a', a'' \in \mathcal{S}^0$ be 0-homogeneous symbols in ξ, and let*

$$a = (pa' + p_t)p + (qa'' + q_t)q. \tag{3.10}$$

Then the cocycle generated by the differential equation

$$b_t = a\big(\chi_t(\omega)\big)b \tag{3.11}$$

leaves the bundles $\mathcal{F}$ and $\mathcal{F}^\perp$ invariant.

Proof. Let us prove that $\mathcal{F}$ is invariant, the argument for $\mathcal{F}^\perp$ being similar. So, suppose $b_0 \in F(\xi_0)$. Then $q(\xi_0)b_0 = 0$, where $q = \mathrm{id} - p$. For the function $(qb)(t) = q(\xi(t; x_0, \xi_0)) \times b(t; x_0, \xi_0, b_0)$ we have the equation

$$(qb)_t = q_t b + q b_t = q_t b + (qp_t p + qa'' q + qq_t q)b.$$

Notice that $q_t = qq_t + q_t q$. So, $qq_t q = 2qq_t q$ and hence $qq_t q = 0$. Similarly, $qp_t p = p_t p$. Thus,

$$(qb)_t = (q_t + p_t p + qa'' q)b.$$

Notice that $q_t + p_t = \mathrm{id}_t = 0$. So,

$$(qb)_t = (q_t q + qa'' q)b = (q_t + qa'')(qb).$$

An application of Grönwall's lemma as in the proof of Lemma 3.1 finishes the argument. □

Lemma 3.4. *Let a be as in Lemma 3.3 and $\widetilde{a} \in \mathcal{S}^{-1}$. Let $\mathcal{G} = \{\mathcal{G}_t\}_{t \in \mathbb{R}}$ be the group generated by the operator*

$$\mathcal{L} = -u_0 \cdot \nabla + \mathrm{Op}[a + \widetilde{a}\,].$$

Then $\Pi \mathcal{G}_t (I - \Pi)$ and $(I - \Pi)\mathcal{G}_t \Pi$ are compact operators.

Proof. Indeed, let $f_0 \in D(\mathcal{L})$ and $f(t) = \mathcal{G}_t f_0$. Then $(\Pi f)_t = \Pi \mathcal{L} f$. By the composition formula, see, e.g., [27, Section I.3.6],

$$\Pi \mathcal{L} f = -u_0 \cdot \nabla(\Pi f) + \mathrm{Op}[p_t + pa]f + \mathrm{Op}[\widetilde{a}_1]f,$$

for some $\widetilde{a}_1 \in \mathcal{S}^{-1}$. Notice that $p_t + pa = ap$ and, hence,

$$\Pi \mathcal{L} f = -u_0 \cdot \nabla(\Pi f) + \mathrm{Op}[a](\Pi f) + \mathrm{Op}[\widetilde{a}_2]f.$$

We have established that Πf satisfies Eq. (2.1), up to a compact perturbation, with the initial condition Πf_0. So, by Duhamel's principle, $\Pi \mathcal{G}_t f_0 = \mathcal{G}_t \Pi f_0 + \mathcal{K}_t f_0$, for some compact operators $\mathcal{K}_t$. This implies the conclusion of the lemma. $\square$

Next, we claim that it suffices to prove Theorem 3.2 only in the case when the bundle $\mathcal{F}$ is trivial. Let $\mathcal{B}_t$ be the cocycle generated by (3.11), and let $\mathcal{B}'_t$ and $\mathcal{B}''_t$ denote its restrictions on $\mathcal{F}$ and $\mathcal{F}^\perp$, respectively. Also, we denote $\mathcal{G}'_t = \Pi \mathcal{G}_t \Pi$ and $\mathcal{G}''_t = (I - \Pi)\mathcal{G}_t(I - \Pi)$. Then

$$\Sigma_{\mathcal{B}} = \Sigma_{\mathcal{B}'} \cup \Sigma_{\mathcal{B}''} \tag{3.12}$$

by Lemma (3.3), and from Lemma 3.4 we infer:

$$\sigma_\Phi\big(\mathcal{G}_t; L^2\big) = \sigma_\Phi\big(\mathcal{G}'_t; L^2_{\mathcal{F}}\big) \cup \sigma_\Phi\big(\mathcal{G}''_t; L^2_{\mathcal{F}^\perp}\big). \tag{3.13}$$

Let us fix a large $\Lambda > 0$ and consider the rescaled symbol $a'' - \Lambda \mathrm{id}$ in place of a'' in the construction of Lemma 3.3. Then the corresponding cocycle will rescale to $e^{-\Lambda t}\mathcal{B}''_t$ while the group will rescale to $e^{-\Lambda t}\mathcal{G}''_t$ as seen from (3.9). Let us suppose that Theorem 3.2 holds in the case of the trivial bundle, that is, for the rescaled cocycle on $\Omega \times \mathbb{C}^d$ and the rescaled group of operators on L^2. Then, using the analogs of (3.12) and (3.13) for the rescaled cocycle and group, for every $t \in \mathbb{R}$ one obtains the identity

$$\sigma_\Phi\big(\mathcal{G}'_t; L^2_{\mathcal{F}}\big) \cup e^{-\Lambda t}\sigma_\Phi\big(\mathcal{G}''_t; L^2_{\mathcal{F}^\perp}\big) = \exp\{t\Sigma_{\mathcal{B}'}\} \cup \exp\big\{t(\Sigma_{\mathcal{B}''} - \Lambda)\big\}. \tag{3.14}$$

