Gearhart-Prüss Theorem in stability for wave equations: a survey

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To Jerry Goldstein and Rainer Nagel on the occasion of their 60th birthdays.

Abstract

We give a brief survey of applications of the Gearhart-Prüss spectral mapping theorem for abstract strongly continuous semigroups on Hilbert spaces to the study of stability of solitary waves for a large class of Hamiltonian partial differential equations of mathematical physics including Klein-Gordon, nonlinear Schrödinger, Boussinesq, Benjamin-Bona-Mahoney (regularized long-wave), Korteweg-deVries, and Green-Naghdi.

1 Introduction

The main goal of this survey is to attract the attention of analysts working in the asymptotic theory of strongly continuous semigroups and abstract evolution equations to applications arising in the theory of linearized stability for distinguished solutions (solitary waves, standing waves, etc.) of a number of equations of mathematical physics. This exposition is concentrated around applications of one particularly important abstract result, the *Gearhart-Prüss Theorem*. This result is well known to and widely used by many experts in nonlinear waves. Their beautiful and important work on stability for nonlinear equations appears to be unfamiliar to the semigroup community.

Let A be the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. The position of the spectrum $\sigma(e^{tA})$ of the semigroup is responsible for its stability: if $\sigma(e^{tA}) \subset \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $t \neq 0$, then the semigroup is uniformly asymptotically stable. However, in any actual problem the generator A (and hopefully, its spectrum $\sigma(A)$) is given, not the semigroup $\{e^{tA}\}_{t\geq 0}$. The classical Lyapunov Theorem takes care of this problem: for a wide range of semigroups if $\sigma(A) \subset \mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$, then $\sigma(e^{tA}) \subset \mathbb{D}, t \neq 0$. This class of semigroups includes analytic semigroups, most frequently arising in applications due to their connections to parabolic problems for PDE's.

Examples of strongly continuous semigroups that do not have the spectral mapping property $\sigma(e^{tA})\backslash\{0\} = \overline{\exp(t\sigma(A))}$, $t \neq 0$, are well known and thoroughly studied by analysts (see [EN, vN]). M. Renardy gave an example of the generator A (for which this property does not hold) that appears if one rewrites as an abstract Cauchy problem the following first-order perturbation of the two-dimensional wave equation: $\partial_t^2 u = \triangle u + e^{ix_2}\partial_{x_1} u$, $u = u(t, x_1, x_2)$, $x_1, x_2 \in [0, 2\pi]$ (see [R]). This example is especially exciting, since for the generators arising in one-dimensional hyperbolic PDE's at least the spectral bound $s(A) = \sup\{\operatorname{Re} z : z \in \sigma(A)\}$ is equal to the growth bound $\omega(A) = t^{-1} \log \sup\{|z| : z \in \sigma(e^{tA})\}$, $t \neq 0$ (see [NRL]).

As Renardy's example shows, one has to be very careful applying results of the spectral analysis of A to stability of $\{e^{tA}\}_{t\geq 0}$ when dealing with "hyperbolic" problems. Quite often the generators of this type appear in the linear stability analysis of some "distinguished" solutions (traveling waves, trains, solitary waves, bound states, etc.) for nonlinear equations of fluid mechanics, nonlinear optics and other applications. We provide a list of the equations and the distinguished solutions as well as the corresponding linearizations in the next section. Since the aforementioned Lyapunov Theorem does not generally work, one needs another tool to derive information about the linear stability of the solution from the spectral information about the generator given by the linearized equation. This is where the following Gearhart-Prüss Theorem is used.

Gearhart-Prüss Theorem. *For a strongly continuous semigroup on a Hilbert space* $\omega(A) < 0$ *if and only if* $s(A) < 0$ *and* $\sup\{\|(z - A)^{-1}\| : \operatorname{Re} z > 0\} < \infty$.

This is a fundamental and well-documented result in the literature on strongly continuous semigroups. The original papers are [Ge, Pr] and detailed discussions may be found in [EN, vN, CL]. This result is also well known to the control theory community as the Huang Theorem (see [Hu] and [CZ]). It finds numerous applications in "abstract" control (see e.g. [Ba, JZ, RT, WR]) as well as applications in more specific models (see e.g. [CLL, Li1, Li2]).We will not review all applications of the Gearhart-Prüss Theorem here, but will concentrate on stability of steady-states for nonlinear equations (see [GJLS, KS, L, MW, PW3, S, W3]).

Typically, the Gearhart-Prüss Theorem is applied as follows. Consider a nonlinear equation $\partial_t u = L(\partial_x)u + \mathcal{N}(u)$ from the list given in the next section. Here $L(\cdot)$ is a polynomial and $\mathcal{N}$ is the nonlinear term. Take a "distinguished" solution, e.g., a traveling wave of the form $u(x, t) = Q(x - ct)$, $x \in \mathbb{R}, t \geq 0$, where c is a constant (the speed of the wave). In the moving frame $\xi = x - ct$ the original equation can be recast as $\partial_t u = L(\partial_\xi)u + c\partial_\xi u + \mathcal{N}(u)$, such that Q is its steady state. The linearization of the last equation about the steady state gives the operator $\mathcal{L} = L(\partial_\xi) + c\partial_\xi + D\mathcal{N}(Q)$. The stability of the semigroup generated by $\mathcal{L}$ corresponds to the "stability" of Q. Depending on the type of "stability" of Q that we are interested in, we choose an appropriate space where we want to study $\sigma(\mathcal{L})$. For instance, for the study of orbital asymptotic stability it is natural to consider $\mathcal{L}$ on a space of exponentially weighted functions (see Section 2 for more information).

