A Spectral Mapping Theorem and Invariant Manifolds for Nonlinear Schrödinger Equations

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ABSTRACT. A spectral mapping theorem is proved that resolves a key problem in applying invariant manifold theorems to nonlinear Schrødinger type equations. The theorem is applied to the operator that arises as the linearization of the equation around a standing wave solution. We cast the problem in the context of space-dependent nonlinearities that arise in optical waveguide problems. The result is, however, more generally applicable including to equations in higher dimensions and even systems. The consequence is that stable, unstable, and center manifolds exist in the neighborhood of a (stable or unstable) standing wave, such as a waveguide mode, under simple and commonly verifiable spectral conditions.

1. MAIN RESULTS

The local behavior near some distinguished solution, such as a steady state, of an evolution equation, can be determined through a decomposition into invariant manifolds, that is, stable, unstable and center manifolds. These (locally invariant) manifolds are characterized by decay estimates. While the flows on the stable and unstable manifolds are determined by exponential decay in forward and backward time respectively, that on the center manifold is ambiguous. Nevertheless, a determination of the flow on the center manifold can lead to a complete characterization of the local flow and thus this decomposition, when possible, leads to a reduction of this problem to one of identifying the flow on the center manifold.

This strategy has a long history for studying the local behavior near a critical point of an ordinary differential equation, or a fixed point of a map, and it has gained momentum in the last few decades in the context of nonlinear wave solutions of evolutionary partial differential equations. Extending the ideas to partial differential equations has, however, introduced a number of new issues. In infinite dimensions, the relation between the linearization and the full nonlinear equations is more delicate. This issue, however, turns out to be not so difficult for the invariant manifold decomposition and has largely been resolved, see, for instance, [2], [3]. A more subtle issue arises at the linear level. All of the known proofs for the existence of invariant manifolds are based upon the use of the group (or semigroup) generated by the linearization. The hypotheses of the relevant theorems are then formulated in terms of estimates on the appropriate projections of these groups onto stable, unstable and center subspaces. These amount to spectral estimates that come directly from a determination of the spectrum of the group. However, in any actual problem, the information available will, at best, be of the spectrum of the infinitesimal generator, that is, the linearized equation and not its solution operator. Relating the spectrum of the infinitesimal generator to that of the group is a spectral mapping problem that is often non-trivial.

In this paper, we resolve this issue for nonlinear Schrødinger equations. We formulate the results for the case of space-dependent nonlinearities in arbitrary dimensions. This class of equations is motivated by the one space dimension case that appears in the study of optical waveguides, see [18], and has attracted the attention of many authors. In particular, there is extensive literature on the existence and instability of standing waves, see, for instance, [10, 11, 12, 15, 16] and the references therein. In many instances the questions of the existence of standing waves and the structure of the spectrum of the linearization of the nonlinear equation around the standing wave are well-understood, see [18].

In the case considered in this paper, the interesting examples are known to have the spectrum of their linearization $\mathcal{A}$ enjoying a disjoint decomposition: the essential spectrum is positioned on the imaginary axis, and there are several isolated eigenvalues off the imaginary axis, see [10, 15, 16]. However, as mentioned above, this spectral information about the linearization

$\mathcal{A}$ is not sufficient to guarantee the existence of invariant manifolds. The general theory gives the existence of these manifolds for a semilinear equation with linear part $\mathcal{A}$ only when the spectrum of the operator $e^{t\mathcal{A}}$, $t > 0$, rather than that of $\mathcal{A}$, admits a decomposition into disjoint components. It is not *a priori* clear that the spectrum of the operator $e^{t\mathcal{A}}$ is obtained from the spectrum of $\mathcal{A}$ by exponentiation. Indeed, in the present case, the operator $\mathcal{A}$ does not generate a semigroup for which this property is known (such as for analytic semigroups). Thus, we prove such a spectral mapping theorem (cf. Theorem 1) in this paper. This spectral mapping theorem is derived from a known abstract result in the theory of strongly continuous semigroups of linear operators (see Theorem 3). To apply this abstract result one needs to prove that the norm of the resolvent of $\mathcal{A}$ is bounded along vertical lines in the complex plane. The corresponding proof is based on Lemmas 5 and 6. The main technical step in the proof of these lemmas concerns a result about the high-energy decay of the norm of a Birman-Schwinger-type operator (cf. Proposition 8), a well-known device borrowed from quantum mechanics.

We consider the following Schrødinger equation with space-dependent nonlinearity,

$$(1) \qquad \begin{aligned} iu_t &= \Delta u + f(x,|u|^2)u + \beta u, \\ & \qquad u = u(x,t) \in \mathbb{C},\ x \in \mathbb{R}^n,\ t \geq 0,\ \beta \in \mathbb{R}, \end{aligned}$$

where $\Delta = \sum_{j=1}^{n} \partial^2/\partial x_j$ denotes the Laplacian, $n \in \mathbb{N}$, and f is real-valued. Rewriting (1) in terms of its real and imaginary parts, $u = v + iw$, one obtains

$$(2) \qquad \begin{aligned} v_t &= \Delta w + f(x, v^2 + w^2)w + \beta w, \\ w_t &= -\Delta v - f(x, v^2 + w^2)v - \beta v. \end{aligned}$$

A standing wave of frequency β for (1) is a time-independent real-valued solution $\boldsymbol{u} = \boldsymbol{u}(x)$ of (2). Suppose the standing wave $\boldsymbol{u}$ is given *a priori*. Consider the linearization of (2) around $\boldsymbol{u}$ (recalling $\boldsymbol{w} = 0$),

$$\begin{aligned} p_t &= \Delta q + f(x, \boldsymbol{u}^2)q + \beta q, \\ q_t &= -\Delta p - f(x, \boldsymbol{u}^2)p - 2\partial_2 f(x, \boldsymbol{u}^2)\boldsymbol{u}^2 p - \beta p, \end{aligned}$$

where $\partial_2 f(x,y) = f_y(x,y)$. Thus, the linearized stability of the standing wave is determined by the operator

$$\mathcal{A} = \begin{bmatrix} 0 & -L_R \\ L_I & 0 \end{bmatrix}, \tag{3}$$

where $L_R = -\Delta - \beta + Q_1$, $L_I = -\Delta - \beta + Q_2$, and the potentials Q_1 and Q_2 are explicitly given by the formulas

$$\begin{aligned} Q_1(x) &= -f(x, \mathfrak{u}^2(x)), \\ Q_2(x) &= -f(x, \mathfrak{u}^2(x)) - 2\partial_2 f(x, \mathfrak{u}^2(x))\mathfrak{u}^2(x). \end{aligned}$$

