

# Hyperbolicity of semigroups and Fourier multipliers

Yuri Latushkin and Roman Shvydkoy

**Abstract.** We present a characterization of hyperbolicity for strongly continuous semigroups on Banach spaces in terms of Fourier multiplier properties of the resolvent of the generator. Hyperbolicity with respect to classical solutions is also considered. Our approach unifies and simplifies the M. Kaashoek–S. Verduyn Lunel theory and multiplier-type results previously obtained by S. Clark, M. Hieber, S. Montgomery-Smith, F. Rübiger, T. Randolph, and L. Weis.

## 1. Introduction

Suppose  $X$  is a complex Banach space and  $\mathbf{T} = (T_t)_{t \geq 0}$  is a strongly continuous semigroup of operators on  $X$ . Let  $A$  denote its infinitesimal generator.

An autonomous version of a well-known result that goes back to O. Perron says the following: a homogeneous differential equation  $\dot{u} = Au$  admits exponential dichotomy on  $\mathbb{R}$  if and only if the inhomogeneous equation  $\dot{u} = Au + f$  has a unique mild solution  $u \in F(\mathbb{R}; X)$  for each  $f \in F(\mathbb{R}; X)$ , see [DK] or [LZ], and [CL], and the literature therein. Here  $F(\mathbb{R}; X)$  is a space of  $X$ -valued functions, for instance,  $F(\mathbb{R}; X) = L_p(\mathbb{R}; X)$ ,  $1 \leq p < \infty$ . The exponential dichotomy for  $\dot{u} = Au$  means that the semigroup generated by  $A$  is hyperbolic, that is, condition  $\sigma(T_t) \cap \{|z| = 1\} = \emptyset$ ,  $t \neq 0$ , holds for the spectrum  $\sigma(\cdot)$ .

Passing, formally, to the Fourier transforms in the equation  $\dot{u} = Au + f$  we have that the solution  $u$  is given by  $u = Mf$ , where  $M : f \mapsto [R(i\cdot; A)\hat{f}]^\vee$ ,  $R(\lambda; A)$  is the resolvent operator, and  $\wedge, \vee$  are the Fourier transforms. Thus, heuristically, the above-mentioned Perron-type theorem could be reformulated to state that the hyperbolicity of the semigroup is equivalent to the fact that the function  $s \mapsto R(is; A)$  is a Fourier multiplier on  $L_p(\mathbb{R}; X)$ ,  $1 \leq p < \infty$ , see, e.g., [A, H1] for the definition of Fourier multipliers. One of the objectives of the current paper is to systematically study the connections of hyperbolicity and  $L_p$ -Fourier multiplier properties of the resolvent.

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The use of Fourier multipliers for stability and hyperbolicity for strongly continuous semigroups has a fairly long history. To put our paper in this context, we briefly review relevant results. Probably, the first Fourier multiplier type result was obtained in the important paper [KVL] by M. Kaashoek and S. Verduyn Lunel. These authors used scalar functions (“matrix elements” of the resolvent) defined by

$$r_\rho(s, x, x^*) = \langle x^*, R(\rho + is; A)x \rangle, \quad \rho \in \mathbb{R}, \quad s \in \mathbb{R}, \quad x \in X, \quad x^* \in X^*.$$

They proved that  $\mathbf{T}$  is hyperbolic if and only if the following two conditions holds:

- (i)  $|\langle r_\rho, \Phi \rangle| \leq K \|x\| \|x^*\| \|\check{\Phi}\|_{L^1}$  for some  $K > 0$ ,  $\rho_0 > 0$  and all  $\rho$  with  $|\rho| < \rho_0$  and all  $\Phi \in \mathcal{S}$ , the Schwartz class of scalar functions on  $\mathbb{R}$ ;
- (ii) the Césaro integral

$$\begin{aligned} G_0 x &= \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is; A) x ds \\ &= \frac{1}{2\pi} \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N \int_{-l}^l R(is, A) x ds dl \end{aligned}$$

converges for all  $x \in X$ .

Remark, that one of the results of the current paper (Theorem 2.7) shows that condition (ii), in fact, follows from (i).

L. Weis in [W2] used Fourier multiplier properties of the resolvent on Besov spaces to give an alternative proof of the fact that the supremum  $\omega_\alpha(\mathbf{T})$  of the growth bounds of “ $\alpha$ -smooth” solutions  $T_t x$  are majorated by the boundedness abscissa  $s_0(A)$  of the resolvent. Remark, that in Section 3 of the present paper we derive a formula (Theorem 3.6) for  $\omega_\alpha(\mathbf{T})$  in terms of Fourier multipliers on  $L_p$ . Moreover, in Sections 4 and 5 we use Fourier multipliers to study an analogue of dichotomy (hyperbolicity) for the smooth solutions.

A similar formula for  $\omega_0(\mathbf{T})$  in terms of the resolvent of the generator was obtained in [CLRM], see also [LMS] and formula (5.20) in [CL]. Formally, Fourier multipliers have not been used in [LMS] and [CLRM]. The hyperbolicity of  $\mathbf{T}$  was characterized in [LMS] and [CLRM], see also [CL], in terms of the invertibility of generator  $\Gamma$  of the evolution semigroup  $\{E^t\}$  defined on  $L_p(\mathbb{R}; X)$  as  $(E^t f)(\tau) = e^{tA} f(\tau - t)$ . However, a simple calculation (see Remark 2.2 below) shows that  $\Gamma^{-1} = -M$ . Thus, formula (5.20) in [CL] for the growth bound of  $\mathbf{T}$  is, in fact, a Fourier multiplier result that is generalized in Theorem 3.6 below.

Via completely different approach based on an explicit use of Fourier multipliers, M. Hieber [H2] gave a characterization of uniform stability for  $\mathbf{T}$  in terms of Fourier multiplier properties of the resolvent. Also, he proved a formula for  $\omega_0(\mathbf{T})$  that is contained in Theorem 3.6 when  $\alpha = 0$ . An important ingredient of his proof was the use of well-known Datko-van Neerven Theorem saying that  $\mathbf{T}$  is uniformly stable if and only if the convolution with  $\mathbf{T}$  is a bounded operator on  $L_p(\mathbb{R}; X)$ . Since the resolvent is the Fourier transform of  $\mathbf{T}$ , the Fourier multipliers characterization of uniform stability follows.

Among other things, this result with a different proof was given in [LR], where Datko-van Neerven Theorem was also used. In fact, Theorem 3.6 was proved in [LR] for  $\alpha = 0$  or  $\alpha = 1$ . Also, a spectral mapping theorem from [LMS] was explained in [LR] using Fourier multipliers instead of evolution semigroups. In addition, a particular case of Theorem 4.1 of the current paper (with a different proof) was established in [LR]. Thus, in the present paper we use new technique to “tie the ends”, and give a universal treatment for the results in [KVL, CLRM, H2, LR] in a more general context.

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## 2. Characterization of hyperbolicity

Let us fix some notation:

- $\mathbf{T} = (T_t)_{t \geq 0}$  is a strongly continuous semigroup on a Banach space  $X$  with the generator  $A$ ;
- $\mathcal{L}(X)$  – the set of bounded linear operators on  $X$ ;
- $R(\lambda, A) = R(\lambda)$  is the resolvent of  $A$ ;
- $\omega_0 = \omega_0(\mathbf{T})$  denotes the growth bound of  $\mathbf{T}$ , i.e.  $\omega_0(\mathbf{T}) = \inf\{\omega \in \mathbb{R} : \|T_t\| \leq M_\omega e^{\omega t}\}$ ;
- $s_0(A)$  denotes the abscissa of uniform boundedness of the resolvent, i.e.  $s_0(A) = \inf\{s \in \mathbb{R} : \sup\{\|R(\lambda)\| : \operatorname{Re} \lambda > s\} < \infty\}$ ;
- $r_\rho(s, x, x^*) = r_\rho(s) = \langle x^*, R(is + \rho)x \rangle$ ;  $s \in \mathbb{R}$ ,  $x \in X$ ,  $x^* \in X^*$ ,  $\rho \in \mathbb{R}$ ;
- $\hat{f}(t) = \int_{\mathbb{R}} f(s) e^{-ist} ds$ ;  $\check{f}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} f(s) e^{ist} ds$ ;
- $\mathcal{S}$  stands for the class of Schwartz functions;
- $\langle r, \Phi \rangle$  denotes the value of a distribution  $r$  on  $\Phi \in \mathcal{S}$ .

**Definition 2.1.** *We say that the semigroup  $\mathbf{T}$  is hyperbolic if there is a bounded projection  $P$  on  $X$ , called splitting, such that  $PT_t = T_tP$  for all  $t > 0$  and there exist positive numbers  $\omega$  and  $M$  such that*

1.  $\|T_t x\| \leq K e^{-\omega t} \|x\|$ , for all  $t > 0$  and  $x \in \operatorname{Im} P$ ,
2.  $\|T_t x\| \geq K e^{\omega t} \|x\|$ , for all  $t < 0$  and  $x \in \operatorname{Ker} P$ .

*The semigroup  $\mathbf{T}$  is called uniformly exponentially stable if  $P = I$ .*

In other words, conditions 1 and 2 say that  $(T_t)_{t \geq 0}$  is uniformly exponentially stable on  $\operatorname{Im} P$ ; all the  $T_t$ 's are invertible on  $\operatorname{Ker} P$  and the semigroup  $(T_{-t})_{t \geq 0}$  is uniformly exponentially stable there.

**Definition 2.2.** *The function*

$$G(t) = \begin{cases} T_t P, & t > 0 \\ -T_t(I - P), & t < 0 \end{cases}$$

is called the Green's function corresponding to the hyperbolic semigroup  $\mathbf{T}$ .

Definition 2.1 allows an equivalent reformulation in terms of spectral properties of  $\mathbf{T}$ . Namely,  $\mathbf{T}$  is hyperbolic if and only if the unit circle  $\mathbb{T}$  lies in the resolvent set of  $T_t$  for one/all  $t$  (see [EN, Proposition V.1.15]).

Let us recall the following inversion result.

**Lemma 2.3.** *Suppose  $\rho > s_0(A)$  and  $x \in X$ , then*

$$F_t(x) = \frac{1}{2\pi i}(C, 1) \int_{\operatorname{Re} \lambda = \rho} e^{\lambda t} R(\lambda) x d\lambda, \quad t \in \mathbb{R},$$

where  $F_t$  is defined as

$$F_t(x) = \begin{cases} T_t x, & t > 0 \\ \frac{1}{2}x, & t = 0 \\ 0, & t < 0 \end{cases}.$$

In particular,  $\check{r}_\rho(t, x, x^*) = e^{-\rho t} \langle x^*, F_t(x) \rangle$ .