The spectral analysis of the semigroup $\{e^{t\mathcal{L}}\}_{t\geq 0}$ consists of two parts which are outlined in Sections 3 and 4, respectively. First, the point spectrum $\sigma_p(\mathcal{L})$ is discussed. The main tool of this analysis is the *Evans function*. This is a Wronskian-type function whose zeros correspond to the eigenvalues of $\mathcal{L}$, see excellent reviews [GJK, S] and the literature therein. Note that the definition of the Evans function is based on the notion of exponential dichotomy and Palmer-Ben-Artzi-Gohberg type results, well known in the field of abstract evolution equations (see, e.g. [EN, p.480] or [CL]). Second, resolvent estimates for $\mathcal{L}$ are done to make sure that the resolvent is bounded in the right half plane and to therefore derive stability from the Gearhart-Prüss Theorem.

Besides the stability analysis, the Gearhart-Prüss Theorem is used to establish the existence of local invariant manifolds in a neighborhood of a steady-state solution for a nonlinear equation (see [GJLS, KS, S]). General results on the existence of the manifolds (see, e. g., [BJ1]) are proved under the assumptions that $\sigma(e^{tA})$ has "spectral gaps"; here A is the operator obtained by the linearization about a steady-state. Since only the information on $\sigma(A)$ is available, one needs to make sure that the spectral mapping property $\sigma(e^{tA})\backslash\{0\} = \exp(t\sigma(A))$ holds for the generator in hand. For this, it is often convenient to use the Gearhart-Prüss Theorem in the following form: $\sigma(e^{tA}) \setminus \{0\}$ consists of all points $e^{\lambda t}$ such that either $\lambda_k = \lambda + 2\pi ik/t \in \sigma(A)$ for some $k \in \mathbb{Z}$, or the sequence $\{\|(\lambda_k - A)^{-1}\|\}_{k\in\mathbb{Z}}$ is unbounded.

Finally, we remark that there are recent generalizations of the Gearhart-Prüss Theorem (for the case of the semigroups acting on Banach spaces) given in terms of Fourier multipliers properties of the resolvent of A (see [AB, H, LR, LS]). A typical result in this direction says that a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ on a *Banach* space X is uniformly exponentially stable if and only if the resolvent of the generator is bounded on $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and, in addition, the function $s \mapsto (is - A)^{-1}$ is an $L_p(\mathbb{R}; X)$–Fourier multiplier, $1 \leq p < \infty$. We do not know if the generalizations can be applied to questions of stability for solutions of nonlinear equations.

2 Equations and linearizations

First, we will briefly discuss a general setup for the linearization of nonlinear equations about traveling waves, see [S]. Consider a partial differential equation of the form

$$\partial_t u = L(\partial_x)u + \mathcal{N}(u), \quad u = u(x,t), \quad x \in \mathbb{R}, \quad t \geq 0. \tag{1}$$

Here $L(\cdot)$ is a polynomial, u may be a vector-valued function, and $\mathcal{N}$ is a nonlinear term. In the moving coordinate frame $\xi = x - ct$ this equation reads:

$$\partial_t v = L(\partial_\xi)v + c\partial_\xi v + \mathcal{N}(v), \quad v = v(\xi,t), \quad \xi \in \mathbb{R}, \quad t \geq 0. \tag{2}$$

This means that if v and u are related via

$$v(\xi,t) = u(\xi + ct, t), \quad \xi = x - ct, \tag{3}$$

then $u = u(x,t)$ satisfies (1) if and only if $v = v(\xi,t)$ satisfies (2). A solution $u(x,t) = Q_c(x - ct)$ of (1), or the function Q_c, is called a *traveling wave* for (1). In other words, Q_c is a traveling wave for (1) if and only $v(\xi,t) = Q_c(\xi)$, related to $u(x,t) = Q_c(x - ct)$ via (3), is a time-independent solution of (2):

$$0 = L(\partial_\xi)Q_c + c\partial_\xi Q_c + \mathcal{N}(Q_c). \tag{4}$$

The linearization of (2) about the steady-state Q_c is given by

$$\partial_t u = \mathcal{L}u, \quad \text{where} \quad \mathcal{L}u = L(\partial_\xi)u + c\partial_\xi u + D\mathcal{N}(Q_c)u, \tag{5}$$

and $D\mathcal{N}$ is the derivative of the nonlinear term. Thus, $\mathcal{L}$ is a variable coefficient differential operator since $Q_c = Q_c(\xi)$ is ξ-dependent.