We impose the following conditions on f, β, and the standing wave $\mathfrak{u}$ (see [10, 15, 16]):

(H1) $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is C^3 and all derivatives are bounded on a set of the form $\mathbb{R}^n \times U$, where U is a neighborhood of $0 \in \mathbb{R}$;
(H2) $f(x,0) \to 0$ exponentially as $|x| \to \infty$;
(H3) $\beta < 0$;
(H4) $|\mathfrak{u}(x)| \to 0$ exponentially as $|x| \to \infty$.

As a result, the potentials Q_1 and Q_2 exponentially decay at infinity. The operator $\mathcal{A}$ is considered on $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$; the domain $\mathcal{D}(-\Delta)$ is chosen to be the standard Sobolev space $H^2(\mathbb{R}^n)$, and the domain of $\mathcal{A}$ is then $H^2(\mathbb{R}^n) \oplus H^2(\mathbb{R}^n)$. Note that

$$\begin{bmatrix} 0 & \Delta \\ -\Delta & 0 \end{bmatrix}$$

generates a strongly continuous group on $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$. Thus, its bounded perturbation $\mathcal{A}$ generates a strongly continuous group $\{e^{t\mathcal{A}}\}_{t\in\mathbb{R}}$ as well.

It was proved in [10, Thm. 3.1] that $\sigma_{\text{ess}}(\mathcal{A}) = \{i\xi \mid \xi \in \mathbb{R},\ |\xi| \geq -\beta\}$. In addition, it was proved in [10, 15, 16] that, under the above hypotheses, $\sigma(\mathcal{A}) \setminus \sigma_{\text{ess}}(\mathcal{A})$ consists of finitely many eigenvalues, symmetric with respect to both coordinate axes.

We prove the following result that relates the spectrum of the semigroup $\{e^{t\mathcal{A}}\}_{t\geq 0}$ and the spectrum of its generator.

***Theorem* 1** (Spectral Mapping Theorem)**.** *For each $n \in \mathbb{N}$, one has*

$$\sigma(e^{t\mathcal{A}}) = e^{t\sigma(\mathcal{A})} \quad \text{for all } t > 0.$$

Recall that the spectral inclusion $e^{t\sigma(\mathcal{A})} \subset \sigma(e^{t\mathcal{A}})$ always holds. Also, since $\{e^{t\mathcal{A}}\}_{t\in\mathbb{R}}$ is a group, $0 \notin \sigma(e^{t\mathcal{A}})$. See [8, 9, 22, 26, 28] for a discussion of the spectral mapping theorems for strongly continuous semigroups and examples where the spectral mapping property as in Theorem 1 fails. A spectral mapping theorem was proved in [19] for a nonlinear Schrødinger equation with a specific potential and in the case $n = 1$. Their proof also uses Theorem 3 that is the key to our result. To the best of our knowledge the work of Kapitula and Sandstede [19] was the first to use the Gearhart-Greiner-Herbst-Prüss Theorem 1 in this context. Theorem 3 was also used by Miller and Weinstein in [21] to prove asymptotic stability of solitary waves for the regularized long-wave equation (see also related work [23] by Pego and Weinstein and the bibliography in these papers).

Since the spectral mapping theorem always holds for the point spectrum, Theorem 1 implies, in particular, that $\sigma_{\text{ess}}(e^{t\mathcal{A}}) \subseteq \mathbb{T} = \{|z| = 1\}$. It follows that there will be only finitely many eigenvalues of $e^{t\mathcal{A}}$ off the unit circle and therefore general results on the existence of invariant manifolds for semilinear equations can be invoked (see, e.g., [3] and compare also with [4] and the literature cited in [20, p. 4]).

The (local) stable manifold is defined as the set of initial data whose solutions stay in the prescribed neighborhood and tend to $\mathfrak{u}$ exponentially as $t \to +\infty$. The unstable manifold is defined analogously but in backward time. The center manifold is complementary to these two and contains solutions with neutral decay behavior (although they can decay, they will not do so exponentially). In particular, the center manifold contains all solutions that stay in the neighborhood in both forward and backward time. For details see, for instance, [3].

Concerning the equations under consideration here, we have the following main theorem.

Theorem 2. *Assuming* (H1)-(H4), *in a neighborhood of the standing wave solution* $\mathfrak{u}$ *of* (1) *there are locally invariant stable, unstable, and center manifolds. Moreover, the stable and unstable manifolds are of* (*equal*) *finite dimension, and the center manifold is infinite-dimensional.*

All of the examples of standing waves considered in [17] and [18] satisfy the hypotheses given here and thus enjoy a local decomposition of the flow by invariant manifolds. Some of these waveguide modes are stable, while others are unstable. In the unstable cases, the above results show that the instabilities are controlled by finite-dimensional (mostly, just one-dimensional) unstable manifolds. A natural question is whether the waveguide modes are stable relative to the flow on the center manifold. Such a result was shown for the case of nonlinear Klein-Gordon equations in [4] using an energy argument. Whether such an argument will work for nonlinear Schrødinger equations is open. It is more than of academic interest, as stability on the center manifold has the consequence that the center manifold is unique, see [3], and armed with such a result, a complete description of the local flow can be legitimately claimed. Cases of standing waves in higher dimensions are given in [16]. Some of these are unstable and the above considerations again apply.

We also wish to stress that the spectral mapping theorem developed here is not restricted to a single equation. Indeed, the results formulated here are easily adaptable to systems of nonlinear Schrødinger equations. This is particularly important as such systems arise in, among other problems, second harmonic generation in waveguides and wave-division multiplexing in optical fibers. The case of systems is considered in Section 4.