The proof can be found in [vN, Theorem 1.3.3]. See also Corollary 3.5.

Below we establish some algebraic properties of the distributions  $\check{r}_\rho$  for small  $|\rho|$  without any additional assumptions on  $s_0(A)$ . The reader will easily recognize the semigroup properties, in the case  $s_0(A) < 0$ .

In order to be able to treat  $r_\rho$ 's as distributions and to justify some computations, we assume that the function  $s \mapsto \|R(is)\|$  is bounded on  $\mathbb{R}$ , though the proofs below require merely that this function grows not faster than a power of  $|s|$ .

**Lemma 2.4.** *If  $\tau > 0$ , then*

$$\check{r}_0(t - \tau, T_\tau x, x^*) = \check{r}_0(t, x, x^*) - \langle x^*, T_t x \rangle \chi_{[0, \tau]}(t), \quad t \in \mathbb{R}.$$

*Proof.* Let us take arbitrary  $\Phi \in \mathcal{S}$ . Then

$$\begin{aligned} \langle \check{r}_0(\cdot - \tau, T_\tau x, x^*), \Phi \rangle &= \langle \check{r}_0(\cdot, T_\tau x, x^*), \Phi(\cdot + \tau) \rangle \\ &= \langle r_0(\cdot, T_\tau x, x^*), e^{-i\tau \cdot} \check{\Phi} \rangle = \int_{-\infty}^{+\infty} \langle x^*, R(is) e^{-is\tau} T_\tau x \rangle \check{\Phi}(s) ds. \end{aligned}$$

Note that

$$e^{-is\tau} R(is) T_\tau x = R(is) x - \int_0^\tau T_r x \cdot e^{-isr} dr. \quad (1)$$

Continuing the line of equalities, we obtain:

$$\begin{aligned} \langle \check{r}_0(\cdot - \tau, T_\tau x, x^*), \Phi \rangle &= \int_{-\infty}^{+\infty} \langle x^*, R(is) x \rangle \check{\Phi}(s) ds \\ &\quad - \int_{-\infty}^{+\infty} \left\langle x^*, \int_0^\tau T_r e^{-isr} x dr \right\rangle \check{\Phi}(s) ds = \langle \check{r}_0, \Phi \rangle \\ &\quad - \int_0^\tau \langle x^*, T_r x \rangle \Phi(r) dr = \langle \check{r}_0 - \langle x^*, T_\cdot x \rangle \chi_{[0, \tau]}(\cdot), \Phi \rangle. \end{aligned}$$

□

**Lemma 2.5.**  $\check{r}_0(t, T_\tau x, x^*) = \check{r}_0(t, x, T_\tau^* x^*)$ ,  $\tau > 0$ ,  $t \in \mathbb{R}$ .

The proof is obvious.

**Lemma 2.6.**  $\check{r}_0(t) = e^{\rho t} \check{r}_\rho(t)$  for all  $\rho$  with  $|\rho| < \rho_0$  and  $t \in \mathbb{R}$ .

*Proof.* We choose  $\rho_0$  such that  $\sup\{\|R(is + \rho)\| : s \in \mathbb{R}, |\rho| < \rho_0\}$  is finite. Suppose  $\Phi \in \mathcal{S}$  has compact support. Then  $\check{\Phi}$  is an entire function. Moreover,

$$\lim_{\alpha \rightarrow \infty} \check{\Phi}(\alpha + i\beta) = 0, \quad (2)$$

uniformly for all  $\beta$  from some finite interval  $[a, b]$ . It is an immediate consequence of the following equality:

$$\check{\Phi}(\alpha + i\beta) = \int_{\mathbb{R}} e^{\beta x} \Phi(x) e^{i\alpha x} dx = -\frac{1}{i\alpha} \int_{\mathbb{R}} [\beta e^{\beta x} \Phi(x) + e^{\beta x} \Phi'(x)] e^{i\alpha x} dx.$$

Now using Cauchy's theorem and (2) we get

$$\begin{aligned} \langle \check{r}_\rho, \check{\Phi} \rangle &= \langle r_\rho, \check{\Phi} \rangle = \int_{\mathbb{R}} \langle x^*, R(is + \rho)x \rangle \check{\Phi}(s) ds \\ &= \int_{\mathbb{R}} \langle x^*, R(i(s - i\rho))x \rangle \check{\Phi}(s) ds = \int_{\mathbb{R} - i\rho} \langle x^*, R(i\lambda)x \rangle \check{\Phi}(\lambda + i\rho) d\lambda \\ &= \int_{\mathbb{R}} \langle x^*, R(is)x \rangle \check{\Phi}(s + i\rho) ds \\ &\quad + 2i \lim_{\alpha \rightarrow \pm\infty} \int_0^\rho \langle x^*, R(i(\alpha + i\beta))x \rangle \check{\Phi}(\alpha + i(\beta + \rho)) d\beta \\ &= \int_{\mathbb{R}} \langle x^*, R(is)x \rangle \check{\Phi}(s + i\rho) ds = \langle r_0, \check{\Phi}(\cdot + i\rho) \rangle \\ &= \langle \check{r}_0, e^{-\rho \cdot} \check{\Phi} \rangle, \end{aligned}$$

and the result follows. □

Now we are in a position to prove our main theorem. Let us denote by  $M_\rho$  the operator acting by the rule

$$M_\rho : f \mapsto [R(i \cdot + \rho) \hat{f}]^\vee.$$

Recall that a function  $m \in L_\infty(\mathbb{R}; \mathcal{L}(X))$  is called a *Fourier multiplier* on  $L_p(\mathbb{R}; X)$  if the operator  $M : f \mapsto [m(\cdot) \hat{f}]^\vee$  is a bounded operator on  $L_p(\mathbb{R}; X)$ . Let  $L_{1,\infty}(\mathbb{R}; X)$  denote the weak- $L_1$  space with values in  $X$  (see, e.g., [T, 1.18.6]), that is, the set of all  $X$ -valued strongly continuous functions  $f$  with the finite norm

$$\|f\|_{L_{1,\infty}} := \sup_{\sigma > 0} \left\{ \sigma \text{mes} \left( \{s \in \mathbb{R} : \|f(s)\| \geq \sigma\} \right) < \infty \right\}.$$

Note that  $L_{1,\infty}(\mathbb{R}; X) \subset L_1(\mathbb{R}; X)$ .

**Theorem 2.7.** For a strongly continuous semigroup  $\mathbf{T}$  on  $X$  the following conditions are equivalent:

- 1)  $\mathbf{T}$  is hyperbolic;
- 2)  $R(i\cdot)$  is a Fourier multiplier on  $L_p(\mathbb{R}, X)$  for some/all  $p$ ,  $1 \leq p < \infty$ ;
- 3) There exists a  $\rho_0 > 0$  such that for all  $\rho$  with  $|\rho| < \rho_0$ ,  $M_\rho$  maps  $L_1(\mathbb{R}, X)$  into  $L_{1,\infty}(\mathbb{R}, X)$ ;
- 4) There exists a  $\rho_0 > 0$  such that for all  $\rho$  with  $|\rho| < \rho_0$  and all  $\Phi \in \mathcal{S}$  we have  $|\langle r_\rho, \Phi \rangle| \leq K_\rho \|x\| \|x^*\| \|\check{\Phi}\|_1$ .

Furthermore, if one of these properties holds, then for every  $t \in \mathbb{R}$  and  $x \in X$  the integral

$$G(t)x = \frac{1}{2\pi}(C, 1) \int_{\mathbb{R}} R(is)x e^{ist} ds$$

converges and represents the Green's function of  $\mathbf{T}$ . Moreover,  $M_0 f = G * f$  for  $f \in L_1(\mathbb{R}, X)$ , and the splitting projection is given by the formula

$$P = \frac{1}{2}I + G(0). \quad (3)$$

*Proof.* 1) $\Rightarrow$ 4). This is a part of Theorem 0.2 from [KVL].

4) $\Rightarrow$ 2). It follows from 4) that  $\check{r}_\rho \in L_\infty$ ,  $|\rho| < \rho_0$  and  $\|\check{r}_\rho\|_\infty \leq K_\rho \|x\| \|x^*\|$ . By Lemma 2.6,  $\check{r}_0(t) = e^{-\rho t} \check{r}_{-\rho}(t)$  a.e. and  $\check{r}_0(t) = e^{\rho t} \check{r}_\rho(t)$  a.e. for some  $\rho > 0$ . So,  $|\check{r}_0(t)| \leq e^{-\rho|t|} K \|x\| \|x^*\|$  a.e. for every  $x \in X$  and  $x^* \in X^*$ , where  $K = \max\{K_\rho, K_{-\rho}\}$ . Now let us fix  $p$ ,  $1 \leq p < \infty$ , and consider a function  $\hat{\Phi} = \sum_{k=1}^n x_k \otimes \Phi_k$ , where  $\Phi_k \in \mathcal{S}$  and  $\{\Phi_k\}$  have disjoint supports. Then  $\|\hat{\Phi}\|_{L_p}^p = \sum_{k=1}^n \|x_k\|^p \|\Phi_k\|_{L_p}^p$ . So, we get the following estimates:

$$\begin{aligned} \|M_0(\hat{\Phi})\|_{L_p}^p &= \int_{\mathbb{R}} \|M_0(\hat{\Phi})(t)\|^p dt = \frac{1}{2\pi} \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} R(is) \hat{\Phi}(s) e^{ist} ds \right\|^p dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \sup_{\|x^*\| \leq 1} \left| \sum_{k=1}^n \int_{\mathbb{R}} r_0(s, x_k, x^*) \hat{\Phi}_k(s) e^{ist} ds \right|^p dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \sup_{\|x^*\| \leq 1} \left| \sum_{k=1}^n \int_{\mathbb{R}} \check{r}_0(\tau, x_k, x^*) \Phi_k(t - \tau) d\tau \right|^p dt \\ &\leq K^p \int_{\mathbb{R}} \left( \sum_{k=1}^n \int_{\mathbb{R}} e^{-\rho|\tau|} \|x_k\| |\Phi_k(t - \tau)| d\tau \right)^p dt \\ &= K^p \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-\rho|\tau|} \left( \sum_{k=1}^n \|x_k\| |\Phi_k(t - \tau)| \right) d\tau \right)^p dt \\ &\leq C_\rho K^p \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\rho|\tau|} \left( \sum_{k=1}^n \|x_k\| |\Phi_k(t - \tau)| \right)^p d\tau dt \\ &= C_\rho K^p \int_{\mathbb{R}} e^{-\rho|\tau|} \int_{\mathbb{R}} \left( \sum_{k=1}^n \|x_k\| |\Phi_k(t - \tau)| \right)^p dt d\tau \end{aligned}$$

$$= C_\rho K^p \int_{\mathbb{R}} e^{-\rho|\tau|} \sum_{k=1}^n \|x_k\|^p \|\Phi_k\|_{L_p}^p d\tau = C'_\rho K^p \|\Phi\|_{L_p}^p.$$

Since the functions  $\Phi$  are dense in  $L_p(\mathbb{R}, X)$ , the proof of 4) $\Rightarrow$ 2) is finished.