Passing to the limit as $\Lambda \to \infty$, we eliminate the pair $(\mathcal{B}'', \mathcal{G}'')$ from (3.14), and obtain the formula

$$\sigma_\Phi\big(\mathcal{G}'_t; L^2_{\mathcal{F}}\big) = \exp\{t\Sigma_{\mathcal{B}'}\}.$$

Applying the above construction for $a' = a'' = a_0$ and $\widetilde{a} = a_1$, we find that $\mathcal{B}'_t = B_t$, and $\mathcal{G}'_t = G_t + K_t$, where the operator K_t is compact. Indeed, by the original assumption $\mathcal{G}_t$ leaves $L^2_{\mathcal{F}}$ invariant, and thus

$$\mathcal{G}'_t = \Pi \mathcal{G}_t \Pi = \mathcal{G}_t \Pi = G_t + (I - \Pi)\mathcal{G}_t \Pi = G_t + K_t$$

by Lemma 3.4. We have justified the claim and, thus, reduced the proof of Theorem 3.2 to the case of the trivial bundle.

3.2. *Proof of Theorem 3.2*

In view of the preceding subsection, we can assume that G_t is acting on the entire L^2, B_t is acting on the trivial bundle $\Omega \times \mathbb{C}^d$, and

$$G_t f = \big(\mathrm{Op}[B_t] f\big) \circ \varphi_{-t} + K_t f, \tag{3.15}$$

where K_t is a compact operator.

Let μ be the product of the Lebesgue and surface measures on $\Omega = \mathbb{T}^n \times \mathbb{S}^{n-1}$. Let $\mathfrak{A}$ be the C^*-algebra of operators on $L^2(\Omega, \mu)$ acting by multiplication by continuous $(d \times d)$-matrix-valued functions. Let us fix $t \in \mathbb{R}$. We define the shift operator

$$T_m u = |\det \partial \chi_{-mt}|^{\frac{1}{2}} u \circ \chi_{-mt}, \quad m \in \mathbb{Z}.$$

Notice that T_m is a unitary operator. Thus, the weighted shift operator $E_{mt} = T_m B_{mt}$ is an element of the C^*-algebra $\mathfrak{B} = C^*(\mathfrak{A}, T, \mathbb{Z})$ generated by the operators from $\mathfrak{A}$, and the operators T_m, $m \in \mathbb{Z}$. Since C^*-subalgebras are saturated, E_m has the same spectra as an element of $\mathfrak{B}$ and as an operator on $L^2(\Omega, \mu)$.

The next lemma relates the dynamical spectrum of the cocycle B and the spectrum of the operator E_t, cf. [22], and also [9,16] and [8, Corollary 6.45].

Lemma 3.5. $|\sigma(E_t)| = e^{t\Sigma_B}$, $t \in \mathbb{R}$.

Proof. Our short argument is based on the use of Mañe sequences, see [29] and [8, Definition 7.16]. Suppose $\lambda \in \Sigma_B$. Then there is a Mañe sequence either for B_t or for its adjoint (see [29]). Let us assume that there is a Mañe sequence $\{(\omega_n, b_n)\}$ for $B = \{B_t\}_{t \in \mathbb{R}}$. So, for some constants $c, C > 0$ we have

$$\left\| B_{nt}(\omega_n) b_n \right\| > c e^{\lambda n t}, \tag{3.16}$$

$$\left\| B_{2nt}(\omega_n) b_n \right\| < C e^{2\lambda n t}. \tag{3.17}$$

Let U_n be a small open neighborhood of ω_n. Let us consider the normalized sequence of functions

$$f_n(\omega) = \mu(U_n)^{-1/2} I_{U_n}(\omega) b_n,$$

where I_{U_n} is a smoothed characteristic function of the neighborhood. A direct calculation shows that

$$\|E_{nt} f_n\| > \frac{c}{2} e^{\lambda n t}, \tag{3.18}$$

$$\|E_{2nt} f_n\| < 2C e^{2\lambda n t}, \tag{3.19}$$

provided U_n's are sufficiently small. According to [29, Lemma 3.1] we conclude that $e^{\lambda t} \in |\sigma(E_t)|$.

If there is a Mañe sequence for the adjoint of B_t, we can run the same argument to show that $e^{\lambda t} \in |\sigma(E_t^*)|$, which yields the same result.

Conversely, let $\lambda \notin \Sigma_B$. Then there exists a continuous projection-valued function $P(\omega)$: $\mathbb{C}^d \to \mathbb{S}(\omega)$ so that $\mathbb{C}^d = \mathbb{S}(\omega) \oplus \mathbb{U}(\omega)$ and

$$\left\| B_t(\omega) b \right\| \leqslant C e^{(\lambda - \varepsilon)t} \|b\|, \quad b \in \mathbb{S}(\omega), \tag{3.20}$$

$$\left\| B_t(\omega) b \right\| \geqslant c e^{(\lambda + \varepsilon)t} \|b\|, \quad b \in \mathbb{U}(\omega). \tag{3.21}$$

This creates the splitting of $L^2(\Omega)$ into the direct sum of subspaces

$$X = \{ f \in L^2 \colon f(\omega) \in \mathbb{S}(\omega) \text{ a.e. } \}, \tag{3.22}$$

$$Y = \{ f \in L^2 \colon f(\omega) \in \mathbb{U}(\omega) \text{ a.e. } \}. \tag{3.23}$$

Clearly, $\|E_t|_X\| \leqslant Ce^{(\lambda-\varepsilon)t}$ and $\|E_t f\| \geqslant ce^{(\lambda+\varepsilon)t}$, $f \in Y$. This implies that the group $\{e^{-\lambda t} E_t\}_{t\in\mathbb{R}}$ is hyperbolic, and concludes the proof of the lemma. □