In the next section, we give the basic set-up that will be used and formulate the necessary lemmas for proving Theorem 1. The proofs are given in Section 3.

2. BASIC LEMMAS

To prove Theorem 1, we will use the following abstract result known as the Gearhart-Greiner-Herbst-Prüss theorem, see, e.g., [22, p. 95].

Theorem 3. *Let $\mathcal{A}$ be a generator of a strongly continuous semigroup on a complex Hilbert space. Then for each $t > 0$, the following spectral mapping theorem is valid:*

$$\sigma(e^{t\mathcal{A}}) \setminus \{0\} = \Big\{ e^{\lambda t} \mid \text{either } \mu_k := \lambda + \frac{2\pi i k}{t} \in \sigma(\mathcal{A}) \text{ for some } k \in \mathbb{Z}, \text{ or the sequence } \{\|(\mu_k - \mathcal{A})^{-1}\|\}_{k\in\mathbb{Z}} \text{ is unbounded} \Big\}.$$

According to the results in [10, Thm.3.1] and [15, 16], the essential spectrum of the generator $\mathcal{A}$ in (3) is given by $\sigma_{\text{ess}}(\mathcal{A}) = \{i\xi \mid \xi \in \mathbb{R},\ |\xi| \geq -\beta\}$. Moreover, $\sigma(\mathcal{A}) \setminus \sigma_{\text{ess}}(\mathcal{A})$ consists of finitely many eigenvalues. In particular, $\{z \in \mathbb{C} : |z| = 1\} \subset e^{t\sigma_{\text{ess}}(\mathcal{A})}$. Also, for sufficiently large $|\tau|$ and each $a \in \mathbb{R} \setminus \{0\}$ we have $a + i\tau \notin \sigma(\mathcal{A})$. We claim that Theorem 1 is implied by the following assertion: *For each $a \in \mathbb{R} \setminus \{0\}$, the function $\tau \mapsto \|(a + i\tau - \mathcal{A})^{-1}\|$ is bounded as $|\tau| \to \infty$.* Indeed, let us suppose that for some $\lambda \in \mathbb{C}$ there exists $e^{t\lambda} \in \sigma(e^{t\mathcal{A}}) \setminus e^{t\sigma(\mathcal{A})}$. Let $a = \operatorname{Re}\lambda$. Since $\{z \in \mathbb{C} : |z| = 1\} \subset e^{t\sigma(\mathcal{A})}$, we conclude that $a \neq 0$. If $\mu_k := \lambda + 2\pi i k/t$, then

$e^{t\mu_k} = e^{t\lambda} \notin e^{t\sigma(\mathcal{A})}$ and hence $\mu_k \notin \sigma(\mathcal{A})$ for all $k \in \mathbb{Z}$. Since $e^{t\lambda} \in \sigma(e^{t\mathcal{A}})$, Theorem 3 implies that the sequence $\{\|(\mu_k - \mathcal{A})^{-1}\|\}_{k\in\mathbb{Z}}$ is unbounded as $k \to \infty$ or $k \to -\infty$. But $\mu_k = a + i\tau$ for $\tau = \operatorname{Im}(\lambda) + 2\pi k/t$, and we arrive at a contradiction with the above assertion.

Therefore, in what follows we will fix $a \in \mathbb{R} \setminus \{0\}$, let $\xi = a + i\tau$ for sufficiently large $|\tau|$ such that $\xi \notin \sigma(\mathcal{A})$, and will show that the function $\tau \mapsto \|(\xi - \mathcal{A})^{-1}\|$ is bounded as $|\tau| \to \infty$.

We denote $D = -\Delta - \beta$ and recall that $\beta < 0$ by (H3). Moreover, we have

$$\sigma(D) = \sigma(-\Delta) - \beta = [-\beta, \infty).$$

We note that D^2 with domain $H^4(\mathbb{R}^n)$ is a self-adjoint operator. Thus, for $\xi = a + i\tau$ with $\tau \neq 0$ one has $-\xi^2 \notin \sigma(D^2)$. Moreover, we write

$$\begin{aligned} \xi - \mathcal{A} &= \begin{pmatrix} \xi & L_R \\ -L_I & \xi \end{pmatrix} = \begin{pmatrix} \xi & D \\ -D & \xi \end{pmatrix} + \begin{pmatrix} 0 & Q_1 \\ -Q_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \xi & D \\ -D & \xi \end{pmatrix} \left[I + \begin{pmatrix} \xi & D \\ -D & \xi \end{pmatrix}^{-1} \begin{pmatrix} 0 & Q_1 \\ -Q_2 & 0 \end{pmatrix} \right], \end{aligned} \tag{4}$$

where, by a direct computation with operator-valued matrices,

$$\begin{pmatrix} \xi & D \\ -D & \xi \end{pmatrix}^{-1} = \begin{pmatrix} \xi[\xi^2 + D^2]^{-1} & -[\xi^2 + D^2]^{-1}D \\ [\xi^2 + D^2]^{-1}D & \xi[\xi^2 + D^2]^{-1} \end{pmatrix}. \tag{4a}$$

***Lemma* 4.** *For $\xi = a + i\tau$, $a \in \mathbb{R} \setminus \{0\}$, $\tau \in \mathbb{R}$, the norm of the operator* (4a) *remains bounded as $|\tau| \to \infty$.*

The elementary proof of this lemma is given in the next section.

Next, we denote

$$\begin{aligned} T(\xi) &= \begin{pmatrix} \xi & D \\ -D & \xi \end{pmatrix}^{-1} \begin{pmatrix} 0 & Q_1 \\ -Q_2 & 0 \end{pmatrix} \\ &= \begin{bmatrix} [\xi^2 + D^2]^{-1}DQ_2 & \xi[\xi^2 + D^2]^{-1}Q_1 \\ -\xi[\xi^2 + D^2]^{-1}Q_2 & [\xi^2 + D^2]^{-1}DQ_1 \end{bmatrix}. \end{aligned} \tag{5}$$

The main step in the proof of Theorem 1 is contained in the next two lemmas. They imply that the norm of the operator $(I + T(\xi))^{-1}$ is bounded as $|\tau| \to \infty$ (in the case $n = 1$ we give two proofs of this fact). Assume P and Q are real-valued continuous potentials exponentially decaying at infinity and let $\xi = a + i\tau$.