2) $\Rightarrow$ 1). Suppose 2) holds for some  $p$ ,  $1 \leq p < \infty$ . Then, by the transference principle (see, for example, [SW, Thm. VII.3.8]),  $\{R(ik + i\xi)\}_{k \in \mathbb{Z}}$  is a multiplier in  $L_p(\mathbb{T}, X)$  for all  $\xi \in \mathbb{R}$ , where  $\mathbb{T}$  is the unit circle. So, using results from [LMS, Theorem 2.3] or [LR, Theorem 1], we conclude that  $e^{2\pi i\xi} \in \rho(T_{2\pi})$  for all  $\xi \in \mathbb{R}$ . Thus,  $\mathbb{T} \subset \rho(T_{2\pi})$  and hence  $\mathbf{T}$  is hyperbolic.

This completes the proof of 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 4).

2) $\Rightarrow$ 3). It is easy to see using the resolvent identity, that there exists a  $\rho_0 > 0$  such that  $R(i \cdot + \rho)$  is a  $L_1(\mathbb{R}, X)$ -multiplier for all  $\rho$  such that  $|\rho| < \rho_0$ .

3) $\Rightarrow$ 4). Without loss of generality, assume  $\rho = 0$ . Denote

$$\mu = \sup_{0 \leq \tau \leq 1} \|T_\tau\|$$

and fix  $x \in X$ ,  $x^* \in X^*$ ,  $\|x\| = \|x^*\| = 1$ . Let us take a function  $\Phi \in \mathcal{S}$ . By condition 3) we have

$$\|M_0(\check{\Phi} \otimes x)\|_{1, \infty} \leq K \|\hat{\Phi}\|_1.$$

So,  $\text{mes}\{\tau : \|M_0(\hat{\Phi} \otimes x)(\tau)\| > 2K \|\hat{\Phi}\|_1\} \leq \frac{1}{2}$ . This implies that there is a  $\tau$ ,  $-1 < \tau < 0$ , such that

$$\|M_0(\check{\Phi} \otimes x)(\tau)\| \leq 2K \|\hat{\Phi}\|_1.$$

Let us apply the functional  $T_{-\tau}^* x^*$  to the left-hand side of the inequality. Then we have:

$$\left| \frac{1}{2\pi} \langle T_{-\tau}^* x^*, [R \cdot \Phi \otimes x]^\vee(\tau) \rangle \right| \leq 2\mu K C \|\hat{\Phi}\|_1.$$

By Lemma 2.4 and 2.5 the expression under the absolute value sign is equal to

$$\begin{aligned} \check{r}_0(\cdot, x, T_{-\tau}^* x^*) * \check{\Phi}(\tau) &= \langle \check{r}_0(\cdot, T_{-\tau} x, x^*), \check{\Phi}(\tau - \cdot) \rangle \\ &= \langle \check{r}_0(\cdot + \tau, T_{-\tau} x, x^*), \hat{\Phi} \rangle = \langle \check{r}_0(\cdot, x, x^*), \hat{\Phi} \rangle - \int_0^{-\tau} \langle x^*, T_t x \rangle \hat{\Phi}(t) dt. \end{aligned}$$

By the triangle inequality, we have

$$|\langle r_0(\cdot, x, x^*), \Phi \rangle| \leq 2K\mu \|\hat{\Phi}\|_1 + \mu \|\hat{\Phi}\|_1 \leq 3K\mu \|\check{\Phi}\|_1,$$

which is what we wanted.

Now we turn to the second part of the theorem. First, we prove an auxiliary Fejér-type lemma (probably, well-known).

**Lemma 2.8.** *If  $f \in L_1(\mathbb{R}, X)$ , then the integral*

$$\frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} \hat{f}(s) e^{ist} ds$$

converges to  $f(t)$  a.e. Moreover,

$$f = \frac{1}{2\pi} L_1 - \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N \int_{-\ell}^{\ell} \hat{f}(s) e^{is \cdot} ds.$$

*Proof.*

$$\begin{aligned} \frac{1}{2\pi} \frac{1}{N} \int_0^N \int_{-\ell}^{\ell} \hat{f}(s) e^{ist} ds &= \frac{1}{2\pi} \int_{-N}^N \hat{f}(s) e^{ist} \left(1 - \frac{|s|}{N}\right) ds \\ &= \frac{1}{2\pi} \int_{-N}^N \int_{-\infty}^{+\infty} f(r) e^{isr} dr \cdot e^{ist} \left(1 - \frac{|s|}{N}\right) ds \\ &= \int_{-\infty}^{+\infty} f(r) \frac{1}{2\pi} \int_{-N}^N e^{is(t-r)} \left(1 - \frac{|s|}{N}\right) ds dr. \end{aligned}$$

The inner integral is equal to  $K_N(t-r) = \frac{1}{\pi N(t-r)^2} [1 - \cos N(t-r)]$ . One can easily check that  $K_N$  is a positive kernel in  $L_1$ , that is,  $(K_N * f)(\cdot)$  tends to  $f(\cdot)$  a.e. and in  $L_1$  as  $N \rightarrow \infty$ .  $\square$

Suppose  $f \in L_1(\mathbb{R})$ . Then by 2) we have that  $M_0(f \otimes x) \in L_1(\mathbb{R}, X)$ . By Lemma 2.8, there is a  $\tau \in (-1, 0)$  such that

$$M_0(f \otimes x)(\tau) = \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is) x \hat{f}(s) e^{is\tau} ds.$$

Let us apply the operator  $T_{-\tau}$ . Then using (1) we obtain:

$$\begin{aligned} T_{-\tau}([R\hat{f} \otimes x]^\vee(\tau)) &= \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is) T_{-\tau} x e^{is\tau} \hat{f}(s) ds \\ &= \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} \left[ R(is) x \hat{f}(s) \right. \\ &\quad \left. - \int_0^{-\tau} T_r x e^{-isr} dr \cdot \hat{f}(s) \right] ds. \end{aligned}$$

Since

$$f(-r) = L_1 - \lim_{N \rightarrow \infty} \frac{1}{2\pi} \frac{1}{N} \int_0^N \int_{-\ell}^{\ell} \hat{f}(s) e^{-isr} ds dl$$

and  $V\varphi = \int_0^{-\tau} T_r x \cdot \varphi(r) dr$  is a bounded linear operator from  $L_1(\mathbb{R})$  to  $X$ , we conclude that the  $(C, 1)$ -integral of the second summand converges and equals  $\int_0^{-\tau} T_r x \cdot f(-r) dr$ . This means, in particular, that  $\frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is) x \hat{f}(s) ds$  converges. Let us denote it by  $G(0, f)$ . Also let

$$G(t, f) = G(0, f(\cdot - t)) = \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is) x \hat{f}(s) e^{ist} ds$$

for  $f \in L_1$ ,  $t \in \mathbb{R}$ ,  $x \in X$ . Now we introduce the following operators:

$$\begin{aligned} S_N^t(f, x) &= \frac{1}{2\pi} \frac{1}{N} \int_0^N \int_{-\ell}^{\ell} R(is) x \hat{f}(s) e^{ist} ds dl; \\ I_N^t(x) &= \frac{1}{2\pi} \frac{1}{N} \int_0^N \int_{-\ell}^{\ell} R(is) x e^{ist} ds dl. \end{aligned} \quad (4)$$

It is easy to see that  $\|S_N^t(f, x)\| \leq C_N \|f\|_{L_1} \|x\|$ . On the other hand, we have just proved that  $G(t, f)x = \lim_{N \rightarrow \infty} S_N^t(f, x)$  exists for all  $f \in L_1$ ,  $x \in X$ . So, by the boundedness principle for bilinear operators,  $\|S_N^t\| \leq C$ , where  $C$  does not depend on  $N$  and  $t$ .

Let  $f_\epsilon$ ,  $\epsilon > 0$ , be a kernel in  $L_1(\mathbb{R})$ , that is,  $f_\epsilon * \Phi \rightarrow \Phi$  as  $\epsilon \rightarrow 0$  for each  $\Phi \in L_1(\mathbb{R})$ . Then  $I_N^t(x) = \lim_{\epsilon \rightarrow 0} S_N^t(f_\epsilon, x)$  and hence,  $\|I_N^t\| \leq C$ .