In many cases, $\mathcal{L}$ is "asymptotically autonomous", that is, the ξ-dependent coefficients of $\mathcal{L}$ have limits as $|\xi| \to \infty$. The corresponding constant coefficient differential operator $\mathcal{L}^\infty$ often has spectrum in $i\mathbb{R}$, and therefore the essential spectrum of $\mathcal{L}$ is located on $i\mathbb{R}$, since $\mathcal{L}$ is a relatively compact perturbation of $\mathcal{L}^\infty$. In addition, $\mathcal{L}$ might have point spectrum in $\mathbb{C}_+$. These "modes" are called unstable and cause instabilities for the nonlinear equation. In some cases $\mathcal{L}$ might have essential spectrum in $\mathbb{C}_+$ and this also causes instability (see an important general result on instability for nonlinear equations in [ShS2]).

Quite often a nonlinear equation has a two-parametric family

$$\mathcal{M} = \{Q_c(\cdot - ct + \gamma) : c > 0, \gamma \in \mathbb{R}\} \tag{6}$$

of solitary waves. It is called *orbitally stable* if a solution of the nonlinear equation that is initially close to a solitary wave $Q_c(\cdot - ct)$ in some (say, Sobolev $H^1(\mathbb{R})$)norm will remain close to the set $\mathcal{M}$ of translates $\{Q_c(\cdot - ct + \gamma)\}$ of the wave, that is if $\inf_\gamma \|u(\cdot, t) - Q_c(\cdot - ct + \gamma)\|$ is small for all $t > 0$ provided it is small for $t = 0$ (see [B, BSS, W1, PW2]). A more refined stability notion is that of *orbital asymptotic stability* (or *convective stability*), see [PW2, PW3, W3]. In this case one wants to show that if u at $t = 0$ is a small perturbation of a given solitary wave $Q_c(\cdot - ct + \gamma)$, then $u(\cdot, t) - Q_{c_+}(\cdot - c_+ t + \gamma_+) \to 0$ as $t \to +\infty$ for some $c_+ \approx c$ and $\gamma_+ \approx \gamma$. Note that convergence here is *not* expected in the sense of a translation-invariant norm (like $L^2(\mathbb{R})$ on $H^1(\mathbb{R})$). This happens because in the coordinate frame moving together with a "large" (and therefore fast) solution Q_c the smaller solitary waves do not decay in the translation—invariant norm, but are being outrun by the large wave (see a very clear explanation of this point and more references in [PW2], and Fig.1).

Therefore, stability is studied in the spaces of exponentially weighted functions

$$L^2_a = \{u : e^{(\cdot)a} u(\cdot) \in L^2(\mathbb{R})\} \quad \text{and} \quad H^1_a = \{u : e^{(\cdot)a} u(\cdot) \in H^1(\mathbb{R})\}. \tag{7}$$

Note that the transition from H^1 to H^1_a and L^2 to L^2_a "shifts" the essential spectrum of $\mathcal{L}$ from $i\mathbb{R}$ into $\mathbb{C}_-$ [L, MW, PW2, W3].

We cite a typical "orbital asymptotic stability" result from [PW2]. Let $Q_c(x - ct + \gamma)$ be a solitary wave solution of the KdV equation (see the equation below). Fix $a \in (0, \sqrt{c/3})$ and $b \in (0, a(c - a^2))$. For constants $C > 0$ and $\epsilon > 0$ sufficiently small consider the initial value problem with $u(x, 0) = Q_c(x + \gamma) + v_0(x)$ where $v_0 \in H^2 \cap H^1_a$ and $\|v_0\|_{H^1} + \|v_0\|_{H^1_a} < \epsilon$. Then there exist $c_+ > 0$ and $\gamma_+ \in \mathbb{R}$ such that $|c - c_+| < C\epsilon$, $|\gamma - \gamma_+| < C\epsilon$ and for all $t \geq 0$ we have:

$$\|u(\cdot, t) - Q_{c_+}(\cdot - c_+ t + \gamma_+)\|_{H^1} \leq C\epsilon \quad \text{and} \quad \|u(\cdot, t) - Q_{c_+}(\cdot - c_+ t + \gamma_+)\|_{H^1_a} \leq C\epsilon e^{-bt}.$$

More advanced results of this type for the Boussinesq, BBM, and Green-Naghdi equations are obtained in [PW2, MW] and [L].

Next, we will give a brief inventory of equations of interest. For each equation, where appropriate, we will include: (a) the general form of the equation, (b) a formula for the nonlinear term most commonly used, (c) the equation written in the "moving frame" $\xi = x - ct$, (d) a formula for the "distinguished" solution (solitary

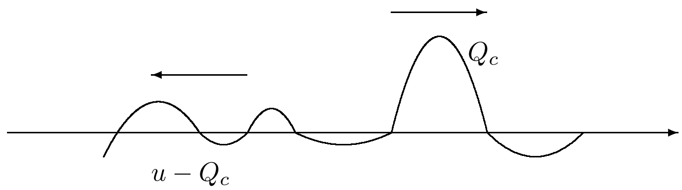


Fig. 1: Smaller waves move to the left relative to the large wave Q_c. Their L^2 and H^2 norms do not change, but L^2_a and H^2_a norms decay exponentially, if the frame moving with Q_c is considered.

wave or standing wave), (e) the equation linearized about the "distinguished" solution, and (f) the formula for the linearized operator $\mathcal{L}$. In what follows, we assume that $u = u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$, $t \geq 0$, unless explicitly mentioned otherwise. We cite sources containing detailed historical remarks and bibliography.