Now let $\mathfrak{A}'$ denote the C^*-algebra generated by PDO's with 0-homogeneous symbols from $\mathfrak{A}$. Let $\mathfrak{K}$ denote the class of compact operators on $L^2(\mathbb{T}^n)$. By a well-known result of Seeley [26], the map

$$J : a \to \mathrm{Op}[a] + \mathfrak{K}$$

defines an isometric isomorphism between $\mathfrak{A}$ and $\mathfrak{A}'/\mathfrak{K}$. Moreover, we have

$$\mathrm{Op}[a \circ \chi_{-t}](f \circ \varphi_{-t}) = (\mathrm{Op}[a] f) \circ \varphi_{-t} + \mathfrak{K}.$$

In terms of the shift operator $T'_m f = f \circ \varphi_{-mt}$ we can express the latter fact as

$$J(T_m a T_{-m}) = T'_m J(a) T'_{-m}.$$

We see that all conditions of the Isomorphism Theorem A.1 are fulfilled. According to the theorem, J extends to the isomorphism of the algebras $\mathfrak{B}$ and

$$\mathfrak{B}' = C^*(\mathfrak{A}', T', \mathbb{Z})/\mathfrak{K} = C^*(\mathfrak{A}'/\mathfrak{K}, T', \mathbb{Z}).$$

In view of (3.15) we have the equality $J(E_t) = G_t$ as elements of $\mathfrak{B}'$. This implies the equivalence of the spectra

$$\sigma(E_t) = \sigma(G_t, \mathfrak{B}') = \sigma_\Phi(G_t). \tag{3.24}$$

An application of Lemma 3.5 finishes the proof of (3.5).

3.3. Extension of Theorem 3.2 to Sobolev spaces

Let us fix a smoothness parameter $m \in \mathbb{R}$, not necessarily an integer, and consider the corresponding frequency constrained Sobolev space

$$H^m_{\mathcal{F}} = \{ u \in H^m \colon \hat{u}(k) \in F(k), \text{ for all } k \in \mathbb{Z}^n \}. \tag{3.25}$$

As on $L^2_{\mathcal{F}}$, Eq. (2.1) generates a group $\{G^m_t\}_{t\in\mathbb{R}}$ on $H^m_{\mathcal{F}}$ as well, and it is natural to ask what is its Fredholm spectrum. Using a construction presented in [28, Section 3.5], one can reduce the Sobolev case back to L^2. According to that construction, we consider the isomorphism $I_m = |\nabla|^m : H^m_{\mathcal{F}} \to L^2_{\mathcal{F}}$. Let L_m denote the generator of the group $\{G^m_t\}_{t\in\mathbb{R}}$ on $H^m_{\mathcal{F}}$. We consider a new generator $L = I_m L_m I_m^{-1}$ on L^2. Clearly, if $G_t = e^{tL}$, then $G_t = I_m G^m_t I_m^{-1}$, and so the spectrum of G_t on $L^2_{\mathcal{F}}$ is equal to the spectrum of G^m_t on $H^m_{\mathcal{F}}$. By the composition formula for symbols, see, e.g., [27, Section I.3.6], the operator L has the same advective structure as the original PDE (2.1) with its principal symbol given by

$$a_m(x,\xi) = a_0(x,\xi) - m\big(\partial u_0^\top(x)\xi, \xi\big)|\xi|^{-2}\mathrm{id}.$$

One can also see that the corresponding b-cocycle $\mathcal{B}^m$ is given by

$$B_t^m(x,\xi) = \big|\partial\varphi_t^{-\top}(x)\xi/|\xi|\big|^m B_t(x,\xi), \quad (x,\xi) \in \Omega. \tag{3.26}$$

A direct application of Theorem 3.2 yields the following result.

Corollary 3.6. *The following identity holds*

$$\big|\sigma_\Phi\big(G_t; H^m_{\mathcal{F}}\big)\big| = e^{t\Sigma_{B^m}}, \tag{3.27}$$

for all $t \in \mathbb{R}$.

Clearly, the spectrum of the new cocycle $\mathcal{B}^m$ can feel the effect of an exponential stretching of trajectories in the original flow u_0. If the stretching occurs, then an analysis presented in [28, Theorem 6.2] shows that for $|m|$ large enough Σ_{B^m} becomes a solid closed segment between the smallest and largest Lyapunov exponents of $\mathcal{B}^m$. Thus the structure of the Fredholm spectrum $\sigma_\Phi(G_t; H^m_{\mathcal{F}})$ in this case is particularly simple.

4. Rotational invariance of the spectrum

Under certain topological conditions on u_0 and the level sets of the map $\omega \to a_0(\omega)$ one can establish rotational invariance of $\sigma_\Phi(G_t)$. We first explore the case of trivial frequency bundle and then extend it to an arbitrary case via the construction of Section 3.1.