***Lemma* 5.** *If* $n = 1$, *then*

(a) $\|Q[\xi^2 + D^2]^{-1}D\| \to 0$, *and*

(b) $\|\xi[\xi^2 + D^2]^{-1}Q\| \to 0$

as $|\tau| \to \infty$.

***Lemma* 6.** *If* $n \geq 1$, *then*

$$\|P[\xi^2 + D^2]^{-1}DQ\| \to 0, \quad \text{and} \quad \|P\xi[\xi^2 + D^2]^{-1}Q\| \to 0 \tag{6}$$

as $|\tau| \to \infty$.

The proof of Lemmas 5 and 6 are given in the next section. We proceed finishing the proof of Theorem 1.

Proof of Theorem 1. In the case $n = 1$, by passing to the adjoint operator of $\xi[\xi^2 + D^2]^{-1}Q_1$ and $[\xi^2 + D^2]^{-1}DQ_2$, Lemma 5 implies that the norm of each of the four block-operators in the right-hand side of (5) is strictly less than 1 for $|\tau|$ sufficiently large. Thus, $\|T(\xi)\| < 1$. By (4), one infers

$$\begin{aligned} \|(a + i\tau - \mathcal{A})^{-1}\| &= \left\| (I + T(\xi))^{-1} \begin{pmatrix} \xi & D \\ -D & \xi \end{pmatrix}^{-1} \right\| \\ &\leq \frac{1}{1 - \|T(\xi)\|} \left\| \begin{pmatrix} \xi & D \\ -D & \xi \end{pmatrix}^{-1} \right\|. \end{aligned}$$

Using Lemma 4, we have that $\|(a + i\tau - \mathcal{A})^{-1}\|$ remains bounded as $|\tau| \to \infty$, and hence Theorem 3 implies the result.

In the case $n \geq 1$ we will use Lemma 6. For $j = 1, 2$ denote

$$|Q_j|^{1/2}(x) = |Q_j(x)|^{1/2} \quad \text{and} \quad Q_j^{1/2}(x) = |Q_j|^{1/2}\operatorname{sgn}(Q(x)), \tag{7}$$

so that $Q_j = Q_j^{1/2}|Q_j|^{1/2}$ for the potentials Q_j, $j = 1, 2$, in (4). Also, for $T(\xi)$ defined in (5) we write $T(\xi) = A(\xi)B$, where

$$\begin{aligned} A(\xi) &= \begin{pmatrix} \xi & D \\ -D & \xi \end{pmatrix}^{-1} \begin{pmatrix} 0 & Q_1^{1/2} \\ -Q_2^{1/2} & 0 \end{pmatrix}, \\ B &= \begin{pmatrix} |Q_2|^{1/2} & 0 \\ 0 & |Q_1|^{1/2} \end{pmatrix}. \end{aligned} \tag{8}$$

Recall the following elementary fact: If A and B are bounded operators, then $I + AB$ is invertible provided $I + BA$ is invertible; moreover,

$$(I + AB)^{-1} = I - A(I + BA)^{-1}B.$$

By a direct calculation using (4a), one obtains

$$BA(\xi) = \begin{bmatrix} |Q_2|^{1/2}[\xi^2 + D^2]^{-1}DQ_2^{1/2} & |Q_2|^{1/2}\xi[\xi^2 + D^2]^{-1}Q_1^{1/2} \\ -|Q_1|^{1/2}\xi[\xi^2 + D^2]^{-1}Q_2^{1/2} & |Q_1|^{1/2}[\xi^2 + D^2]DQ_1^{1/2} \end{bmatrix}.$$

By Lemma 6, applied to each of the four blocks in the right-hand side of this identity, we obtain $\|BA(\xi)\| \to 0$ as $|\tau| \to \infty$. Thus, for some $\tau_0 > 0$, one infers $\|BA(\xi)\| < 1$ and $\sup_{|\tau|\geq\tau_0} \|(I + BA(\xi))^{-1}\| < \infty$. Since $\sup_{\tau\geq\tau_0} \|A(\xi)\| < \infty$ by Lemma 4, we conclude that

$$\sup_{|\tau|\geq\tau_0} \|(I + T(\xi))^{-1}\| = \sup_{|\tau|\geq\tau_0} \|(I + A(\xi)B)^{-1}\| < \infty. \qquad \square$$

3. PROOFS OF LEMMAS 4-6

In this section we give the proofs of Lemmas 4-6. The proof of Lemma 5 is based on a direct estimate of the trace norms. We will give two proofs of Lemma 6. The first proof is applicable to all $n \geq 1$ and uses estimates for the norm of the resolvent on weighted spaces of L^2-functions. The second proof works for the cases $n = 1, 2, 3$, and is based on explicit estimates for the integral kernel of the resolvent of the Laplacian.

The main tool in the proof of Lemma 5 is the following well-known result. Denote by $\mathcal{J}_q(L^2(\mathbb{R}^n))$ the set of bounded linear operators $A \in \mathcal{L}(L^2(\mathbb{R}^n))$, $n \geq 1$, such that $\|A\|_{\mathcal{J}_q(L^2(\mathbb{R}^n))} = (\mathrm{tr}\,(|A|^q))^{1/q} < \infty$, $q \geq 1$, where $\mathrm{tr}\,(\cdot)$ denotes the trace of operators in $L^2(\mathbb{R}^n)$. We recall that $\|A\| \leq \|A\|_{\mathcal{J}_q(L^2(\mathbb{R}^n))}$ for all $q \geq 1$.

Theorem 7. (*see, e.g.,* [24, Theorem XI.20].) *Suppose* $2 \leq q < \infty$ *and let* $f, g \in L^q(\mathbb{R}^n)$. *Then* $f(\cdot)g(-i\nabla) \in \mathcal{J}_q(L^2(\mathbb{R}^n))$, *and*

$$\|f(\cdot)g(-i\nabla)\|_{\mathcal{J}_q(L^2(\mathbb{R}^n))} \leq (2\pi)^{-n/2}\|f\|_{L^q(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}.$$

We will apply Theorem 7 to the exponentially decaying $Q = f \in L^q(\mathbb{R}^n)$, $q > 1$, and an appropriate choice of g. Throughout, c is a generic constant.