Let us show that  $G(t)x = \lim_{N \rightarrow \infty} I_N^t(x)$  exists for all  $x \in D(A^2)$ . This will be enough to prove that  $G(t)x$  exists for all  $x \in X$ . Fix  $x \in D(A^2)$  and notice that

$$\begin{aligned} I_N^t(x) &= \frac{1}{2\pi} \frac{1}{N} \int_0^N \int_{-\ell}^{\ell} R(is) x e^{ist} ds dl \\ &= \frac{1}{2\pi} \frac{1}{N} \int_0^1 \int_{-\ell}^{\ell} R(is) x e^{ist} ds dl + \frac{1}{2\pi} \frac{1}{N} \int_1^N \int_{|s| \leq 1} R(is) x e^{ist} ds dl \\ &\quad + \frac{1}{2\pi} \frac{1}{N} \int_1^N \int_{1 \leq |s| \leq \ell} R(is) x e^{ist} ds dl \\ &= \frac{1}{2\pi} \frac{1}{N} \int_0^1 \int_{-\ell}^{\ell} R(is) x e^{ist} ds dl + \frac{1}{2\pi} \frac{N-1}{N} \int_{|s| \leq 1} R(is) x e^{ist} ds \\ &\quad + \frac{1}{2\pi} \frac{1}{N} \int_1^N \int_{1 \leq |s| \leq \ell} \left[ -\frac{R(is)A^2x}{s^2} + \frac{x}{is} - \frac{Ax}{s^2} \right] e^{ist} ds dl. \end{aligned}$$

So,

$$\begin{aligned} \lim_{N \rightarrow \infty} I_N^t(x) &= \frac{1}{2\pi} \int_{|s| \leq 1} R(is) x e^{ist} ds \\ &\quad - \frac{1}{2\pi} \int_{|s| \geq 1} \left[ \frac{Ax}{s^2} + \frac{R(is)A^2x}{s^2} \right] e^{ist} ds \\ &\quad + \frac{x}{2\pi i} \int_t^{+\infty} \frac{\sin(s)}{s} ds \cdot \chi_{\mathbb{R} \setminus \{0\}}(t). \end{aligned} \quad (5)$$

Finally, it is only left to verify that  $G(t)$  is indeed the Green's function. Let us prove the first equality in Definition 2.2, the second one being analogous. We

have

$$\begin{aligned}
T_\tau Px &= \frac{1}{2}T_\tau x + \frac{1}{2\pi}(C, 1) \int_{\mathbb{R}} R(is)xT_\tau x ds \\
&= \frac{1}{2}T_\tau x + \frac{1}{2\pi}(C, 1) \int_{\mathbb{R}} R(is)xe^{ist} ds \\
&\quad - \frac{1}{2\pi}(C, 1) \int_{\mathbb{R}} \int_0^\tau T_r x e^{-isr} dr \cdot e^{ist} ds \\
&= G(\tau)x + \frac{1}{2}T_\tau x - \frac{1}{2}T_\tau x = G(\tau)x,
\end{aligned}$$

where we use the ordinary Fejér's theorem.

It follows from the above that  $G(\tau)$  is an exponentially decaying function. So,  $f \mapsto G * f$  is a bounded operator on  $L_1$ . On the other hand,  $M_0\Phi = G * \Phi$  for all  $\Phi \in \mathcal{S}$ . Hence,  $M_0f = G * f$  for all  $f \in L_1$ .

The proof of (3) can be found in [KVL].  $\square$

**Corollary 2.9** ([G, H]). *Suppose  $X$  is a Hilbert space. Then the semigroup  $\mathbf{T}$  is hyperbolic if and only if the resolvent  $R(\lambda, A)$  is bounded in some strip containing the imaginary axes.*

**Remark 2.10.** *Condition 3) can be considerably weakened in the following way. Suppose  $F$  is a space of functions on  $\mathbb{R}$  with the following property: for any  $f \in F$  there is a  $t \in [-1, 0]$  such that  $|f(t)| \leq c\|f\|_F$ . Many quasi-normed function spaces have this property, for example,  $L_{q,r}(\mathbb{R})$ ,  $H_p(\mathbb{R})$ ,  $C_0(\mathbb{R})$ , or any function lattice with  $\|\chi_{[-1,0]}\| \neq 0$ . Denote by  $F(X)$  the space of all strongly measurable functions  $f$  with values in  $X$  such that  $\|f(\cdot)\| \in F$ . Our proof shows that it is enough to require that  $M_\rho$  maps  $L_1(\mathbb{R}, X)$  into  $F(X)$  (see also the proof of Theorem 3.1).*

**Remark 2.11.** *Recall that the generator  $\Gamma$  of the evolution semigroup  $(E^t)_{t \geq 0}$ , defined on  $L_p(\mathbb{R}, X)$  by  $(E^t f)(s) = T_t f(s-t)$ , is the closure of the operator  $-d/dt + A$  on the domain  $D(-d/dt) \cap D(A)$ . It is known, see [CL, Thm.2.39], that  $\mathbf{T}$  is hyperbolic if and only if the operator  $\Gamma$  is invertible on one/all  $L_p(\mathbb{R}, X)$ ,  $1 \leq p < \infty$ . This result immediately implies that conditions 1) and 2) in Theorem 2.7 are equivalent. Indeed, if  $x \in D(A)$  and  $\Phi \in \mathcal{S}$ , then  $\Gamma(\Phi \otimes x) = -\Phi' \otimes x + \Phi \otimes Ax$ . Using elementary properties of Fourier transform, we have that*

$$M_0\Gamma(\Phi \otimes x) = -\Phi \otimes x, \quad x \in D(A)$$

and

$$\Gamma M_0(\Phi \otimes x) = -\Phi \otimes x, \quad x \in X,$$

and the result follows.

It is worth noting that in the special case  $s_0(A) < 0$ , by Lemma 2.3, the splitting projection turns into the identity and our theorem gives the characterization of uniform exponential stability observed in [LR].

There is a Mihlin-type sufficient condition due to M. Hieber [H1] for an operator-valued symbol to be  $L_1$ -multiplier. Applied to the resolvent it yields the

following: if there exists a  $\delta > \frac{3}{4}$  such that  $\sup\{|s|^\delta \|R(is)\|\} < \infty$ , then  $R(i\cdot)$  is a multiplier.

Yet another condition for operator-valued symbol to be a multiplier is recently developed in [W1]. It works if  $X$  is a UMD-space and says that if the families  $\{R(is)\}_{s \in \mathbb{R}}$  and  $\{sR^2(is)\}_{s \in \mathbb{R}}$  are R-bounded, then  $R(i\cdot)$  is a multiplier.

### 3. Extension to the case $\alpha > 0$

It turns out that many arguments from Section 2 work in a more general situation, when the resolvent multiplier is restricted to  $L_p(\mathbb{R}, X_\alpha)$ , where  $X_\alpha$  is the domain of the fractional power  $(A - \omega)^\alpha$ , endowed with the norm  $\|x\|_\alpha = \|(A - \omega)^\alpha x\|$ . In this section we show that  $R(i\cdot + \rho)$  is a multiplier from  $L_p(\mathbb{R}, X_\alpha)$  to  $L_p(\mathbb{R}, X)$  for small values of  $\rho$  if and only if the following modified Kaashoek - Verduyn Lunel inequality holds:  $|\langle r_\rho, \Phi \rangle| \leq K \|x\|_\alpha \|x^*\| \|\Phi\|_{L_1}$ . Also in this case  $G(t)x$  exists for all  $x \in X_\alpha$  and is exponentially decaying as  $|t| \rightarrow \infty$ . As a by-product of this results we obtain the following relationship between the fractional growth bound  $\omega_\alpha(\mathbf{T})$  and its spectral analogue  $s_\alpha(A)$  (see (10) for the definitions):  $\omega_\alpha(\mathbf{T})$  is the infimum of all  $\omega > s_\alpha(A)$  such that  $R(i\cdot + \omega)$  is a multiplier from  $L_p(\mathbb{R}, X_\alpha)$  to  $L_p(\mathbb{R}, X)$ . In the particular case, when  $X$  is a Hilbert space, the latter condition will be shown to hold for all  $\omega > s_\alpha(A)$ . So,  $s_\alpha(A) = \omega_\alpha$ , which gives a different proof of G. Weiss's [Ws] result for arbitrary  $\alpha \geq 0$ , also obtained by L. Weis and V. Wrobel in [WW]. The main result in this section is an extension of Theorem 2.7 to the case of arbitrary  $\alpha > 0$ . To be more precise, we treat only conditions 2)-4), as hyperbolicity is ambiguous in this situation and therefore it is postponed to the next section.

One can notice that most of the proof of Theorem 2.7 work for all  $\alpha > 0$  if one replaces all  $X$ -norms by  $X_\alpha$ -norms. However, the "some/all" part of condition 2), being an easy consequence of results in [LMS] and the spectral characterization of hyperbolicity in case  $\alpha = 0$ , requires some additional duality argument.

Before we state our main theorem, let us recall the notion of fractional power of  $A$ . Suppose  $\omega > \max\{\omega_0 + 3, 3\}$ . Denote  $A - \omega$  by  $A_\omega$ . Let  $\gamma$  be the path consisting of two rays  $\Gamma_1 = \{-1 + te^{i\theta} : t \in [0, +\infty)\}$  and  $\Gamma_2 = \{-1 - te^{i\theta} : t \in [0, +\infty)\}$  going upwards. We assume that  $\theta$ ,  $\theta < \frac{\pi}{6}$ , is small enough to ensure the inequality  $\|R(\mu + \omega)\| \leq C \frac{1}{1+|\mu|}$  in the sector generated by  $\gamma$ . For any  $\alpha > 0$  we define  $A_\omega^\alpha$  as the inverse to the operator  $A_\omega^{-\alpha}$  acting on  $X$  by the rule

$$A_\omega^{-\alpha}(x) = \frac{1}{2\pi i} \int_\gamma \mu^{-\alpha} R(\mu + \omega) x d\mu.$$

Let us denote by  $X_\alpha$  the domain of  $A_\omega^\alpha$  endowed with the norm  $\|x\|_\alpha = \|(A - \omega)^\alpha x\|$ . Then  $X_\alpha$  is a Banach space and it does not depend on the particular choice of  $\omega$ ,  $\omega > \omega_0$ , see [EN] for more information concerning fractional powers.

**Theorem 3.1.** *Assume that there exists a  $\rho_0 > 0$  such that*

$$\sup \left\{ \frac{\|R(\lambda)\|}{1 + |\lambda|^\alpha} : |\operatorname{Re} \lambda| < \rho_0 \right\} < \infty. \quad (6)$$

*Then the following conditions are equivalent:*

- 1)  $R(i \cdot + \rho)$  is a multiplier from  $L_p(\mathbb{R}, X_\alpha)$  to  $L_p(\mathbb{R}, X)$ , for some/all  $p$ ,  $1 \leq p < \infty$ , and all  $\rho$ ,  $|\rho| < \rho_0$ ;
- 2)  $R(i \cdot + \rho)$  is a multiplier from  $L_p(\mathbb{R}, X_\alpha)$  to  $E(X)$ , for some  $p$ ,  $1 \leq p < \infty$ , and all  $\rho$ ,  $|\rho| < \rho_0$ , where  $E$  is a rearrangement invariant quasi-Banach lattice;
- 3)  $R(i \cdot + \rho)$  is a multiplier from  $L_1(\mathbb{R}, X_\alpha)$  to  $F(X)$  for all  $\rho$ ,  $|\rho| < \rho_0$ , where  $F$  is some rearrangement invariant quasi-Banach lattice;
- 4)  $|\langle r_\rho, \Phi \rangle| \leq K \|x\|_\alpha \|x^*\| \|\hat{\Phi}\|_{L_1}$  for all  $\Phi \in \mathcal{S}$ ,  $x \in X_\alpha$ ,  $x^* \in X^*$  and  $|\rho| < \rho_0$ .