In view of (3.24) we can restate the issue of rotational invariance entirely in terms of the Mather semigroup $\{E_t\}_{t\in\mathbb{R}}$, see [22] and [8, Chapter 6] for the terminology. So, let $\mathcal{B}$ be the cocycle generated by the differential equation

$$b_t = a\big(\chi_t(\omega)\big)b \tag{4.1}$$

over the flow χ_t on Ω. Let us consider the corresponding Mather semigroup on $L^2(\Omega,\mu)$:

$$E_t f(\omega) = \big|\det\partial\chi_{-t}(\omega)\big|^{\frac{1}{2}} B_t\big(\chi_{-t}(\omega)\big) f\big(\chi_{-t}(\omega)\big).$$

We recall from [8, Section 6.2] that a point $\omega_0 \in \Omega$ and a vector $b_0 \in \mathbb{C}^d$ are called the Mañe pair for $\mathcal{B}$ if

$$\sup\big\{\big|B_t(\omega_0)b_0\big|:\ t \in \mathbb{R}\big\} < \infty.$$

It is known that for every $\lambda \in \Sigma_B$ there exists a Mañe pair either for the cocycle $e^{-\lambda t}B_t$ or for its adjoint $e^{-\lambda t}B^*_{-t}$, see [29] and [8, Chapter 6]. The classical construction of Mather [22] allows one to establish rotational invariance of the spectrum if the underlying transformation is aperiodic. We recall that a transformation $\chi : \Omega \to \Omega$ is called aperiodic if the set of points with arbitrary long orbits for χ is dense in Ω, that is, if for every $N > 0$ and open $U \subset \Omega$ there exists a point $x \in U$ with χ-period no less than N. In the following lemma we adopt Mather's argument to the case of L^2 (see also Lemma 6.28 and Corollary 6.45 in [8] and the literature cited therein).

Lemma 4.1. *Suppose that for a fixed t the diffeomorphism $\chi_t : \Omega \to \Omega$ is aperiodic. Then*

$$\sigma(E_t) = \mathbb{T} \cdot e^{t\Sigma_B}. \tag{4.2}$$

Proof. Let $\lambda \in \Sigma_B$ and let us fix an arbitrary $\alpha \in [0, 2\pi]$. We need to show that $z = e^{t(\lambda+i\alpha)} \in \sigma(E_t)$. Suppose there exists a Mañe pair (ω_0, b_0) for the cocycle $\{e^{-\lambda t} B_t\}_{t\in\mathbb{R}}$ (the case of a pair for the adjoint cocycle is treated by duality) so that

$$\sup\{\left|e^{-\tau\lambda} B_\tau(\omega_0) b_0\right|: \tau \in \mathbb{R}\} = M < \infty.$$

Let us fix a natural $N > 0$ and find a point ω_1, close enough to ω_0, with period greater than $2N + 1$, and such that

$$\sup\{\left|e^{-\tau\lambda} B_\tau(\omega_1) b_0\right|: \tau \in [-Nt, Nt]\} < 2M.$$

Define a bump function $h(\omega)$ near ω_1 so that

$$\frac{1}{2}\left|B_{tj}(\omega_1) b_0\right| \leqslant \left\|B_{tj}(\cdot) b_0 h(\cdot)\right\|_2 \leqslant 2\left|B_{tj}(\omega_1) b_0\right|,$$

for all $j = -N, \ldots, N$, and

$$\operatorname{supp}(h \circ \chi_{-tj}) \cap \operatorname{supp}(h \circ \chi_{-ti}) = \emptyset,$$

for all $j \neq i$, $i, j \in [-N, N]$. Let $\gamma(j) = (N - |j|)/N$ for $j = -N, \ldots, N$ and $\gamma(j) = 0$ otherwise. Define the following function:

$$g = \sum_{j=-N}^{N} z^{-j} \gamma(j) E_{tj}(b_0 h).$$

By construction,

$$\|g\|_2^2 = \sum_{j=-N}^{N} \left|\gamma(j)\right|^2 e^{-2\lambda tj} \left\|B_{tj}(\cdot) b_0 h(\cdot)\right\|_2^2 \geqslant \frac{1}{2}.$$

A direct computation shows

$$E_t g = z \sum_{j=-N}^{N} z^{-j} \gamma(j-1) E_{tj}(b_0 h).$$

So,

$$\|E_t g - zg\|_2^2 \leqslant 2\frac{e^{2\lambda t}}{N^2}\sum_{j=-N}^{N} e^{-2\lambda tj}\big\|B_{tj}(\cdot)b_0h(\cdot)\big\|_2^2 \leqslant \frac{8Me^{2\lambda t}}{N}.$$

Thus,

$$\frac{\|E_t g - zg\|_2^2}{\|g\|_2^2} \leqslant C/N,$$

where N can be chosen arbitrarily large. This finishes the proof. □

Lemma 4.2. *Suppose that the interior of the set of stationary points of the flow $\{\chi_t\}_{t\in\mathbb{R}}$ is empty. Then for all but countably many $t > 0$ the transformation $\chi_t : \Omega \to \Omega$ is aperiodic.*

Proof. Let $\pi(\omega)$ denote the prime period function of the flow $\{\chi_t\}_{t\in\mathbb{R}}$, i.e.,

$$\pi(\omega) = \inf\big\{t \geqslant 0\colon\ \chi_t(\omega) = \omega\big\}.$$

Clearly, $\pi^{-1}(0)$ is closed and, by assumption, has empty interior. Let us consider the set

$$S = \big\{s \in \mathbb{R}^+ \cup \{\infty\}\colon\ \pi^{-1}(s) \text{ is somewhere dense}\big\}.$$

The somewhere dense condition means that there is an open set U such that $\pi^{-1}(s)$ is dense in U. By Zorn's lemma for every $s \in S$ there exists a maximal such set (which is also the largest). We denote the maximal set by U_s.