Proof of Lemma 5. We proceed with the proof of (a) in Lemma 5 for any $n \geq 1$, to demonstrate where the argument in the proof breaks down for $n > 1$ (that is why Lemma 6 is needed). We recall that $\beta < 0$ by (H3).

First, let $g(x) = (|x|^2 - \beta)(\xi^2 + (|x|^2 - \beta)^2)^{-1}$. Then, for $n \geq 1$, one has $g(-i\nabla) = [\xi^2 + (-\Delta - \beta)^2]^{-1}(-\Delta - \beta) = [\xi^2 + D^2]^{-1}D$. For $r = |x|$, one infers

$$\begin{aligned} \|g\|_{L^q(\mathbb{R}^n)}^q &= c\int_0^\infty r^{n-1}(r^2-\beta)^q[(a^2-\tau^2+(r^2-\beta)^2)^2+4a^2\tau^2]^{-q/2}\,dr \\ &= I_1 + I_2, \end{aligned}$$

denoting

$$\begin{aligned} I_1 &= c(2|a\tau|)^{-q}\int_{|\beta|^{1/2}}^\infty r^{n-1}(r^2-\beta)^q\left[\left(\frac{(r^2-\beta)^2-(\tau^2-a^2)}{2|a\tau|}\right)^2+1\right]^{-q/2}dr, \\ I_2 &= c(2|a\tau|)^{-q}\int_0^{|\beta|^{1/2}} r^{n-1}(r^2-\beta)^q\left[\left(\frac{(r^2-\beta)^2-(\tau^2-a^2)}{2|a\tau|}\right)^2+1\right]^{-q/2}dr. \end{aligned}$$

Note that

$$I_2 \leq c(2|a\tau|)^{-q}\int_0^{|\beta|^{1/2}} r^{n-1}(r^2-\beta)^q\,dr \leq c|\tau|^{-q} \to 0 \quad \text{as } |\tau| \to \infty,$$

for any $q > 0$. Thus, it suffices to focus on the integral I_1. The change of variable $y = ((r^2-\beta)^2 - (\tau^2-a^2))/(2|a\tau|)$ shows that

$$\begin{aligned} I_1 &= k(\tau)\int_{(2\beta^2/|a\tau|)-\tau'}^\infty (y^2+1)^{-q/2}\left(\frac{1}{\tau'}y+1\right)^{(q-1)/2} \\ &\qquad \times\left[\left(\frac{1}{\tau'}y+1\right)^{1/2}+\frac{\beta}{(\tau^2-a^2)^{1/2}}\right]^{(n-2)/2}dy, \end{aligned}$$

where

$$k(\tau) = \frac{1}{4}(2|a\tau|)^{1-q}(\tau^2-a^2)^{(q-1)/2+(n-2)/4} = O(|\tau|^{(n-2)/2}) \quad \text{as } |\tau| \to \infty,$$

and

$$\tau' = \frac{\tau^2-a^2}{2|a\tau|} = O(|\tau|) \quad \text{as } |\tau| \to \infty.$$

We claim that, for all $n \geq 1$,

$$I_1 \leq c|\tau|^{(n-2)/2} I(-\tau', \infty), \tag{9}$$

denoting

$$I(a,b) = \int_a^b (y^2+1)^{-q/2} \left(\frac{1}{\tau'}y + 1\right)^{(q-1)/2+(n-2)/4} dy.$$

If $n \geq 2$, then (9) holds by our assumption $\beta < 0$. To prove the claim (9) for $n = 1$, we observe that

$$\frac{1}{2}\left(\frac{1}{\tau'}y+1\right)^{1/2} - \frac{|\beta|}{(\tau^2 - a^2)^{1/2}} \geq 0 \quad \text{for } y \geq \frac{2\beta^2}{|a\tau|} - \tau'.$$

Thus,

$$\left(\frac{1}{\tau'}y+1\right)^{1/2} + \frac{\beta}{(\tau^2-a^2)^{1/2}} \geq \frac{1}{2}\left(\frac{1}{\tau'}y+1\right)^{1/2},$$

and (9) follows. Next, one observes that $|\tau|^{(n-2)/2} I(\tau', \infty) \to 0$ as $|\tau| \to \infty$, for $q > 2n$. Indeed, for $y \geq \tau'$, one has

$$\begin{aligned} |\tau|^{(n-2)/2} I(\tau', \infty) &\leq |\tau|^{(n-2)/2} \int_{\tau'}^{\infty} (y^2)^{-q/2} \left(2\frac{y}{\tau'}\right)^{(2q+n-4)/4} dy \\ &= c|\tau|^{(n-2)/2} \tau'^{(1-q)} \to 0, \quad \text{as } |\tau| \to \infty. \end{aligned}$$

To prove part (a) of Lemma 5 we remark that, for $n = 1$, one has

$$|\tau|^{(n-2)/2} I(-\tau', \tau') \to 0 \quad \text{as } |\tau| \to \infty.$$

We stress that this assertion does not hold for $n \geq 2$; that is why one needs Lemma 6. Indeed, assuming $n = 1$, one concludes that for $y \leq \tau'$,

$$|\tau|^{-1/2} I(-\tau', \tau) \leq 2^{(2q-3)/4} |\tau|^{-1/2} \int_{-\infty}^{\infty} (y^2+1)^{-q/2} \, dy \to 0 \quad \text{as } |\tau| \to \infty.$$

Therefore, for $n = 1$, one infers that $\|g\|_{L^q(\mathbb{R})} \to 0$ as $|\tau| \to \infty$ and, using Theorem 7, that

$$\begin{aligned} \|Q[\xi^2 + D^2]^{-1} D\| &\leq \|Q(\cdot) g(-i\nabla)\|_{\mathcal{J}_q(L^2(\mathbb{R}))} \\ &\leq c\|Q\|_{L^q(\mathbb{R})} \|g\|_{L^q(\mathbb{R})} \to 0 \quad \text{as } |\tau| \to \infty. \end{aligned}$$

Thus, part (a) of Lemma 5 is proved.