The generalized Korteweg-deVries (KdV) equation (see [B, BSS, PW1]):

$$\begin{aligned}
&\text{(a)} \quad \partial_t u + \partial_x f(u) + \partial_x^3 u = 0; \quad \text{(b)} \quad f(u) = u^{p+1}/(p+1);\\
&\text{(c)} \quad \partial_t v + v^p \partial_\xi v + \partial_\xi^3 v - c\partial_\xi v = 0;\\
&\text{(d)} \quad Q(\xi) = [c(p+1)(p+2)/2]^{\frac{1}{p}} \operatorname{sech}^{\frac{2}{p}}\left(p\sqrt{c}\xi/2\right);\\
&\text{(e)} \quad \partial_t u + [Q^p \partial_\xi + \partial_\xi Q^p] u + \partial_\xi^3 u - c\partial_\xi u = 0;\\
&\text{(f)} \quad \mathcal{L} = -[Q^p \partial_\xi + \partial_\xi Q^p] - \partial_\xi^3 + c\partial_\xi.
\end{aligned}$$

The generalized Benjamin-Bona-Mahoney (BBM) equation (and regularized long-wave (RLW) equation, see [PW1] and [MW, W3]):

$$\begin{aligned}
&\text{(a)} \quad \partial_t u + \partial_x u + \partial_x f(u) - \partial_x^2 \partial_t u = 0; \quad \text{(b)} \quad f(u) = u^{p+1}/(p+1);\\
&\text{(c)} \quad \partial_t v + \partial_\xi v + v^p \partial_\xi v - c\partial_\xi v - \partial_\xi^2 \partial_t v + c\partial_\xi^3 v = 0;\\
&\text{(d)} \quad Q(\xi) = \alpha \operatorname{sech}^{\frac{2}{p}}(\beta x),\ \alpha = [(c-1)(p+1)(p+2)/2]^{\frac{1}{p}},\ \beta = p(\frac{c-1}{c})^{\frac{1}{2}}/2;\\
&\text{(e)} \quad (I - \partial_\xi^2)\partial_t u = \partial_\xi(-c\partial_\xi^2 + (c-1) - Q^p)u;\\
&\text{(f)} \quad \mathcal{L} := (I - \partial_\xi^2)^{-1}(\partial_\xi(-c\partial_\xi^2 + (c-1) - Q^p)).
\end{aligned}$$

The generalized regularized Boussinesq equation (see [PW1, PW3]):

$$\begin{aligned}
&\text{(a)} \quad \partial_t^2 u - \partial_x^2 u - \partial_x^2 f(u) - \partial_x^2 \partial_t^2 u = 0; \quad \text{(b)} \quad f(u) = u^{p+1}/(p+1);\\
&\text{(c)} \quad (\partial_t - c\partial_\xi)^2 (I - \partial_\xi^2) v - \partial_\xi^2 v - \partial_\xi(v^p \partial_\xi v) = 0;\\
&\text{(d)} \quad Q(\xi) = \alpha \operatorname{sech}^{\frac{2}{p}}(\beta x),\ \alpha = \left[(c^2-1)(p+1)(p+2)/2\right]^{\frac{1}{p}},\ \beta = p(\frac{c^2-1}{c^2})^{\frac{1}{2}}/2;\\
&\text{(e)} \quad (\partial_t - c\partial_\xi)^2 (I - \partial_\xi^2) u = \partial_\xi^2 (u + Q^p u).
\end{aligned}$$

One can re-write ([PW3, p.359]) the second order in t equation as a first order system and obtain:

$$\text{(f)}\quad \mathcal{L} = \begin{bmatrix} c\partial_\xi & S \\ S(I+Q^p) & c\partial_\xi \end{bmatrix}, \quad S = \partial_\xi (I - \partial_\xi^2)^{-\frac{1}{2}}.$$

Green-Naghdi system of equations (see [L]):

$$\text{(a)}\quad \partial_t \eta + \partial_x w = 0, \quad \partial_t w = -\partial_x \left(w^2/\eta\right) - \eta \partial_x \eta + \frac{1}{3}\partial_x \left(\eta^2 \frac{d}{dt} \left(\eta \partial_x \left(w/\eta\right)\right)\right),$$

$$w = u\eta, \quad u = u(x,t), \quad \eta = \eta(x,t), \quad \frac{d}{dt} = \partial_t + u\partial_x;$$

$$\text{(d)}\quad w_c = c(c^2-1)\operatorname{sech}^2((3(c^2-1))^{\frac{1}{2}}(x-ct)/(2c)), \quad \eta_c = 1 + w/c;$$

$$\text{(f)}\quad \mathcal{L}\begin{bmatrix} w \\ \eta \end{bmatrix} = \begin{bmatrix} (J_1 w + J_2 w' + J_3 w'' + J_4 \eta + J_5 \eta' + J_6 \eta'')' \\ c\eta' - w' \end{bmatrix},$$

where $' = \partial_\xi, \quad \xi = x - ct,$

$$J_1 = c - 2w_c/\eta_c - 2\eta_c' w_c'/3 - 2w_c\eta_c''/3 + 2\eta_c w_c''/3 + 2w_c(\eta_c')^2/(3\eta_c)$$

$$J_2 = c\eta_c\eta_c'/3 - 2\eta_c' w_c/3, \quad J_3 = 2\eta_c w_c/3 - c\eta_c^2/3,$$

$$J_4 = -2c\eta_c w_c''/3 + w_c^2/\eta_c^2 - \eta_c + c\eta_c' w_c'/3 + 2w_c w_c''/3 - w_c^2(\eta_c')^2/(3\eta_c^2),$$

$$J_5 = c\eta_c w_c'/3 - 2w_c w_c'/3 + 2w_c^2\eta_c'/(3\eta_c), \quad J_6 = -w_c^2/3.$$