We will now select, by an iterative procedure, an at most countable subcollection $U_{s_1}, \ldots,$ $U_{s_n}, \ldots$ such that $V = \bigcup_{n=1}^{\infty} U_{s_n}$ absorbs all other sets of the collection $\{U_s\}_{s\in S}$ up to a negligible leftover. In other words, for every $s \in S$ one has

$$\mu(U_s \setminus V) = 0. \tag{4.3}$$

Let us denote

$$m_0 = \sup_{s\in S}\mu(U_s).$$

If $m_0 = 0$, then $V = \emptyset$ and we are done. If $m_0 > 0$ let us choose $s_1 \in S$ such that

$$\mu(U_{s_1}) > m_0/2.$$

Denote

$$m_1 = \sup_{s\in S}\mu(U_s \setminus U_{s_1}).$$

If $m_1 = 0$ we stop, otherwise we select $s_2 \in S$ such that

$$\mu(U_{s_2} \setminus U_{s_1}) > m_1/2.$$

Suppose the sets $U_{s_1}, \dots, U_{s_n}$ have been selected. We denote

$$m_n = \sup_{s\in S} \mu\big(U_s \setminus (U_{s_1} \cup \cdots \cup U_{s_n})\big).$$

If $m_n = 0$ we stop, otherwise find a $s_{n+1} \in S$ such that

$$\mu\big(U_{s_{n+1}} \setminus (U_{s_1} \cup \cdots \cup U_{s_n})\big) > m_n/2.$$

We have

$$\infty > \mu(V) = \mu(U_{s_1}) + \sum_{n=1}^{\infty} \mu\big(U_{s_{n+1}} \setminus (U_{s_1} \cup \cdots \cup U_{s_n})\big) > \frac{1}{2}\sum_{n=1}^{\infty} m_n.$$

Thus, $m_n \to 0$ as $n \to \infty$. To prove (4.3), we observe that if there exists $s \in S$ such that

$$\mu(U_s \setminus V) > 0,$$

then one can choose n large enough so that

$$\mu(U_s \setminus V) > m_n.$$

Hence,

$$\mu\big(U_s \setminus (U_{s_1} \cup \cdots \cup U_{s_n})\big) \geqslant \mu(U_s \setminus V) > m_n,$$

which contradicts the definition of m_n, thus proving (4.3).

We now claim that the set

$$E = \{s_n \cdot r\colon\ s_n \text{ finite},\ n \in \mathbb{N},\ r \in \mathbb{Q}\}$$

is exceptional, i.e., for every $t \notin E$ the map χ_t is aperiodic. To this end, let us fix $t \notin E$, an open set $U \subset \Omega$, and let us show that U contains points of arbitrarily long χ_t-orbits.

First, let us assume that $U \cap V \neq \emptyset$. If $U \cap U_\infty \neq \emptyset$, we are done. If $U \cap U_{s_n} \neq \emptyset$ for a finite s_n, then, since $\pi^{-1}(s_n)$ is dense in U_{s_n}, one can find $\omega \in U \cap \pi^{-1}(s_n)$. If ω were χ_t-periodic then for some $m \in \mathbb{N}$ one has $\chi_{mt}(\omega) = \omega = \chi_{s_n}(\omega)$ yielding $mt = ks_n$ for some $k \in \mathbb{N}$, cf. [6, Theorem II.2.12]. Thus, in view of the choice of t, ω is non-periodic and we are done in this case as well.

Next, let us assume that $U \cap V = \emptyset$, and there is a cap on the periods of points in U. In other words, there is a finite $N \in \mathbb{N}$ such that for any $\omega \in U$ there is $n \leqslant N$ such that $\chi_{nt}(\omega) = \omega$. Let $M = N!$. Then $\chi_{Mt}(\omega) = \omega$ for all $\omega \in U$. Since $\pi(\omega)$ is the smallest period of ω, by [6, Theorem II.2.12] we have

$$\pi(\omega) = \frac{Mt}{n(\omega)},$$

for every $\omega \in U$ and corresponding $n(\omega) \in \mathbb{N} \cup \{\infty\}$. This shows that the sets $\{\pi^{-1}(Mt/n)\}_{n\in\mathbb{N}\cup\{\infty\}}$ cover U. By the Baire category theorem one of these sets, say for

$s^* = Mt/n^*$, must be somewhere dense in U. Hence, $s^* \in S$, and by the maximality of U_{s^*}, one has $U_{s^*} \cap U \neq \emptyset$. Thus, we obtain

$$0 < \mu(U_{s^*} \cap U) \leqslant \mu(U_{s^*} \setminus V),$$

which contradicts property (4.3) of V. □

Notice that the above lemma holds if Ω is replaced by any complete metric space endowed with a finite Borel measure, which is not vanishing on any open set. In particular, it holds on any factor-space of Ω.

Let us denote by V the interior of the set $\pi^{-1}(0)$, and by W the interior of the set of stationary points of u_0. Notice that $V = W \times \mathbb{S}^{n-1}$. So, the condition of Lemma 4.2 is equivalent to W being empty. Combining Lemmas 4.1 and 4.2, we obtain a rotational invariance result which does not require any additional assumptions on the symbol a. So, it holds immediately for the original group $\{G_t\}_{t\in\mathbb{R}}$ of operators on the space $L^2_{\mathcal{F}}$ with constraints.