To prove part (b) of Lemma 5, let $g(x) = \xi(\xi^2 + (|x|^2 - \beta)^2)^{-1}$. Then, for $n = 1$, one obtains $g(-i\nabla) = \xi[\xi^2 + (-\Delta - \beta)^2]^{-1}$. If $\xi = a + i\tau$ and $r = |x|$, then

$$\begin{aligned} \|g\|^q_{L^q(\mathbb{R})} &= 2(a^2+\tau^2)^{q/2}\int_0^\infty ([a^2-\tau^2+(|x|^2-\beta)^2]^2+4a^2\tau^2)^{-q/2}\,dx \\ &= 2(a^2+\tau^2)^{q/2}(2|a\tau|)^{-q}\int_0^\infty \left(\left[\frac{(r^2+|\beta|)^2}{2|a\tau|}-\tau'\right]^2+1\right)^{-q/2} dr. \end{aligned}$$

The change of variables $y = (r^2 + |\beta|)^2/(2|a\tau|) - \tau'$ shows that

$$\|g\|^q_{L^q(\mathbb{R})} \le c|\tau|^{-1/2}(J_1 + J_2),$$

denoting

$$J_1 = \int_{|\beta|^2/2|a\tau|-\tau'}^{-1/2\tau'} \left(\left(\frac{1}{\tau'}y+1\right)^{1/2} - \frac{|\beta|}{(\tau^2-a^2)^{1/2}}\right)^{-1/2}\left(\frac{1}{\tau'}y+1\right)^{-1/2} \times (y^2+1)^{-q/2}\,dy,$$

$$J_2 = \int_{-1/2\tau'}^{\infty} \left(\left(\frac{1}{\tau'}y+1\right)^{1/2} - \frac{|\beta|}{(\tau^2-a^2)^{1/2}}\right)^{-1/2}\left(\frac{1}{\tau'}y+1\right)^{-1/2} \times (y^2+1)^{-q/2}\,dy.$$

For the integral J_1, we note that $y^2 + 1 \ge (\frac{1}{4}\tau')^2$, and

$$\frac{|\beta|^2}{(\tau^2-a^2)} \le \frac{1}{\tau'}y+1 \le \frac{1}{2} \quad \text{for } \frac{|\beta|^2}{2|a\tau|}-\tau' \le y \le -\frac{1}{2}\tau'.$$

Therefore, for large $|\tau|$ we have

$$\begin{aligned} |\tau|^{-1/2}J_1 &= |\tau|^{-1/2}\int_{|\beta|^2/2|a\tau|-\tau'}^{-1/2\tau'} \left(\left(\frac{1}{\tau'}y+1\right)^{1/2} + \frac{|\beta|}{(\tau^2-a^2)^{1/2}}\right)^{1/2} \\ &\quad \times \left(\frac{1}{\tau'}y+1\right)^{-1/2}(y^2+1)^{-q/2}\left(\left(\frac{1}{\tau'}y+1\right) - \frac{|\beta|^2}{\tau^2-a^2}\right)^{-1/2} dy \\ &\le c|\tau|^{-q}\int_{|\beta|^2/2|a\tau|-\tau'}^{-1/2\tau'}\left(\frac{1}{\tau'}y+1-\frac{|\beta|^2}{\tau^2-a^2}\right)^{-1/2} dy. \end{aligned}$$

One more change of variables $z = \tau'/y + 1 - |\beta|^2/(\tau^2 - a^2)$ then yields

$$|\tau|^{-1/2} J_1 \le c|\tau|^{-q+1} \int_0^{1/2} z^{-1/2}\, dz = O(|\tau|^{-q+1}) \to 0 \quad \text{as } |\tau| \to \infty.$$

For the integral J_2, we note that $y \ge -\frac{1}{2}\tau'$ implies

$$\left(\frac{1}{\tau'}y + 1\right)^{1/2} - \frac{|\beta|}{(\tau^2 - a^2)^{1/2}} \ge c, \quad \text{and} \quad \frac{1}{\tau'}y + 1 \ge \frac{1}{2}.$$

Hence,

$$|\tau|^{-1/2} J_2 \le c|\tau|^{-1/2} \int_{-\infty}^{\infty} (y^2 + 1)^{-q/2}\, dy \to 0 \quad \text{as } |\tau| \to \infty.$$

Therefore, for $n = 1$, one infers $\|g\|_{L^q(\mathbb{R})} \to 0$ as $|\tau| \to \infty$, and using Theorem 7,

$$\|Q\xi[\xi^2 + (-\Delta - \beta)^2]^{-1}\| \le \|Q(\cdot)g(-i\nabla)\|_{\mathcal{J}_q(L^2(\mathbb{R}))} \le c\|Q\|_{L^q(\mathbb{R})}\|g\|_{L^q(\mathbb{R})} \to 0,$$

as $|\tau| \to \infty$. Thus, part (b) of Lemma 5 is proved. ❐

The proof of Lemma 6 is based on the following proposition.

***Proposition* 8.** *Assume P and Q are continuous on $\mathbb{R}^n$ and exponentially decaying at infinity. Then for $n \ge 1$ and* $\operatorname{Im}(\omega) > 0$*, one infers*

$$\|P(-\Delta - \omega^2)^{-1}Q\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \to 0 \quad \text{as } |\operatorname{Re}(\omega^2)| \to \infty. \tag{10}$$

We postpone the proof of the proposition and proceed with the proof of Lemma 6.

Proof of Lemma 6. Recall that $D = -\Delta - \beta$ and $\xi = a + i\tau$. Choose ω_1 and ω_2 such that $\operatorname{Im}(\omega_1) > 0$, $\operatorname{Im}(\omega_2) > 0$, and $\omega_1^2 = \beta - i\xi$, $\omega_2^2 = \beta + i\xi$. Then $|\operatorname{Re}(\omega_j^2)| \to \infty$ as $|\tau| \to \infty$ for $j = 1, 2$.