Klein-Gordon equation (see [BJ1, BJ2, BL, JK, Sh, ShS1, St]):

(a) $\partial_t^2 u = \Delta u + f(u), \quad u = u(x,t), \quad x \in \mathbb{R}^n, \quad n \geq 3,$

(b) $f(u) = |u|^\gamma u - m^2 u, \quad \gamma > 0;$ or $f(u) = u(|u|^2 - |u|^4 - 1)$ [Sh];

(d) $Q(\cdot) =$ any radially symmetric stationary solution [BL, JK, St].

As usual, one can re-write the equation as a first order system as follows:

$$\text{(a)}\quad \partial_t \begin{bmatrix} u \\ v \end{bmatrix} = \mathcal{L}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f(u+Q) - f(Q) - f'(Q)u \end{bmatrix}, \quad \partial_t u = v$$

$$\text{(f)}\quad \mathcal{L} = \begin{bmatrix} 0 & I \\ \Delta + f'(Q) & 0 \end{bmatrix}.$$

Nonlinear Schrödinger (NLS) equation (see [G1, G2, GJLS, GSS, ShS1, SW1, SW2, W2]:

(a) $iu_t = -\Delta u - f(x,|u|^2)u + \beta u, \quad u = u(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{R},$

(b) $f(x,|u|^2) = 4|u|^2,$

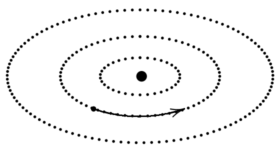
(d) $Q(x) = \sqrt{\beta/2}\operatorname{sech}(\sqrt{\beta}x).$

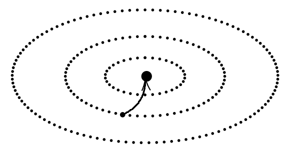
Writing $u = v + iw$, one can re-write this equation as a system as follows:

$$\text{(a)}\quad \partial_t \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \Delta w + f(x, v^2+w^2)w + \beta w \\ -\Delta v - f(x, v^2+w^2)v - \beta v \end{bmatrix}, \quad v = \operatorname{Re} u, \quad w = \operatorname{Im} u$$

$$\text{(f)}\quad \mathcal{L} = \begin{bmatrix} 0 & -L_R \\ L_I & 0 \end{bmatrix}, \quad L_R = -\Delta - \beta + \partial_2 f(x, Q^2(x)),$$

$$L_I = -\Delta - \beta - f(x,Q^2(x)) - 2\partial_2 f(x,Q^2(x))Q^2(x).$$





$\partial_t u = JDE(u)$ $\qquad\qquad$ $\partial_t u = -DE(u)$

Fig. 2: Hamiltonian versus gradient systems (dotted lines are the level surfaces of the energy functional E).

We conclude this section by the following important remark. All equations above have a Hamiltonian structure, and can be written as $\partial_t u = JDE(u)$, where E is an energy functional on an appropriate function space and J is a skew-symmetric linear operator. This fact is the basic fact for the stability/instability analysis, see [G1, G2, GSS, SS, ShS1, W2, W3] and the literature therein. The energy E is a conserved quantity. Thus, the solution $u(\cdot, t)$ moves along the level surfaces of E. Note the contrast with gradient systems of the form $\partial_t u = -DE(u)$, where solutions cross the level surfaces (see Fig. 2).

For many Hamiltonian equations of interest the energy E is composed from two conserved quantities, the Hamiltonian H and charge (or impulse) N. The distinguished solutions (solitary waves or standing waves) are critical points for E. Their stability or instability is, formally, determined by the fact that the Hessian D^2E is sign definite. Since there are two conserved quantities present, the solitary or standing wave Q_c could be a constrained minimum of E. For example, for the generalized KdV we have ([PW2]):

$$E(u) = H(u) + cN(u), \quad H(u) = \int_{\mathbb{R}} \left[\frac{1}{2}(\partial_x u)^2 - F(u)\right] dx, \quad N(u) = \frac{1}{2}\|u\|^2_{L^2},$$

where $F'(u) = f(u)$, $F(0) = 0$, and for the NLS $E(u) = H(u) - \beta N(u)$ ([W1, W3]). A general abstract theory for stability of critical points in Hamiltonian equations is developed in [GSS].