Proposition 4.3. *If the interior of the set of stationary points of the field u_0 is empty, then for all but countably many $t \in \mathbb{R}$ one has*

$$\sigma_\Phi\big(G_t; L^2_{\mathcal{F}}\big) = \mathbb{T} \cdot e^{t\Sigma_B}.$$

If V is not empty, then the conclusion of Lemma 4.2 of course does not hold. However we can still obtain the rotational invariance of the spectrum for a variety of most natural equations. To this end we will use a factorization procedure that eliminates V under the assumption that the level sets of $a(\omega)$ and hence $B_t(\omega)$ cross through the boundary of V. So, for the matrix-valued map

$$\omega \to a(\omega) \in M^{d\times d}$$

let us denote by F_a the level set

$$F_a = \big\{\omega \in \Omega\colon a(\omega) = a\big\}.$$

Let $\overline{V}$ denote the closure of V, and ∂V the boundary of V. By our original assumption $u_0 \neq 0$, so V does not cover the entire $\mathbb{T}^n$.

Lemma 4.4. *Suppose that for every matrix $a \in M^{d\times d}$ either $F_a \cap \overline{V} = \emptyset$ or else $F_a \cap \partial V \neq \emptyset$. Then the identity*

$$\sigma(E_t) = \mathbb{T} \cdot e^{t\Sigma_B} \tag{4.4}$$

holds for all but countably many $t \in \mathbb{R}$.

Proof. Let us consider the decomposition of Ω into conjugacy classes:

$$\Omega = \bigcup_{a\in M^{d\times d}} \{F_a \cap \overline{V}\} \bigcup_{\omega\notin\overline{V}} \{\omega\}.$$

Let $\widetilde{\Omega}$ denote the corresponding factor-space, and $j : \Omega \to \widetilde{\Omega}$ be the factor map. Notice that the diffeomorphism χ_t maps points of one conjugacy class into a single class (indeed, for every $\omega \notin \overline{V}$ the corresponding to $\chi_t(\omega)$ class is a singleton and the statement is trivial; on the other hand, $\chi_t(\omega) = \omega$ for every $\omega \in F_a \cap \overline{V}$). Thus, the factor flow given by

$$\widetilde{\chi}_t\big(j(\omega)\big) = j\big(\chi_t(\omega)\big)$$

is well defined on $\widetilde{\Omega}$. Notice that the map $\omega \to a(\omega)$ is constant on every conjugacy class and hence (4.1) produces the cocycle B_t with the same property. Let us denote by $\widetilde{B}_t$ the corresponding factor-cocycle over $\widetilde{\chi}_t$.

We claim that the interior of the set of stationary points of the flow $\{\widetilde{\chi}_t\}_{t\in\mathbb{R}}$ is empty. Indeed, let $\widetilde{U} \subset \widetilde{\Omega}$ by an open set. Then $U = j^{-1}(\widetilde{U})$ is open in Ω and with each point contains its entire conjugacy class. In the case when $\omega \notin \overline{V}$, $\omega \in U$, the map $t \to \widetilde{\chi}_t(\{\omega\})$ is not constant, and we are done. So, let us suppose that $\omega \in U \cap F_a \cap \overline{V}$ for some $a \in M^{d\times d}$. Then $F_a \cap \overline{V} \subset U$, and hence by the condition of the lemma there is a point $\omega' \in \partial V \cap U$. This implies that U contains a point in the complement of $\overline{V}$ and we are in the previous case again.

According to Lemma 4.2, the transformation $\widetilde{\chi}_t : \widetilde{\Omega} \to \widetilde{\Omega}$ is aperiodic for all but countably many t. This implies that the set of points with arbitrary long orbits of χ_t is dense in $\Omega \setminus \overline{V}$.

Let us now fix $\lambda \in \Sigma_B$ and suppose that (ω_0, b_0) is a Mañe pair for $e^{-\lambda t} B_t$ (the case of adjoint cocycle is treated similarly). If $\omega_0 \notin \overline{V}$ then, by the previous paragraph, one can approximate ω_0 by points with arbitrary long orbits of χ_t, and the remaining argument repeats that of Lemma 4.1. Suppose $\omega_0 \in F_a \cap \overline{V}$. Since the map $\omega \to a(\omega)$ is constant throughout the class $F_a \cap \overline{V}$ and χ_t is stationary on it, we conclude that the map $\omega \to B_t(\omega)$ is constant on the class as well. Thus, the pair (ω, b_0) with any $\omega \in F_a \cap \overline{V}$ is a Mañe pair. By the condition of the lemma, we can choose $\omega \in F_a \cap \partial V$. Every open neighborhood of ω must intersect $\Omega \setminus \overline{V}$ and hence will contain points with arbitrary long orbits of χ_t. We are back to Lemma 4.1 again. □

We can use Lemma 4.4 to produce sufficient conditions for rotational invariance of the spectrum of the original group G_t. Notice that on V we have $p_t = q_t = 0$, see the proof of Lemma 3.1. So, choosing $a' = a_0$ and $a'' = -\Lambda \mathrm{id}$ in (3.10), we obtain a symbol a which, on V, equals

$$a(x,\xi) = p(\xi)a_0(x,\xi)p(\xi) - \Lambda q(\xi).$$

Suppose that pa_0p is independent of x on each connected component of V, and, hence, so is $a(\omega)$. This means that every level set F_a that crosses $\overline{V}$ at some point (x_0, ξ_0) also contains the entire slice $\overline{W} \times \{\xi_0\}$. As a consequence, the set must contain a point of the boundary $\partial V = \partial W \times \mathbb{S}^{n-1}$, and the condition of Lemma 4.4 is fulfilled. We summarize this discussion in the following result.