By the functional calculus we have

$$\begin{aligned} [\xi^2 + D^2]^{-1}D &= \frac{1}{2}[D + i\xi]^{-1} + \frac{1}{2}[D - i\xi]^{-1}, \\ \xi[\xi^2 + D^2]^{-1} &= \frac{1}{2i}[D - i\xi]^{-1} - \frac{1}{2i}[D + i\xi]^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} P[\xi^2 + D^2]^{-1}DQ &= \frac{1}{2}P[-\Delta - \omega_1^2]^{-1}Q + \frac{1}{2}P[-\Delta - \omega_2^2]^{-1}Q, \\ P\xi[\xi^2 + D^2]^{-1}Q &= \frac{1}{2i}P[-\Delta - \omega_2^2]^{-1}Q - \frac{1}{2i}P[-\Delta - \omega_1^2]^{-1}Q, \end{aligned}$$

and (6) in Lemma 6 follows from Proposition 8. ❐

We will give two proofs of Proposition 8. The first proof is based on an estimate for the norm of the resolvent for the Laplacian acting between weighted L^2-spaces. This estimate is given in Lemma 9 below. In fact, Lemma 9 is just a minor refinement of Lemma XIII.8.5 in [25]. Results of this type go back to Agmon [1], Ikebe and Saito [13], and others (cf. the discussion in [25, p. 345-347]). Additional (and more sophisticated) results of this type can be found in the work by Jensen [14] and the bibliography therein. Our second proof for $n = 1, 2, 3$ only uses explicit properties of the integral kernel of $(-\Delta - \omega^2)^{-1}$.

For $n \geq 1$ and $s > 1/2$, let $\rho_s(x) = (1 + |x|^2)^{s/2}$, for $x \in \mathbb{R}^n$, and consider the weighted L^2-spaces

$$\begin{aligned} L^2_s(\mathbb{R}^n) &= \{f : \|f\|_s := \|\rho_s f\|_{L^2(\mathbb{R}^n)} < \infty\}, \quad \text{and} \\ L^2_{-s}(\mathbb{R}^n) &= \{f : \|f\|_{-s} := \|\rho_s^{-1} f\|_{L^2(\mathbb{R}^n)} < \infty\}. \end{aligned}$$

We note the continuous embeddings $L^2_s(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n) \hookrightarrow L^2_{-s}(\mathbb{R}^n)$. Moreover, we denote by $S(\mathbb{R}^n)$ the Schwarz class of rapidly decaying functions on $\mathbb{R}^n$.

***Lemma* 9.** *There exists a constant $d = d(n,s)$, depending only on n and s, such that, for all $\lambda \in \mathbb{C}$ with $\operatorname{Im}(\lambda) \neq 0$ and $|\operatorname{Re}(\lambda)| \geq 1$, and all $\varphi \in S(\mathbb{R}^n)$, the following estimate holds*

$$\|\varphi\|_{-s} \leq d|\operatorname{Re}(\lambda)|^{-1/2}\|(-\Delta - \lambda)\varphi\|_s. \tag{11}$$

Proof. For completeness, we briefly sketch a modification of the proof of Lemma XIII.8.5 in [25] to prove inequality (11). An elementary calculation shows that, for each $\lambda \in \mathbb{C}$ with $\operatorname{Im}(\lambda) \neq 0$, the expression

$$\gamma^{-2} := \inf_{y\in\mathbb{R}}[|y^2 - \lambda|^2 + |y|^2] = \inf_{y\geq 0}[y^2 + (1 - 2\operatorname{Re}(\lambda))y + (\operatorname{Re}(\lambda))^2 + (\operatorname{Im}(\lambda))^2]$$

is equal to $\operatorname{Re}(\lambda) + (\operatorname{Im}(\lambda))^2 - \frac{1}{4}$ for $\operatorname{Re}(\lambda) > \frac{1}{2}$, and to $|\lambda|^2$ for $\operatorname{Re}(\lambda) \leq \frac{1}{2}$. Therefore, $\gamma \leq 2|\operatorname{Re}(\lambda)|^{-1/2}$ for $|\operatorname{Re}(\lambda)| \geq 1$.

For a positive α to be selected below, let $\rho_{1,\alpha}(x) = (1+\alpha|x|^2)^{1/2}$. Arguing as in the proof of Lemma XIII.8.5 in [25] (see the corresponding equations (61)-(62) in [25]), one infers

$$\begin{aligned}\|\rho_{1,\alpha}^{-s}\varphi\|_{L^2(\mathbb{R}^n)} &\leq \gamma\|\rho_{1,\alpha}^{-s}(-\Delta-\lambda)\varphi\|_{L^2(\mathbb{R}^n)} + \gamma(2s\alpha^{1/2}+1)\\ &\qquad \times \sum_{j=1}^{n}\|\rho_{1,\alpha}^{-s}\partial_j\varphi\|_{L^2(\mathbb{R}^n)} + \gamma(d\alpha+n\alpha^{1/2}s)\|\rho_{1,\alpha}^{-s}\varphi\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

where d depends only on n and s. Next, pick $\alpha < 1$ such that $d\alpha + n\alpha^{1/2}s < \frac{1}{4}$. Note that α depends only on n and s. Moreover, $\gamma(d\alpha + n\alpha^{1/2}s) < \frac{1}{2}$ uniformly for $|\mathrm{Re}\,(\lambda)| \geq 1$, since $\gamma \leq 2|\mathrm{Re}\,(\lambda)|^{-1/2}$. Since $s > \frac{1}{2}$, one obtains

$$\frac{1}{2}\|\rho_{1,\alpha}^{-s}\varphi\|_{L^2(\mathbb{R}^n)} \leq \gamma(2s+1)\Big(\|\rho_{1,\alpha}^{-s}(-\Delta-\lambda)\varphi\|_{L^2(\mathbb{R}^n)} + \sum_{j=1}^{n}\|\rho_{1,\alpha}^{-s}\partial_j\varphi\|_{L^2(\mathbb{R}^n)}\Big).$$