*If one of these conditions holds, then the integral*

$$G(t)x = \frac{1}{2\pi}(C, 1) \int_{\mathbb{R}} R(is)x e^{ist} ds$$

*converges for all  $x \in X_\alpha$ , and  $\|G(t)x\| \leq K \|x\|_\alpha e^{-\rho|t|}$  for all  $\rho$ ,  $0 < \rho < \rho_0$ . Moreover,  $M_0 f = G * f$  for  $f \in L_1(\mathbb{R}, X)$ .*

*Proof.* 1) $\Rightarrow$ 2) is evident.

2) $\Rightarrow$ 3). Assume for simplicity that  $\rho = 0$ . First we claim that  $M_0$  maps  $L_p(\mathbb{R}, X_\alpha)$  into  $L_\infty(\mathbb{R}, X)$ . To prove this, let us take an arbitrary function  $\Phi$  of the form  $\sum_{i=1}^n \Phi_i x_i$ , where  $x_i \in X_\alpha$  and  $\Phi_i \in \mathcal{S}$ . Then, by condition 2),

$$\|M_0(\Phi)\|_{E(X)} \leq K \|\Phi\|_{L_p(\mathbb{R}, X_\alpha)}.$$

It implies that for every  $n \in \mathbb{Z}$  there exists a  $t \in [n, n+1]$  such that

$$\|[R\hat{\Phi}]^\vee(t)\|_X \leq \frac{2K}{\varphi(1)} \|\Phi\|_{L_p(\mathbb{R}, X_\alpha)},$$

where  $\varphi$  is the characteristic function of  $E$ . For any fixed  $\tau \in [0, 2]$ , let us apply the operator  $T_\tau$  to the right-hand side of this inequality. Then we get

$$\left\| \frac{1}{2\pi} \int_{\mathbb{R}} T_\tau R(is) \hat{\Phi}(s) e^{ist} ds \right\| \leq C \|\Phi\|_{L_p(\mathbb{R}, X_\alpha)}.$$

Now using equality (1) we obtain the following

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{R}} T_\tau R(is) \hat{\Phi}(s) e^{ist} ds &= \frac{1}{2\pi} \int_{\mathbb{R}} R(is) \hat{\Phi}(s) e^{is(t+\tau)} ds \\ &\quad - \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^\tau T_r \hat{\Phi}(s) e^{is(\tau-r)} dr ds \\ &= M_0(\Phi)(t + \tau) - \frac{1}{2\pi} \int_0^\tau T_r \Phi(\tau - r) dr. \end{aligned}$$

Thus,  $\|M_0(\Phi)(t + \tau)\|_X \leq \tilde{C}\|\Phi\|_{L_p(\mathbb{R}, X_\alpha)}$ . By the choice of  $\tau$  and  $t$  we have the same inequality on the whole real line. Since  $\tau$  was chosen arbitrary, the claim is proved.

Let us observe that the boundedness of  $M_0$  is equivalent to the fact that  $R(\cdot)A_\omega^{-\alpha}$  is an  $L_p(\mathbb{R}, X) - L_\infty(\mathbb{R}, X)$  multiplier.

Denote by  $X^\odot$  the sun dual to  $X$  on which the dual semigroup is strongly continuous (see [EN]). One can easily check, by duality, that for a test function  $\Phi = \sum_{i=1}^n \Phi_i x_i^\odot$  one has

$$\left\| [R^\odot(A^\odot)^{-\alpha} \hat{\Phi}]^\vee \right\|_{L_q(\mathbb{R}, X^\odot)} \leq \tilde{C} \|\Phi\|_{L_1(\mathbb{R}, X^\odot)},$$

where  $\frac{1}{p} + \frac{1}{q} = 1$  and  $A^\odot$  is the generator of the sun dual semigroup. In other words,  $M_0^\odot$  maps  $L_1(\mathbb{R}, X_\alpha^\odot)$  into  $L_q(\mathbb{R}, X^\odot)$ .

By what we just proved,  $M_0^\odot$  is bounded from  $L_1(\mathbb{R}, X_\alpha^\odot)$  to  $L_\infty(\mathbb{R}, X^\odot)$ , and again by duality,  $M_0$  maps  $L_1(\mathbb{R}, X_\alpha)$  into  $L_\infty(\mathbb{R}, X)$ , which proves condition 3) with  $F = L_\infty$ .

The proofs of all other implications are completely analogous to those of Theorem 2.7.

Let us now turn to the second part of our theorem. Although its proof is also essentially the same, some comments will be in order. By Lemma 3.2, proved below, assumption (6) is equivalent to  $\sup \{\|R(\lambda)\|_{X_\alpha \rightarrow X} : |\operatorname{Re} \lambda| < \rho_0\} < \infty$ . So, the operators  $S_N^t$ , introduced in (4), are bounded from  $X_\alpha \times L_1(\mathbb{R})$  to  $X$ . Uniform boundedness follows from the fact that  $\lim_{N \rightarrow \infty} S_N^t(x, f)$  exists for all  $x \in X_\alpha$  and  $f \in L_1(\mathbb{R})$  by Lemma 2.8. Consequently,  $\|I_N^t\|_{X_\alpha \rightarrow X} \leq C$ . Formula (5) still makes sense for all  $x \in X_{\alpha+2}$ , because then  $A^2 x \in X_\alpha$  and all the integrals converge absolutely. So,  $G(t)x$  exists for all  $x \in X_\alpha$ , and it is continuous in  $t$ ,  $t \neq 0$ .

Since  $\langle x^*, G(t)x \rangle = \check{r}_0(t, x, x^*)$ , by condition 3) and Lemma 2.6 we have that  $|\langle x^*, G(t)x \rangle| \leq K e^{-\rho|t|} \|x\|_\alpha \|x^*\|$  almost everywhere and hence, by the continuity of  $G(t)x$ , for all  $t \in \mathbb{R}$ . Thus,  $\|G(t)x\|_X \leq K e^{-\rho|t|} \|x\|_\alpha$  and the proof is finished.  $\square$

**Lemma 3.2.** *Let  $S = \{\lambda \in \mathbb{C} : a < \operatorname{Re} \lambda < b\}$ ,  $a, b \in \mathbb{R}$ , be a subset of  $\rho(A)$ , where  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$ . Then conditions*

$$\sup \left\{ \frac{\|R(\lambda)\|}{1 + |\lambda|^\alpha} : \lambda \in S \right\} < \infty \quad \text{and} \quad \sup \left\{ \|R(\lambda)A_\omega^{-\alpha}\| : \lambda \in S \right\} < \infty$$

*are equivalent.*

*Proof.* Since  $b$  is finite, there are constants  $c > 0$  and  $\varphi_0$ ,  $0 < \varphi_0 < \pi$  such that  $|\mu - e^{i\varphi}| > |\mu| + c$  for all  $\mu \in \gamma$  and  $\varphi_0 < |\varphi| < \pi - \varphi_0$ . Pick  $N > 1$  large enough to satisfy  $\frac{\omega}{N} < \frac{c}{2}$  and such that whenever  $\lambda \in S$  and  $|\lambda| > N$ , then  $\varphi_0 < |\arg \lambda| < \pi - \varphi_0$  and  $\lambda$  does not belong to the sector bounded by the contour  $|\lambda|\gamma$ . For all such  $\lambda$  we have

$$\left| \mu + \frac{\omega}{|\lambda|} - e^{i \arg \lambda} \right| > |\mu| + \frac{c}{2}. \quad (7)$$

Let us consider the following integral:

$$I_\lambda = \int_\gamma \frac{\mu^{-\alpha}}{\mu + \omega - \lambda} d\mu, \quad \lambda \in S, \quad |\lambda| > N.$$

By the choice of  $N$ , the integrand does not have singular points between  $\gamma$  and  $|\lambda|\gamma$ . By the Cauchy Theorem, we have

$$I_\lambda = \int_{|\lambda|\gamma} \frac{\mu^{-\alpha}}{\mu + \omega - \lambda} d\mu = \frac{1}{|\lambda|^\alpha} \int_\gamma \frac{\mu^{-\alpha}}{\mu + \frac{\omega}{|\lambda|} - e^{i \arg \lambda}} d\mu.$$

Inequality (7) implies that the absolute value of the last integral is bounded from above by a constant that does not depend on  $\lambda$ , whenever  $\lambda \in S$ ,  $|\lambda| > N$ . The analogous estimate from below follows from geometric considerations. Thus,

$$\frac{d_1}{|\lambda|^\alpha} \leq |I_\lambda| \leq \frac{d_2}{|\lambda|^\alpha}, \quad (8)$$

for some positive  $d_1$  and  $d_2$ .

Suppose  $x \in X$ . Then

$$\begin{aligned} R(\lambda)A_\omega^{-\alpha}x &= \frac{1}{2\pi i} \int_\gamma \mu^{-\alpha} R(\lambda)R(\mu + \omega)x d\mu \\ &= \frac{1}{2\pi i} I_\lambda R(\lambda)x - \frac{1}{2\pi i} \int_\gamma \frac{\mu^{-\alpha}}{\mu + \omega - \lambda} R(\mu + \omega)x d\mu. \end{aligned}$$

Let us notice that  $|\mu + \omega - \lambda| \geq K(|\mu| + 1)$  for some  $K > 0$  and all  $\mu \in \gamma$ ,  $\lambda \in S$ ,  $|\lambda| > N$ , whereas  $\|R(\mu + \omega)\| \leq C \frac{1}{1+|\mu|}$ . Consequently,

$$\left\| \int_\gamma \frac{\mu^{-\alpha}}{\mu + \omega - \lambda} R(\mu + \omega)x d\mu \right\| \leq K_\alpha \|x\|.$$

In combination with (8) this gives the following estimates:

$$\begin{aligned} \|R(\lambda)A_\omega^{-\alpha}x\| &\geq d_1 \frac{\|R(\lambda)x\|}{|\lambda|} - K_\alpha \|x\|, \\ \|R(\lambda)A_\omega^{-\alpha}x\| &\leq d_2 \frac{\|R(\lambda)x\|}{|\lambda|} + K_\alpha \|x\|, \end{aligned}$$

for all  $\lambda \in S$ ,  $|\lambda| > N$  and  $x \in X$ , which proves the lemma.  $\square$

**Remark 3.3.** In view of Lemma 3.2, assumption (6) in Theorem 3.1, in fact, follows from condition 1) or 3).

**Remark 3.4.** Just as in the proof of Theorem 2.7 one can show the following identities:

$$\begin{aligned} G(t) &= T_t P, \quad t > 0 \\ G(t)T_{-t} &= -(I - P), \quad t < 0 \end{aligned} \quad (9)$$

on  $X_\alpha$ , where  $P$  is defined as  $\frac{1}{2}I + G(0)$ .