Theorem 4.5. *Let W be the interior of the set of stationary points of u_0. Suppose that the matrix $p(\xi)a_0(x,\xi)p(\xi)$ is independent of x on each connected component of $W \times \mathbb{S}^{n-1}$. Then the Fredholm spectrum of G_t is rotationally invariant for all but countably many $t \in \mathbb{R}$, and we have the identities*

$$\sigma_\Phi\big(G_t; L^2_{\mathcal{F}}\big) = \mathbb{T} \cdot e^{t\Sigma_B}, \tag{4.5}$$

$$\sigma_\Phi\big(G_t; H^m_{\mathcal{F}}\big) = \mathbb{T} \cdot e^{t\Sigma_{B^m}}. \tag{4.6}$$

A simple example shows that the "all but countably many" clause cannot be removed. Indeed, let us consider the 2D Euler equation linearized around a constant parallel shear flow $u_0(x, y) = \langle U, 0\rangle$. Then $Lv = -U\partial_x v$ and $G_t v(x, y) = v(x - Ut \bmod 2\pi, y)$. Clearly, for $t \in 2\pi U^{-1}\mathbb{Q}$ the spectrum of G_t is not rotationally invariant.

5. Examples

Many examples of advective equations with their symbols were displayed in [28, Section 3.3]. All of them satisfy the conditions of Theorems 3.2 and 4.5. Here, we mention only a few.

Example 5.1. The Euler equations in velocity form, see [14] and the literature therein, are given by

$$u_t + (u \cdot \nabla)u + \nabla p = 0, \tag{5.1a}$$

$$\operatorname{div} u = 0. \tag{5.1b}$$

Let u_0 be a smooth equilibrium solution of (5.1). The linearized equation takes the form

$$v_t = -(u_0 \cdot \nabla)v - (v \cdot \nabla)u_0 - \nabla q. \tag{5.2}$$

Let us rewrite it as follows:

$$v_t = -(u_0 \cdot \nabla)v + (v \cdot \nabla)u_0 - 2(v \cdot \nabla)u_0 - \nabla q. \tag{5.3}$$

The first two terms form the Lie bracket of u_0 and v, which is divergence-free. Therefore, the Leray projection Π affects only the third term. We thus can write the pseudo differential operator A as

$$Av = (v \cdot \nabla)u_0 - 2\Pi\big[(v \cdot \nabla)u_0\big].$$

So, the principal symbol equals

$$a_0(x, \xi) = \big(q(\xi) - p(\xi)\big)\partial u_0(x), \tag{5.4}$$

where $q(\xi) = |\xi|^{-2}\xi \otimes \xi$. Clearly, it satisfies the assumptions of Theorem 4.5. It is worth mentioning that in two-dimensional case the rotational invariance holds for all $t \in \mathbb{R}$, provided $\lambda_{\max} > 0$, by an explicit construction of a sequence of approximate eigenfunctions presented in [30].

In 3D case the b-cocycle may not have more than one gap in its dynamical spectrum.

Example 5.2. For the Euler equation equipped with the Coriolis force, see [32], one obtains the symbol

$$a_0(x, \xi) = \big(q(\xi) - p(\xi)\big)\partial u_0(x) - 2p(\xi)\Omega \times b, \tag{5.5}$$

where Ω is an angular velocity. Again, on the entire set W, a_0 depends only on ξ. So, Theorem 4.5 applies.

Example 5.3. In the vorticity form of the Euler equations, see [19], the symbol takes the following form:

$$a_0(x,\xi) = \partial u_0(x) - \frac{\omega_0(x)\cdot\xi}{|\xi|^2}\xi\times,$$

where $\omega_0(x) = \nabla \times u_0(x)$, and the conditions of Theorems 3.2 and 4.5 hold.

Example 5.4. The Camasa–Holm model [13] corresponds to the symbol

$$a_0(x,\xi) = -p(\xi)\partial u_0^\top(x) + q(\xi)\partial u_0(x),$$

and the conditions of Theorems 3.2 and 4.5 hold.

Example 5.5. The surface quasi-geostrophic (active scalar) equation [10,15] has a purely complex symbol

$$a_0(x,\xi) = i\frac{\xi^\perp\cdot\nabla\theta_0(x)}{|\xi|},$$

where θ_0 is a stationary temperature, and $u_0 = -\nabla^\perp(-\Delta)^{-1/2}\theta_0$. In this case we have trivially $\Sigma_B = \{0\}$ and there is no unstable continuous spectrum (see [15] for more information).

Appendix A. The Isomorphism Theorem

The Isomorphism Theorem concerns the following situation. Let $\mathfrak{A}$ be a C^*-algebra realized as a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. Let T: $m \to T_m$ be a unitary representation of the group $\mathbb{Z}$ in $\mathcal{B}(\mathcal{H})$ satisfying the property $T_m\mathfrak{A}T_{-m} = \mathfrak{A}$ for each $m\in\mathbb{Z}$ so that it defines the group action $\widehat{T}_m : a \to T_m a T_{-m}$, $a\in\mathfrak{A}$, on $\mathfrak{A}$. We denote by $\mathfrak{B} = C^*(\mathfrak{A}, T, \mathbb{Z})$ the C^*-algebra of operators generated by $\mathfrak{A}$ and T_m, $m\in\mathbb{Z}$. A frequently studied particular case is when $\mathcal{H} = L^2(\mathcal{E})$, the space of sections of a smooth finite dimensional trivial vector bundle $\mathcal{E} = \Omega\times\mathbb{C}^d$ with the compact base Ω, $\mathfrak{A}$ is realized as the algebra of operators of multiplication by continuous matrix-valued functions, and T_m are the unitary shift operators $T_m u = |\det\partial\chi_{-m}|^{1/2} u\circ\chi_{-m}$ induced by a group $\{\chi_m\}_{m\in\mathbb{Z}}$ of diffeomorphisms of Ω.