3 Point spectrum and Evans function

We now return to the general discussion in the beginning of Section 2 and again follow [S]. We want to study the spectral problem $\mathcal{L}w = \lambda w$ for the operator $\mathcal{L}$ as in (5), where $\lambda \in \mathbb{C}$ and $w = w(\xi)$, or equivalently, solutions of $\partial_t u = \mathcal{L}u$ of the form $e^{t\lambda}w(\xi)$. Note that the steady state equation (4) is an ODE that contains higher order derivatives in ξ and thus can be written as a first order system $\partial_\xi V = f(V)$, where $V = V(\xi)$ and f (depending also on c) is a vector field whose dimension depends on the order of $L(\cdot)$. The corresponding eigenvalue problem $\mathcal{L}w = \lambda w$ for $\mathcal{L}$ is also an ODE with higher order derivatives in ξ and can also be recast as $\partial_\xi Y = (Df(V) + \lambda B)Y$ for an appropriate matrix $B = B(\xi)$ and vector-function $Y = Y(\xi)$.

As a result, the linearized stability of a traveling wave for (1) is related to the study of the nonautonomous linear equation $\partial_\xi Y = A(\xi; \lambda)Y$ or the spectral properties of the operator $\Gamma(\lambda) := -\partial_\xi + A(\cdot; \lambda)$ on a space of vector-functions

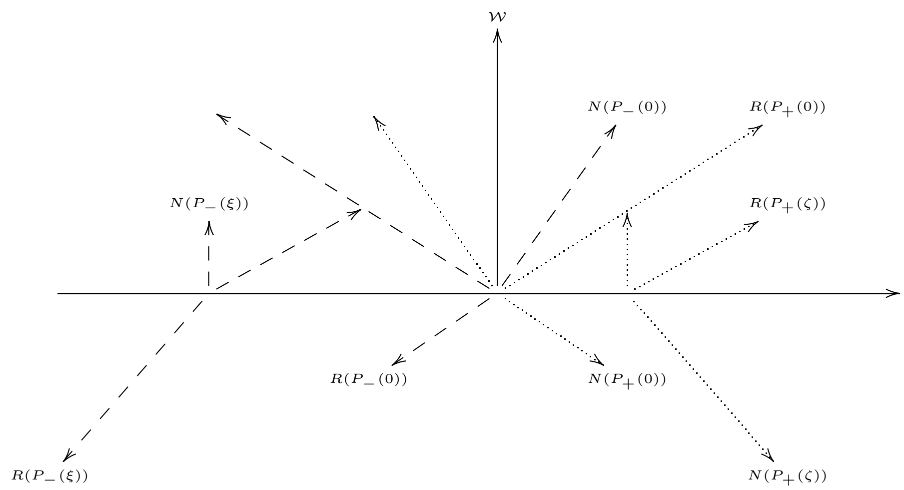


Fig. 3: Dichotomy: $\mathcal{W} = N(P_-(0)) \cap R(P_+(0))$

$Y = Y(\xi)$ (say, $L^2(\mathbb{R};\mathbb{C}^n)$). We will assume that $A(\xi,\lambda) = \tilde{A}(\xi) + \lambda B(\xi)$, where $\tilde{A}, B \in C^\infty(\mathbb{R};\mathbb{R}^{n\times n})$ (see [S]).

Fix $\lambda \in \mathbb{C}$ and let $U(\xi,\zeta)$ denote the evolution family (propagator) for $\partial_\xi Y = A(\xi;\lambda)Y$. Recall ([CL, EN, S]) that $\partial_\xi Y = A(\xi;\lambda)Y$ is said to have exponential dichotomy on $I (= \mathbb{R}^+, \mathbb{R}^-$ or $\mathbb{R})$ if there exists a continuous projection-valued function P and constants $M \geq 1$, $\alpha > 0$ such that, for $\xi, \zeta \in I$, we have $U(\xi,\zeta)P(\zeta) = P(\xi)U(\xi,\zeta)$ and

$$\|U^s(\xi,\zeta)\| \leq Me^{-\alpha(\xi-\zeta)}, \quad \xi \geq \zeta; \qquad \|U^u(\xi,\zeta)\| \leq Me^{\alpha(\xi-\zeta)}, \quad \xi \leq \zeta.$$

Here and below $\mathbf{R}(\cdot)$ and $\mathbf{N}(\cdot)$ denote the range and kernel of an operator, and $U^s(\xi,\zeta) = U(\xi,\zeta) \mid \mathbf{R}(P(\zeta)), U^u(\xi,\zeta) = U(\xi,\zeta) \mid \mathbf{N}(P(\zeta))$ are the corresponding restrictions. The ξ-independent $\dim \mathbf{N}(P(\xi))$ is called the *Morse index*. If $\partial_\xi Y = A(\xi;\lambda)Y$ has exponential dichotomy on $\mathbb{R}_-$ and $\mathbb{R}_+$ with projections P_- and P_+, then we denote $i_\pm(\lambda) = \dim \mathbf{N}(P_\pm(0))$.

A well-known result in [BrGK, BGK, P] relates the Fredholm properties of the operator $\Gamma(\lambda)$ on $L^2(\mathbb{R};\mathbb{C}^n)$ and dichotomy of $\partial_\xi Y = A(\xi;\lambda)Y$ as follows: $\Gamma(\lambda)$ is invertible if and only if there is exponential dichotomy on $\mathbb{R}$; also, $\Gamma(\lambda)$ is Fredholm if and only if there are dichotomies on $\mathbb{R}_-$ and $\mathbb{R}_+$. If this is the case then the spaces $\mathbf{N}(P_-(0)) \cap \mathbf{R}(P_+(0))$ and $\mathbf{N}(\Gamma(\lambda))$ are isomorphic via $Y(0) \mapsto Y(\cdot)$, and the index $\operatorname{ind}\Gamma(\lambda) = i_-(\lambda) - i_+(\lambda)$.