Let us now recall the definition of the *fractional growth bound*  $\omega_\alpha(\mathbf{T})$  and its spectral counterpart  $s_\alpha(A)$ :  $\omega_\alpha(\mathbf{T})$  is the infimum of all  $\omega \in \mathbb{R}$  such that  $\|T_t x\| \leq M_\omega e^{\omega t} \|x\|_\alpha$ , for some  $M_\omega > 0$  and all  $x \in X_\alpha$  and  $t \geq 0$ , and

$$s_\alpha(A) = \inf \left\{ s : \sup \left\{ \frac{\|R(\lambda)\|}{1 + |\operatorname{Im} \lambda|^\alpha} : \operatorname{Re} \lambda > s \right\} < \infty \right\}. \quad (10)$$

As another consequence of Lemma 3.2 we get the following inversion formula (see [vN] for the case  $\alpha = 0$ ).

**Corollary 3.5.** *Let  $x \in X_\alpha$  and  $h > s_\alpha(A)$ . If  $F_t$  is defined as in Lemma 2.3, then*

$$F_t(x) = \frac{1}{2\pi i} (C, 1) \int_{\operatorname{Re} \lambda = h} e^{\lambda t} R(\lambda) x d\lambda$$

for all  $t \in \mathbb{R}$ .

*Proof.* If  $h \geq \omega$ , then our statement is the ordinary inversion formula (see Lemma 2.3). Otherwise, by the resolvent identity, we have

$$R(u + iv)x = (1 - (u - \omega)R(u + iv))A_\omega^{-\alpha} R(\omega + iv)A_\omega^\alpha x,$$

for all  $u, h \leq u \leq \omega$ . So, in view of Lemma 3.2,  $\lim_{v \rightarrow \infty} R(u + iv)x = 0$  uniformly in  $u \in [h, \omega]$ . Then, by the Cauchy Theorem, we get

$$\begin{aligned} \frac{1}{2\pi i} (C, 1) \int_{\operatorname{Re} \lambda = h} e^{\lambda t} R(\lambda) x d\lambda &= \frac{1}{2\pi i} (C, 1) \int_{\operatorname{Re} \lambda = \omega} e^{\lambda t} R(\lambda) x d\lambda \\ &= F_t(x). \end{aligned}$$

□

Let us recall the inequality  $\omega_\alpha(\mathbf{T}) \geq s_\alpha(A)$  (see [Ws]). Now suppose  $\omega > s_\alpha(A)$  and  $R(i \cdot + \omega)$  is a multiplier from  $L_p(\mathbb{R}, X_\alpha)$  to  $L_p(\mathbb{R}, X)$ . One can easily notice that the implication 1)  $\Rightarrow$  4) of Theorem 3.1 was proved individually for every  $\rho$ . Thus,

$$|\langle r_\omega, \Phi \rangle| \leq |\langle \check{r}_\omega, \hat{\Phi} \rangle| \leq K \|x\|_\alpha \|x^*\| \|\hat{\Phi}\|_{L_1}.$$

However, by Corollary 3.5,  $\check{r}_\omega(t) = e^{-\omega t} \langle x^*, T_t x \rangle$ ,  $t > 0$ , which implies  $\|T_t x\| \leq e^{\omega t} \|x\|_\alpha$ . So,  $\omega_\alpha(\mathbf{T}) \leq \omega$ .

On the other hand, if  $\omega > \omega_\alpha(\mathbf{T})$ , then  $\|e^{-\omega t} T_t\|_{X_\alpha \rightarrow X}$  is exponentially decaying. Consequently, the operator  $M_\omega$ , being a convolution with the kernel  $e^{-\omega t} T_t$ , maps  $L_p(\mathbb{R}, X_\alpha)$  into  $L_p(\mathbb{R}, X)$  as a bounded operator. Thus, we have proved the following result.

**Theorem 3.6.** *For any  $C_0$ -semigroup  $\mathbf{T}$  on a Banach space  $X$ ,  $\omega_\alpha(\mathbf{T})$  is the infimum over all  $\omega > s_\alpha(A)$  such that  $R(i \cdot + \omega)$  is a multiplier from  $L_p(\mathbb{R}, X_\alpha)$  to  $L_p(\mathbb{R}, X)$ , for some  $p$ ,  $1 \leq p < \infty$ .*

**Corollary 3.7** ([Ws, WW]). *If  $X$  is a Hilbert space, then  $\omega_\alpha(\mathbf{T}) = s_\alpha(A)$  for any strongly continuous semigroup  $\mathbf{T}$  and  $\alpha \geq 0$ .*

There are many results about properties of the constants  $\omega_\alpha(\mathbf{T})$ ,  $s_\alpha(A)$  and relations between them. We refer the reader to paper [WW] for a detailed exposition of the subject.

We conclude this section by proving an  $\alpha$ -analogue of Perron's Theorem, cf. [LR]. Let us recall the classical result: a  $C_0$ -semigroup  $\mathbf{T}$  with generator  $A$  is hyperbolic if and only if for every  $g \in L_p(\mathbb{R}, X)$  the following integral equation

$$u(\theta) = T_{\theta-\tau}u(\tau) + \int_\tau^\theta T_{\theta-s}g(s)ds, \quad \theta \geq \tau, \quad (11)$$

has unique solution in  $L_p(\mathbb{R}, X)$  (see, e.g. [CL, Theorem 4.33]).

In case of arbitrary  $\alpha \geq 0$ , we are looking for a necessary and sufficient condition on  $\mathbf{T}$ , which provides existence and uniqueness of solution to (11) in  $L_p(\mathbb{R}, X)$  for any given  $g \in L_p(\mathbb{R}, X_\alpha)$ . It turns out that the multiplier property of  $R(is)$  is the condition we need.

**Theorem 3.8.** *Suppose  $i\mathbb{R} \subset \rho(A)$ . Then the following assertions are equivalent:*

- 1)  $R(i\cdot)$  is a multiplier from  $L_p(\mathbb{R}, X_\alpha)$  to  $L_p(\mathbb{R}, X)$ ;
- 2) for every  $g \in L_p(\mathbb{R}, X_\alpha)$  there exists a unique solution of (11) belonging to  $L_p(\mathbb{R}, X)$ .

Before we prove the theorem let us state one auxiliary fact, see [MRS] or [CL, Prop.4.32].

**Lemma 3.9.** *A function  $u$  is a solution of (11) if and only if  $u \in D(\Gamma)$  and  $\Gamma u = -g$ , where  $\Gamma$  is the generator of the associated evolution semigroup.*

*Proof.* 2)  $\Rightarrow$  1). Denote by  $L$  the linear operator that maps  $g \in L_p(\mathbb{R}, X_\alpha)$  to the corresponding solution of (11). By the Closed Graph Theorem,  $L$  is bounded. We prove that actually  $L = M_0$ . Indeed, by Lemma 3.9,  $Lg \in D(\Gamma)$  and  $\Gamma Lg = -g$ , for every  $g \in L_p(\mathbb{R}, X_\alpha)$ . On the other hand, a straightforward computation shows that if  $g$  is a  $C^\infty$ -function with compact support, then  $M_0g \in D(\Gamma)$  and  $\Gamma M_0g = -g$ . Thus,  $\Gamma(M_0g - Lg) = 0$ . However, if  $\Gamma u = 0$  for some  $u \in D(\Gamma)$ , then again by Lemma 3.9,  $u$  is a solution of (11) corresponding to  $g = 0$ . By the uniqueness, we get  $u = 0$ . So,  $M_0g = Lg$  on a dense subspace of  $L_p(\mathbb{R}, X_\alpha)$  and boundedness of  $M_0$  is proved.

1)  $\Rightarrow$  2). Suppose  $M_0$  is bounded from  $L_p(\mathbb{R}, X_\alpha)$  to  $L_p(\mathbb{R}, X)$ . For a fixed  $C^\infty$ -function  $g$  having compact support, we show that  $u = M_0g$  solves (11). Indeed, using (1), we get

$$\begin{aligned} u(\theta) - T_{\theta-\tau}u(\tau) &= \int_{\mathbb{R}} R(is)\hat{g}(s)e^{is\theta}ds - \int_{\mathbb{R}} R(is)T_{\theta-\tau}\hat{g}(s)e^{is\tau}ds \\ &= \int_0^{\theta-\tau} T_r \int_{\mathbb{R}} e^{is(\theta-r)}\hat{g}(s)dsdr \\ &= \int_0^{\theta-\tau} T_r g(\theta-r)dr = \int_\tau^\theta T_{\theta-r}g(r)dr, \end{aligned}$$

which is precisely (11).

Now suppose  $g$  is an arbitrary function from  $L_p(\mathbb{R}, X_\alpha)$ . Let us approximate  $g$  by functions  $(g_n)$  of considered type. Then  $u_n = M_0 g_n$  converge to  $u = M_0 g$  in  $L_p(\mathbb{R}, X)$  and, without loss of generality, pointwise on a set  $E \subset \mathbb{R}$  with  $\text{mes}\{\mathbb{R} \setminus E\} = 0$ . Thus, (11) is true for  $u, g$  and all  $\theta$  and  $\tau$  from  $E$ . To get (11) for all  $\theta$  and  $\tau$ , we will modify  $u$  on the set  $\mathbb{R} \setminus E$ . To this end, let us take a decreasing sequence  $(\tau_n) \subset E$  such that  $\lim \tau_n = -\infty$ . Observe that the functions  $f_n(\theta) = T_{\theta-\tau_n} u(\tau_n) + \int_{\tau_n}^{\theta} T_{\theta-s} g(s) ds$  defined for  $\theta \geq \tau_n$  are continuous. Since  $u = f_n = f_m$  on  $(+\infty, \max(\tau_n, \tau_m)] \cap E$ , we get  $f_n = f_m$  everywhere in the half-line  $(+\infty, \max(\tau_n, \tau_m)]$ . Put  $\tilde{u}$  to be  $f_n$  on  $(+\infty, \tau_n]$ . By the above,  $\tilde{u}$  is a well-defined function on all  $\mathbb{R}$ . Obviously,  $u = \tilde{u}$  on  $E$ . Let us show that  $\tilde{u}$  satisfies (11). Indeed, for any  $\theta \geq \tau$  and  $\tau > \tau_n$  we have

$$\begin{aligned} T_{\theta-\tau} \tilde{u}(\tau) + \int_{\tau}^{\theta} T_{\theta-s} g(s) ds &= T_{\theta-\tau} [T_{\tau-\tau_n} u(\tau_n) + \int_{\tau_n}^{\tau} T_{\tau-s} g(s) ds] \\ &\quad + \int_{\tau}^{\theta} T_{\theta-s} g(s) ds \\ &= T_{\theta-\tau_n} u(\tau_n) + \int_{\tau_n}^{\theta} T_{\theta-s} g(s) ds = \tilde{u}(\theta). \end{aligned}$$

□

Clearly, assertion 1) in Theorem 3.8 is weaker than condition 1) in Theorem 3.1. We do not know if they are equivalent. In case  $\alpha = 0$ , though, we can apply the resolvent identity to argue that if  $R(i \cdot)$  is a multiplier, then  $R(i \cdot + \rho)$  is also a multiplier for small values of  $\rho$ . So, by Theorem 2.7, this is equivalent to hyperbolicity of the semigroup  $\mathbf{T}$ , and our statement turns into classical Perron's Theorem.