A general form of the Isomorphism Theorem states the conditions under which $\mathfrak{B}$ is isomorphic to the cross-product of $\mathfrak{A}$ and the group action $\widehat{T}$, see [1, Theorem 8.8], [4, Corollary 12.16]. In another, more suitable for our purposes version of the Isomorphism Theorem, see Corollary 12.17 and Theorem 16.21 in [4], one considers yet another C^*-algebra $\mathfrak{A}'$ and unitary representation T' of $\mathbb{Z}$ with the property $T'_m\mathfrak{A}'T'_{-m} = \mathfrak{A}'$ for each $m\in\mathbb{Z}$. Moreover, it is assumed that there is an isomorphism

$$J : \mathfrak{A}\to\mathfrak{A}' \quad \text{so that} \quad J(T_m a T_{-m}) = T'_m J(a) T'_{-m} \quad \text{for all } a\in\mathfrak{A},\ m\in\mathbb{Z}.$$

The Isomorphism Theorem states the conditions under which the maps $a\to J(a)$, $T_m\to T'_m$ extend to an isomorphism of the C^*-algebras $\mathfrak{B} = C^*(\mathfrak{A}, T, \mathbb{Z})$ and $\mathfrak{B}' = C^*(\mathfrak{A}', T', \mathbb{Z})$.

It is an application of the latter version of the Isomorphism Theorem that is being used in this paper. Specifically, we use the following special case of the result recorded in Theorem 4.12 and Remark 4.13 of [2], see also Theorem 12.4 and the proof of Theorem 16.1 in [1]. Let M be a C^∞-smooth compact manifold with a fixed Riemann metric which induces a measure on M, thus generating the norm in $L^2(M;\mathbb{C}^d)$. Given a group of diffeomorphisms $\{\varphi_m\}_{m\in\mathbb{Z}}$ on M, preserving the measure, we consider the corresponding representation of $\mathbb{Z}$ in $L^2(M;\mathbb{C}^d)$ by the group of the unitary operators $\{T'_m\}_{m\in\mathbb{Z}}$ defined by $T'_m f = f\circ\varphi_{-m}$. Let $\mathfrak{A}'$ denote the algebra of matrix pseudo differential operators of zero order in $L^2(M;\mathbb{C}^d)$. Let $\mathfrak{K}$ denote the ideal of compact operators in $L^2(M;\mathbb{C}^d)$. We will call an operator of the form

$$\sum_{m=-k}^{k} a'_m T'_m + K, \quad a'_m \in \mathfrak{A}',\ k\in\mathbb{Z},\ K\in\mathfrak{K}, \tag{A.1}$$

the *pseudo differential operator with the shift*, and let $\mathfrak{B}' = C^*(\mathfrak{A}', T', \mathbb{Z})$ denote the algebra of the pseudo differential operators with the shift, that is, the C^*-algebra generated by the operators in (A.1). Further, we let $\Omega = S^*M$ denote the unit cotangent bundle of M, and let $\mathfrak{A}$ denote the C^*-algebra of matrix symbols of the pseudo differential operators from $\mathfrak{A}'$ realized as the algebra of operators of multiplication by matrices in $L^2(\Omega;\mathbb{C}^d)$. By the classical results [26], the map $J: a\to a'+\mathfrak{K}$, $a'=\mathrm{Op}[a]$, is an isomorphism between $\mathfrak{A}$ and $\mathfrak{A}'/\mathfrak{K}$. Finally, let $\{\chi_m\}_{m\in\mathbb{Z}}$ denote the extension of φ_m on Ω defined by the formula

$$\chi_m(\omega) = \partial^*\varphi_m(\omega)/\left|\partial^*\varphi_m(\omega)\right|, \quad \omega\in\Omega,$$

where ∂^* is the co-differential. In local coordinates, $\omega=(x,v)$, $x\in M$, $v\in T^*_x M$, $|v|=1$,

$$\chi_m(x,v) = \left(\varphi_m(x), \partial\varphi_m^{-\top}(x)v/\left|\partial\varphi_m^{-\top}(x)v\right|\right),$$

and $\partial\varphi_m^{-\top}$ is the inverse transpose to the Jacobi matrix. Let us define the group of the shift operators $\{T_m\}_{m\in\mathbb{Z}}$ on $L^2(\Omega;\mathbb{C}^d)$ by $T_m u = |\det\partial\chi_{-mt}|^{\frac12} u\circ\chi_{-mt}$, and denote by $\mathfrak{B}=C^*(\mathfrak{A},T,\mathbb{Z})$ the C^*-algebra generated by the operators from $\mathfrak{A}$ and T_m, $m\in\mathbb{Z}$. The algebra $\mathfrak{B}$ is called the algebra of symbols of the pseudo differential operators with the shift.

Theorem A.1. *(See [2, Theorem 4.12 and Remark 4.13].) The maps*

$$J: a\to a'+\mathfrak{K}, \quad a'=\mathrm{Op}[a], \quad a\in\mathfrak{A}, \qquad T_m\to T'_m, \quad m\in\mathbb{Z},$$

extend to an isomorphism between the algebras $\mathfrak{B}$ and $\mathfrak{B}'/\mathfrak{K} = C^*(\mathfrak{A}'/\mathfrak{K}, T', \mathbb{Z})$. In particular, the Fredholm spectrum of a pseudo differential operator with the shift in $L^2(M;\mathbb{C}^d)$ is equal to the spectrum of its symbol as an operator in $L^2(\Omega;\mathbb{C}^d)$.

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