Suppose $\Omega \subset \mathbb{C}$ is a simply-connected open set such that for each $\lambda \in \Omega$ the operator $\Gamma(\lambda)$ is Fredholm with index zero. Then $P_+ = P_{+,\lambda}$ and $P_- = P_{-,\lambda}$, the dichotomy projections on $\mathbb{R}_+$ and $\mathbb{R}_-$ for $\partial_\xi Y = A(\xi,\lambda)Y$, exist and can be chosen analytically in $\lambda \in \Omega$ (see [PSS]). Also, $k := \dim \mathbf{N}(P_{-,\lambda}(0)) = \dim \mathbf{N}(P_{+,\lambda}(0))$ is constant for $\lambda \in \Omega$. Choose (analytically in λ) bases $Y_1(\lambda), \ldots, Y_k(\lambda)$ of $\mathbf{N}(P_{-,\lambda}(0))$ and $Y_{k+1}(\lambda), \ldots, Y_n(\lambda)$ of $\mathbf{R}(P_{+,\lambda}(0))$ and define the *Evans function* by

$$E(\lambda) = \det[Y_1(\lambda), \ldots, Y_n(\lambda)], \quad \lambda \in \Omega. \tag{8}$$

Remark that $E(\lambda) = 0$ if and only if $\mathbf{N}(P_{-,\lambda}(0)) \cap \mathbf{R}(P_{+,\lambda}(0)) \neq \{0\}$ or, equivalently, $\mathbf{N}(\Gamma(\lambda)) \neq \{0\}$. Going back to the spectral problem $\mathcal{L}w = \lambda w$, we conclude that $\lambda \in \sigma_p(\mathcal{L})$ if and only if $E(\lambda) = 0$ ([GJK, Thm.3.1]).

Let us consider three specific examples of the applications of the Evans function.

First, we shall follow an excellent exposition in [GJK] to sketch the proof of the instability of the scalar reaction-diffusion equation linearized about a pulse solution. The reaction-diffusion equation

$$\partial_t u = \partial_x^2 u - u + 2u^3, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{9}$$

supports a pulse solution of the form $Q(x) = \text{sech} x$. The eigenvalue problem for the linearization about Q is then:

$$w'' - (1 - 6Q^2(x))w = \lambda w, \quad ' = \partial_x \tag{10}$$

We shall indicate how the Evans function may be used to prove the existence of an unstable mode and thus to show that the linearization is unstable. Let the linearized equation be recast as a first-order system for $Y = [w, w']^T$:

$$Y' = (M(\lambda) + R(x))Y, \quad M(\lambda) = \begin{bmatrix} 0 & 1 \\ 1+\lambda & 0 \end{bmatrix}, \quad R(x) = \begin{bmatrix} 0 & 0 \\ -6Q^2(x) & 0 \end{bmatrix}. \tag{11}$$

Now assume that $\text{Re}\,\lambda > -1$. The eigenvalues of $M(\lambda)$ are $\mu^\pm(\lambda) = \pm\sqrt{1+\lambda}$ with eigenvectors $\eta^\pm(\lambda) = [1, \mu^\pm(\lambda)]^T$. Consider solutions $Y^+(\lambda, x)$ and $Y^-(\lambda, x)$ to (11) such that

$$\lim_{x \to +\infty} Y^+(\lambda, x) e^{-\mu^-(\lambda)x} = \eta^-(\lambda), \ \lim_{x \to -\infty} Y^-(\lambda, x) e^{-\mu^+(\lambda)x} = \eta^+(\lambda). \tag{12}$$

Note that $\lim_{x\to+\infty} |Y^+(\lambda, x)| = 0$ and $\lim_{x\to-\infty} |Y^-(\lambda, x)| = 0$. The Evans function is then defined to be: $E(\lambda) = \det[Y^-(\lambda, x), Y^+(\lambda, x)]$ and does not depend upon x (see [GJK]).

By the remark after equation (8), we know that $E(\lambda) = 0$ if and only if λ is an eigenvalue of (10). Note that $\lambda = 0$ is an eigenvalue of (10) with eigenfunction $Q'(x)$. We then know that $E(0) = 0$. It can be shown that $E'(0) > 0$ and that $E(\lambda)$ has a negative limit for large λ (see [GJK]). These facts then prove the existence of a positive root for $E(\lambda) = 0$. This gives the existence of a positive eigenvalue which then implies the instability of the linearized equation for (9).

Secondly, the Evans function may also be used to show that there does not exist a positive real root to $E(\lambda) = 0$ and thus assures the asymptotic stability of the linearized system. Consider the KdV equation as discussed in Section 2. For $p = 1, 2$ the Evans function takes the form (see[PW2])

$$E(\lambda) = \left(\frac{\mu_1(\lambda) + \sqrt{c}}{\mu_1(\lambda) - \sqrt{c}} \right)^2, \tag{13}$$

where $\mu_1(\lambda)$ is the root of $\mu^3 - c\mu + \lambda = 0$ with smallest real part. Then, if $E(\lambda) = 0$ we have that $\mu_1(\lambda) = -\sqrt{c}$ which in turn gives $\lambda = 0$. So for the cases $p = 1$ or $p = 2$, $E(\lambda) = 0$ only for $\lambda = 0$.