#### 4. An $\alpha$ -analogue of hyperbolicity

We begin with a discrete version of Theorem 3.1 in the spirit of [LR, Theorem 5]. Denote by  $\text{Rg } T$  the range of an operator  $T$ .

**Theorem 4.1.** *Suppose  $i\mathbb{Z} \subset \rho(A)$ . Then the following conditions are equivalent:*

- 1)  $X_\alpha \subset \text{Rg}(I - T_{2\pi})$ ;
- 2) The sum  $(C, 1) \sum_{k \in \mathbb{Z}} R(ik)x$  exists in  $X$ -norm for all  $x \in X_\alpha$ ;
- 3)  $\{R(ik)\}_{k \in \mathbb{Z}}$  is a multiplier from  $L_p(\mathbb{T}, X_\alpha)$  to  $L_p(\mathbb{T}, X)$  for some/all  $1 \leq p < \infty$ ;
- 4)  $\{R(ik)\}_{k \in \mathbb{Z}}$  is a multiplier from  $L_1(\mathbb{T}, X_\alpha)$  to  $F(\mathbb{T}, X)$ , where  $F$  is some quasi-normed function lattice;
- 5) There exists a constant  $K > 0$  such that

$$|\langle r_0, \Phi \rangle| = \left| \sum_{k \in \mathbb{Z}} r_0(k, x, x^*) \Phi(k) \right| \leq K \|x\|_\alpha \|x^*\| \|\check{\Phi}\|_{L_1(\mathbb{T})}$$

holds for all  $x \in X_\alpha$ ,  $x^* \in X^*$ , and  $\Phi \in C_\infty(\mathbb{T})$ .

*Proof.* 1) $\Leftrightarrow$ 2). Note that

$$\frac{1}{2\pi}R(ik)(I - T_{2\pi})x = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} T_t x dt,$$

for all  $x \in X$ . So,

$$\frac{1}{2\pi}(C, 1) \sum_{k \in \mathbb{Z}} R(ik)(I - T_{2\pi})x = \frac{1}{2}(I + T_{2\pi})x.$$

Thus, 1) implies 2).

Now assume 1). Denote  $S = \frac{1}{2\pi}(C, 1) \sum_{k \in \mathbb{Z}} R(ik)x$ . Then

$$\left(\frac{1}{2}I + S\right)(I - T_{2\pi})x = (I - T_{2\pi})\left(\frac{1}{2}I + S\right)x = x, \quad (12)$$

for all  $x \in X_\alpha$ , and 1) follows.

Following [LR] we denote by  $K$  the operator of convolution with the semi-group, i.e.

$$Kf(t) = \int_0^{2\pi} T_s f((t-s)[\text{mod } 2\pi]) ds.$$

Clearly,  $K$  is bounded on  $L_p(\mathbb{T}, X)$  for all  $1 \leq p < \infty$  and  $\alpha > 0$ . Now we define the discrete multiplier operator  $L$  by the rule

$$Lf(\theta) = \sum_{k \in \mathbb{Z}} R(ik) \hat{f}(k) e^{ik\theta}, \quad \theta \in [0, 2\pi],$$

where  $f$  is a trigonometric polynomial. One can check the identity

$$K = L(I - T_{2\pi}) = (I - T_{2\pi})L. \quad (13)$$

By the assumption and the spectral mapping theorem for the point spectrum the operator  $(I - T_{2\pi})$  is one-to-one. Suppose 1) holds. Then  $(I - T_{2\pi})$  has the left inverse  $U_1 := (I - T_{2\pi})_{|_{\text{left}}}^{-1}$  defined on  $X_\alpha$ . By the Closed Graph Theorem  $U_1$  is bounded as an operator from  $X_\alpha$  to  $X$ . Then (13) says that  $KU_1 = L$  on trigonometric polynomials with values in  $X_\alpha$ . So,  $L$  maps  $L_p(\mathbb{T}, X_\alpha)$  into  $L_p(\mathbb{T}, X)$  for all  $1 \leq p < \infty$ , which is what is stated in 3).

If the assertion in 3) is true only for some  $p$ , then as in the proof of Theorem 3.1,  $LA_\omega^{-\alpha}$  maps  $L_p(\mathbb{T}, X)$  into  $L_\infty(\mathbb{T}, X)$ . By duality,  $(LA_\omega^{-\alpha})^*$  maps  $L_1(\mathbb{T}, X^\odot)$  into  $L_q(\mathbb{T}, X^\odot)$  and hence into  $L_\infty(\mathbb{T}, X^\odot)$ . So,  $LA_\omega^{-\alpha}$  is a bounded operator from  $L_1(\mathbb{T}, X)$  to  $L_\infty(\mathbb{T}, X)$ , which proves 4) with  $F = L_\infty(\mathbb{T}, X)$ .

Assume 4). Then for every  $f \in C_\infty(\mathbb{T}, X)$  there is a  $\theta \in [0, 2\pi]$  such that

$$\left\| \sum_{k \in \mathbb{Z}} R(ik) \hat{f}(k) e^{ik\theta} \right\| \leq K \|f\|_1.$$

Applying  $T_\theta$  in the above inequality and using (1) we have:

$$\left\| \sum_{k \in \mathbb{Z}} R(ik) \hat{f}(k) \right\| \leq K' \|f\|_1.$$

In particular, for  $f = \check{\Phi} \otimes x$  the last inequality yields 5).

If 5) holds, then taking  $\Phi = \sum_{n=1}^N \Phi_n \otimes x_n$ , with  $\Phi_n \in C_\infty(\mathbb{T})$  having disjoint supports we get the following estimates

$$\begin{aligned} \|L\Phi\|_1 &= \frac{1}{2\pi} \int_0^{2\pi} \sup_{\|x^*\|=1} \left| \sum_{n=1}^N \sum_{k \in \mathbb{Z}} r_0(k, x_n, x^*) \hat{\Phi}_n(k) e^{ik\theta} \right| d\theta \\ &\leq \sum_{n=1}^N \frac{1}{2\pi} \int_0^{2\pi} \sup_{\|x^*\|=1} |\langle r_0, \widehat{\Phi(\cdot + \theta)} \rangle| d\theta \\ &\leq \sum_{n=1}^N \|x_n\|_\alpha \|\Phi_n\|_1 = \|\Phi\|_1. \end{aligned}$$

So, we proved 3).

Finally, similarly to the proof of convergence of the  $(C, 1)$ -integral in Theorem 3.1, we can show that 2) follows from  $L_1$ -boundedness of  $LA_\omega^{-\alpha}$ .  $\square$

In Theorem 2.7 we have proved that conditions 2) through 4), involving multipliers, are equivalent to the hyperbolicity of the semigroup, that is, to a spectral property of  $\mathbf{T}$ . A natural question is to see if the multipliers-type conditions 2) through 4) in Theorem 3.1 are equivalent to a spectral property that could be formulated in terms of  $\mathbf{T}$  acting on the Banach space  $X$ , and *not* in terms of a space of  $X$ -valued functions. Theorem 4.1 suggests that each of the conditions 2) through 4) in Theorem 3.1 implies that the inclusion  $X_\alpha \subset \text{Rg}(zI - T_{2\pi})$  holds for all  $z$  from some annulus  $\mathbb{A}$  containing  $\mathbb{T}$ . It turns out that this inclusion alone is not equivalent to any of the conditions in Theorem 3.1. Below we will find the needed complement, but let us first make some observations.

Assume that  $\mathcal{S}$  is some strip containing the imaginary axes and  $\mathcal{S} \subset \rho(A)$ . Suppose also that  $X_\alpha \subset \text{Rg}(zI - T_{2\pi})$  for all  $z$  from some annulus  $\mathbb{A}$  containing  $\mathbb{T}$ . By the Point Spectrum Mapping Theorem  $(zI - T_{2\pi})$  is one-to-one. Thus, the left inverse operator  $U_z = (zI - T_{2\pi})_{left}^{-1} : X_\alpha \rightarrow X$  exists and is bounded by the Closed Graph Theorem. The family  $\mathbb{U} = \{U_z\}_{z \in \mathbb{A}}$  obeys the resolvent identity on vectors from  $X_{2\alpha}$ . However, to prove analyticity, first of all one needs uniform boundedness of  $\mathbb{U}$ . And that is the condition we are looking for.