Lastly, a similar stability result holds for the RLW equation under certain scaling conditions (see [W3]). The scaled Evans function for the RLW equation converges to the Evans function (13) for the KdV equation. For $(c-1)/c$ positive and sufficiently small, $\lambda = 0$ is the only eigenvalue of the linearized RLW problem for sufficiently small $|\lambda|$. For larger values of $|\lambda|$, one can show that $E(\lambda) \sim 1$. The reader is encouraged to seek details in [PW3, W3].

4 Resolvent estimates for Gearhart-Prüss Theorem and consequences

We now return to the general discussion in the beginning of Section 2 and sketch the abstract scheme that lead to the application of the Gearhart-Prüss Theorem. We follow an excellent exposition in [W3], see also [L, MW, PW2, PW3] for detailed analyses of the KdV, BBM, Boussinesq and GN equations.

Suppose that equation (2) has a two-parametric family $\mathcal{M}$ of fixed points $Q(\cdot, \vec{p})$, where $\vec{p}$ is a parameter (it could be the pair (γ, c) as in (6)). We represent the solution v of (2) in the form $v(\xi,t) = Q(\xi, \vec{p}(t)) + w(\xi, t)$. The evolution of $\vec{p}$ describes the dynamics along $\mathcal{M}$ while w describes the dynamics in the "transversal" to $\mathcal{M}$ direction. Fix a reference fixed point $Q_c \in \mathcal{M}$ of (2). Linearizing (2) about Q_c, and taking in account certain symmetries in the equation (e.g. translation invariance in space or phase), after some substantial work ([MW, W3]) we arrive to the following equations:

$$\partial_t w = \mathcal{L}w + \mathcal{F}(w, \vec{p}), \quad \partial_t \vec{p} = \mathcal{S}(w, \vec{p}(t)).$$

Here $\mathcal{L}$ is given in (5). For instance, for the regularized long-wave equation (BBM with $p = 1$) we have ([MW, W3]):

$$\mathcal{L} = (I - \partial_\xi^2)^{-1} \partial_\xi L_{Q_c}, \quad \text{where } L_{Q_c} = -c\partial_\xi^2 + c - 1 - Q_c, \quad c > 1.$$

Based on the heuristic argument in Section 2 regarding the *asymptotic* orbital stability, we do not expect that $\omega(\mathcal{L}) < 0$ on a function space with a translation-invariant norm. Indeed, the Hamiltonian structure of the system usually implies that the L^2-spectrum of $\mathcal{L}$ belongs to $i\mathbb{R}$. It is natural, however, to consider $\mathcal{L}$ on *weighted* spaces (with an exponential weight for the KdV, RLW, Boussinesq and GN equations, and power weight for the NLS). Speaking about the exponentially weighted spaces, we want to study $\mathcal{L}$ on H^1_a, $a > 0$, see (7), or equivalently, we want to study on $H^1(\mathbb{R})$ the operator $\mathcal{L}_a = e^{(\cdot)a} \mathcal{L} e^{-(\cdot)a}$. Note that under the transformation $\mathcal{L} \mapsto \mathcal{L}_a$ the spectrum of the constant coefficient "asymptotic" operator $\mathcal{L}_a^\infty$ will shift from $i\mathbb{R}$ to form a curve in $\mathbb{C}_-$. Using the Evans function, one can give a (highly nontrivial) argument to show that $\lambda = 0$ is the only eigenvalue for $\mathcal{L}_a$ in $\overline{\mathbb{C}_+}$ with two-dimensional eigenspace $\ker \mathcal{P}$ for the spectral projection $\mathcal{P}$ ([MW, W3]).

Finally, one is ready to apply the Gearhart-Prüss Theorem, and to derive the required estimate

$$\|e^{\mathcal{L}_a t} \mathcal{P} w\|_{H^1} \leq C e^{-bt} \|w\|_{H^1}, \quad t \geq 0.$$

For this, one has to show that the resolvent estimate $\sup\{\|(\lambda - \mathcal{L}_a \mathcal{P})^{-1}\| : \operatorname{Re} \lambda > -b\} < \infty$ holds for some $b > 0$. The proof of the resolvent estimate is based on the identity

$$\lambda - \mathcal{L}_a = (\lambda - \mathcal{L}_a^\infty)(I + K(\lambda)), \quad \text{where } K(\lambda) = (\lambda - \mathcal{L}_a^\infty)^{-1}(\mathcal{L}_a - \mathcal{L}_a^\infty) \qquad (14)$$

is a compact operator. Since $\mathcal{L}_a^\infty$ is a constant coefficient differential operator, its resolvent $(\lambda - \mathcal{L}_a^\infty)^{-1}$ is given explicitly, and is proved to be bounded for $\operatorname{Re}\lambda > -b$. Again, using the Evans function, one can prove that $\|K(\lambda)\| < 1$ for large $|\lambda|$. This proves the resolvent estimate, and therefore the desired exponential decay is established.

Note that an approach similar to (14) was used in [GJLS] to prove the spectral mapping theorem for the linearized NLS equation.

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