**Theorem 4.2.** *Suppose there is a strip  $\mathcal{S}$  such that  $i\mathbb{R} \subset \mathcal{S} \subset \rho(A)$ . Then any of the equivalent conditions of Theorem 3.1 holds if and only if there exists an annulus  $\mathbb{A}$  containing  $\mathbb{T}$  such that  $X_\alpha \subset \text{Rg}(zI - T_{2\pi})$  for all  $z \in \mathbb{A}$ , and  $\sup\{\|U_z\|_{X_\alpha \rightarrow X} : z \in \mathbb{A}\} < \infty$ .*

*Proof.* Suppose that  $M_\rho$  maps  $L_p(\mathbb{R}, X_\alpha)$  into  $L_p(\mathbb{R}, X)$  for  $|\rho| < 2\rho_0$ . By the Uniform Boundedness Principle  $M_\rho$  are uniformly bounded for all  $|\rho| < \rho_0$ . Then, by transference,  $\{R(i(k + \xi) + \rho)\}_{k \in \mathbb{Z}}$  is a multiplier uniformly in  $\xi \in [0, 1)$  and  $|\rho| < \rho_0$ . In view of just proved Theorem 4.1 we get  $X_\alpha \subset Rg(zI - T_{2\pi})$  for all  $z$  from some open annulus  $\mathbb{A}$  containing  $\mathbb{T}$ . In order to show uniform boundedness of  $\mathbb{U}$ , let us look at identity (12) first. It shows, in particular, that  $U_1 = \frac{1}{2} + S$ . Just like in the second part of the proof of Theorem 2.7, one can estimate  $\|S\|_{X_\alpha \rightarrow X}$  by the multiplier norm of  $\{R(ik)\}$ . Rescaling gives the same conclusion for all  $U_z$ . Since norms of the corresponding multipliers are uniformly bounded, the desired result is proved.

Now let us prove the converse statement.

Clearly, the family  $\mathbb{U} = \{U_z\}_{z \in \mathbb{A}}$  obeys the resolvent identity on vectors from  $X_{2\alpha}$ . Since, in addition, it is bounded, the mapping  $z \rightarrow U_z$  is strongly continuous on vectors from  $X_{2\alpha}$  and, hence, on all  $X_\alpha$ . Again by the resolvent identity,  $U_z$  is strongly analytic on  $X_{2\alpha}$ . Since for any  $x \in X_\alpha$ ,  $U_z x$  is the uniform limit of a sequence  $U_z x_n$  with  $x_n \in X_{2\alpha}$ ,  $U_z x$  is analytic.

It suffices to show that the integral  $G(t)x = (C, 1) \int_{\mathbb{R}} R(is)x e^{ist} ds$  converges for all  $x \in X_\alpha$  and there exists a  $\beta > 0$  such that  $\|G(t)x\| \leq K e^{-\beta|t|} \|x\|_\alpha$ .

So, let us fix  $x \in X_\alpha$  and  $t \in \mathbb{R}$ . Then for any  $s \in \mathbb{R}$ ,  $x = (e^{2\pi is} - T_{2\pi})U_{e^{2\pi is}}x$ . Thus

$$\begin{aligned} R(is)x e^{ist} &= R(is)(I - e^{-2\pi is} T_{2\pi})U_{e^{2\pi is}}x e^{ist} e^{2\pi is} \\ &= e^{(2\pi+t)is} \int_0^{2\pi} e^{-ris} T_r U_{e^{2\pi is}} x dr. \end{aligned}$$

From this we get

$$\begin{aligned} G(t)x &= \lim_{N \rightarrow \infty} \int_0^{2\pi} T_r \int_{-N}^N U_{e^{2\pi is}} x e^{(2\pi+t-r)is} \left(1 - \frac{|s|}{N}\right) ds dr \\ &= \lim_{N \rightarrow \infty} \int_0^{2\pi} T_r \int_0^1 U_{e^{2\pi is}} x e^{(2\pi+t-r)is} \sum_{n=-N}^N e^{(t-r)in} \left(1 - \frac{|s+n|}{N}\right) ds dr \\ &= \lim_{N \rightarrow \infty} \int_0^{2\pi} T_r \int_0^1 U_{e^{2\pi is}} x e^{(2\pi+t-r)is} \sum_{n=-N}^N e^{(t-r)in} \left(1 - \frac{|n|}{N}\right) ds dr \\ &= \lim_{N \rightarrow \infty} \int_0^{2\pi} T_r \int_0^1 U_{e^{2\pi is}} x e^{(2\pi+t-r)is} \mathcal{F}_N(-r+t \bmod 2\pi) ds dr, \end{aligned}$$

where  $\mathcal{F}_N$  is the Fejér kernel. Passing to limit inside the integral we get

$$\begin{aligned} G(t) &= \frac{1}{2} \left[ T_{2\pi} \int_0^1 U_{e^{2\pi is}} x e^{ist} ds + \int_0^1 U_{e^{2\pi is}} x e^{is(2\pi+t)} ds \right] \\ &= \frac{1}{2} \left[ T_{2\pi} \frac{1}{2\pi i} \int_{\mathbb{T}} z^t U_z x dz + \frac{1}{2\pi i} \int_{\mathbb{T}} z^{2\pi+t} U_z x dz \right], \end{aligned}$$

if  $t = 0 \pmod{2\pi}$ . And

$$\begin{aligned} G(t) &= T_{t \pmod{2\pi}} \int_0^1 U_{e^{2\pi is}} x e^{(2\pi+t-t \pmod{2\pi})is} ds \\ &= T_{t \pmod{2\pi}} \frac{1}{2\pi i} \int_{\mathbb{T}} z^{2\pi+t-t \pmod{2\pi}} U_z x dz, \end{aligned}$$

otherwise. In either case, replacing  $\mathbb{T}$  by  $(1+\varepsilon)\mathbb{T}$ , if  $t < 0$ , or by  $(1-\varepsilon)\mathbb{T}$ , otherwise, we get the desired exponential decay.  $\square$

## 5. Strong $\alpha$ -hyperbolicity

In this section we introduce yet another notion of  $\alpha$ -hyperbolicity for strongly continuous semigroups. The spectral property we considered in the previous section, though strong enough, fails to produce any splitting projection, which is so natural in the case  $\alpha > 0$ . Therefore, we investigate a notion of strong  $\alpha$ -hyperbolicity, in which we force such a projection to exist.

**Definition 5.1.** A  $C_0$ -semigroup  $\mathbf{T} = (T_t)_{t \geq 0}$  is said to be strongly  $\alpha$ -hyperbolic if there exists a projection  $P$  on  $X$ , called *splitting*, such that  $PT_t = T_t P$ ,  $t \geq 0$  and the following two conditions hold:

1.  $\omega_\alpha(\mathbf{T}|_{\text{Im } P}) < 0$ ;
2. the restriction of  $\mathbf{T}$  on  $\text{Ker } P$  is a group, and  $\omega_\alpha(\mathbf{T}^{-1}|_{\text{Ker } P}) < 0$ , where  $\mathbf{T}^{-1} = (T_{-t}|_{\text{Ker } P})_{t \geq 0}$ .

The function  $G(t)$  defined as in Definition 2.2 is called the *Green's function* corresponding to the  $\alpha$ -hyperbolic semigroup  $\mathbf{T}$ .

It is an immediate consequence of the definition that Green's function exponentially decays at infinity on vectors from  $X_\alpha$ .

Now we prove an analogue of Theorem 2.7 for  $\alpha$ -hyperbolic semigroups.

**Theorem 5.2.** A semigroup  $\mathbf{T}$  is  $\alpha$ -hyperbolic if and only if one of the equivalent conditions of Theorem 3.1 is satisfied and the operator

$$G(t)x = \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is) x e^{ist} ds$$

has a continuous extension to all of  $X$  for each  $t \in \mathbb{R}$ .

If this is the case,  $G(t)$  represents the *Green's function*. Furthermore, the splitting projection is unique and given by

$$P = \frac{1}{2} I + G(0). \quad (14)$$

*Proof.* Let us prove necessity.

If  $\mathbf{T}$  is  $\alpha$ -hyperbolic, then there is a splitting projection  $P$ . Suppose  $x \in (\text{Im } P)_\alpha$ . Then by Corollary 3.5 applied to the semigroup  $\mathbf{T}|_{\text{Im } P}$ , we have  $F_t(x) = G(t)x$ . In particular,  $Px = x = \frac{1}{2}x + G(0)x$ . On the other hand, if  $x \in (\text{Ker } P)_\alpha$ ,

then by the same reason,  $\tilde{F}_t(x) = \frac{1}{2\pi}(C, 1) \int_{\mathbb{R}} R(is, -A)xe^{ist}ds = -G(-t)x$ , where  $\tilde{F}_t(x)$  is defined by

$$\tilde{F}_t(x) = \begin{cases} T_{-t}x, & t > 0 \\ \frac{1}{2}x, & t = 0. \\ 0, & t < 0 \end{cases}$$

So,  $Px = 0 = \frac{1}{2}x + G(0)x$ . Since  $X_\alpha = (\text{Im } P)_\alpha + (\text{Ker } P)_\alpha$  is dense in  $X$ , this shows that  $G(t)$  continuously extends to all of  $X$  and equality (14) is true. The uniqueness of  $P$  follows automatically from (14).

Since the  $(X_\alpha \rightarrow X)$ -norm of  $G(t)$  is exponentially decreasing and  $M_0(\Phi) = G * \Phi$  for all  $\Phi \in \mathcal{S}$ ,  $M_0$  is bounded from  $L_1(\mathbb{R}, X_\alpha)$  to  $L_1(\mathbb{R}, X)$ . To show boundedness of  $M_\rho$ , it is enough to notice that if  $\mathbf{T}$  is  $\alpha$ -hyperbolic, then the scaled semigroup  $e^{\rho \cdot} \mathbf{T}$  is also  $\alpha$ -hyperbolic, for small values of  $\rho$ .

Now we prove sufficiency.

Let us introduce the operator  $P = \frac{1}{2}I + G(0)$ . Since Theorem 2.7 is valid, and hence formulas (9) in Remark 3.4 are true, the norm of  $T_t$  on  $P(X_\alpha)$  is exponentially decaying. Consequently, by the ordinary inversion formula for Laplace transform, we get  $G(0)x = \frac{1}{2}x$ , for all  $x \in P(X_\alpha)$ . This implies  $P^2 = P$  on all  $X$ , in view of the continuity of  $P$ . So,  $P$  is a projection.

Obviously,  $PT_t = T_tP$ . On the other hand, since  $P(X_\alpha) = (\text{Im } P)_\alpha$ , we have  $\omega_\alpha(\mathbf{T}|_{\text{Im } P}) < 0$  and condition 1 of Definition 5.1 is proved.

To show invertibility of  $T_t$  on  $\text{Im}(I - P)$ , we apply formula (9). It implies that  $\|G(t)\| \|T_{-t}x\| \geq \|x\|$ , for  $x \in \text{Im}(I - P)$ , and hence,  $T_{-t}|_{\text{Im}(I - P)}$  is invertible. Another application of (9) and the second part of Theorem 2.7 proves condition 2 in Definition 5.1.  $\square$

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Department of Mathematics,  
University of Missouri,  
Columbia, MO 65211, USA

*E-mail address:* yuri@math.missouri.edu, RIS1DB@mizzou.edu