This and the inequality $\rho_s^{-s} \leq \rho_{1,\alpha}^{-s} \leq \alpha^{-s/2}\rho_s^{-s}$ show that

$$\begin{aligned}\|\varphi\|_{-s} &= \|\rho_s^{-s}\varphi\|_{L^2(\mathbb{R}^n)} \leq \|\rho_{1,\alpha}^{-s}\varphi\|_{L^2(\mathbb{R}^n)}\\ &\leq \gamma(2s+1)\alpha^{-s/2}\Big(\|(-\Delta-\lambda)\varphi\|_{-s} + \sum_{j=1}^{n}\|\partial_j\varphi\|_{-s}\Big).\end{aligned}$$

Now (11) follows from the inequalities $\|\cdot\|_{-s} \leq \|\cdot\|_s$ and $\|\partial_j\varphi\|_{-s} \leq C\|(-\Delta-\lambda)\varphi\|_s$, where C is an absolute constant (see Lemma XIII.8.4 in [25]). ❐

If $\mathrm{Im}\,(\lambda) \neq 0$, then the resolvent $(-\Delta-\lambda)^{-1}$ is a bounded operator on $L^2(\mathbb{R}^n)$. Consider its restriction $R_{s,-s}(\lambda) := (-\Delta-\lambda)^{-1}|_{L_s^2(\mathbb{R}^n)}$ as an operator from $L_s^2(\mathbb{R}^n)$ to $L_{-s}^2(\mathbb{R}^n)$. Inequality (11) shows that $R_{s,-s}(\lambda)$ is a bounded operator from $L_s^2(\mathbb{R}^n)$ to $L_{-s}^2(\mathbb{R}^n)$ and that

$$\|R_{s,-s}(\lambda)\|_{\mathcal{L}(L_s^2,L_{-s}^2)} \leq d|\mathrm{Re}\,(\lambda)|^{-1/2} \to 0 \quad \text{as } |\mathrm{Re}\,(\lambda)| \to \infty. \tag{12}$$

This relation with $\lambda = \omega^2$ will be used in the first proof of Proposition 8.

First proof of Proposition 8. Define the multiplication operators $(M_s f)(x) = \rho_s(x)f(x)$ and $(M_{-s}f)(x) = \rho_s^{-1}(x)f(x)$. If P and Q are continuous potentials exponentially decaying at infinity, then the operators $PM_{-s}^{-1} = PM_s$ and M_sQ are bounded operators on $L^2(\mathbb{R}^n)$. On the other hand, $M_s : L_s^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $M_{-s} : L_{-s}^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are isometric isomorphisms. This fact, and the identity

$$P(-\Delta - \omega^2)^{-1}Qf = [PM_{-s}^{-1}][M_{-s}R_{s,-s}(\omega^2)M_s^{-1}][M_sQ]f$$

for $f \in L^2(\mathbb{R}^n)$ yield the following estimate

$$\begin{aligned} &\|P(-\Delta - \omega^2)^{-1}Q\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \\ &\quad \le \|PM_{-s}^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\|M_{-s}R_{s,-s}(\omega^2)M_s^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\|M_sQ\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \\ &\quad \le \|PM_{-s}^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\|R_{s,-s}(\omega^2)\|_{\mathcal{L}(L_s^2(\mathbb{R}^n),L_{-s}^2(\mathbb{R}^n))}\|M_sQ\|_{\mathcal{L}(L^2(\mathbb{R}^n))}. \end{aligned}$$

Using relation (12) for $\lambda = \omega^2$, one concludes that (10) holds. ❐

The main tool in the second proof of Proposition 8 in the case $n = 3$ is Theorem I.23 in [27], inspired by previous results of Zemach and Klein [33]. Assume $n = 3$ and suppose that the potentials P and Q satisfy the following (Rollnik) condition (see [27, p. 3])

$$\int\!\!\int_{\mathbb{R}^6} \frac{|P(x)|^2|Q(y)|^2}{|x-y|^2}\,dx\,dy < \infty. \tag{13}$$

Note that (13) trivially holds for exponentially decaying continuous P and Q. Consider on $L^2(\mathbb{R}^3)$ the operator $\mathcal{R}_\omega$ with integral kernel

$$R_\omega(x,y) = P(x)\frac{e^{i\omega|x-y|}}{4\pi|x-y|}Q(y), \quad x,\ y \in \mathbb{R}^3,\ x \ne y,\ \mathrm{Im}\,(\omega) \ge 0. \tag{14}$$

***Theorem* 10.** ([27, Theorem I.23].) *Assume* (13). *Then* $\mathcal{R}_k$, $k \in \mathbb{R}$, *is a Hilbert-Schmidt operator in* $L^2(\mathbb{R}^3)$ *with* $\lim_{|k|\to\infty}\|\mathcal{R}_k\| = 0$, $k \in \mathbb{R}$. *Moreover,*

$$\mathrm{tr}\,(\mathcal{R}_k^*\mathcal{R}_k\mathcal{R}_k^*\mathcal{R}_k) \to 0 \quad \text{as } |k| \to \infty,\ k \in \mathbb{R}. \tag{15}$$

Relation (15) follows from a direct calculation of $\mathrm{tr}\,(\mathcal{R}_k^*\mathcal{R}_k\mathcal{R}_k^*\mathcal{R}_k)$, resulting in

$$\begin{aligned} &\iiiint_{\mathbb{R}^{12}} e^{-ik(|x-y|-|y-z|+|z-w|-|w-x|)} \\ &\qquad \times \frac{|Q(x)|^2\,|P(y)|^2\,|P(w)|^2\,|Q(z)|^2}{(4\pi)^4|x-y|\,|y-z|\,|z-w|\,|w-x|}\,dx\,dy\,dz\,dw, \end{aligned}$$

see [27, p. 24] (and compare also with (20) below), and from the Riemann-Lebesgue lemma. The first assertion in Theorem 10 follows from (15) and the estimate $\|\mathcal{R}_k\|^4 = \|\mathcal{R}_k^*\mathcal{R}_k\|^2 \leq \|\mathcal{R}_k^*\mathcal{R}_k\|_{\mathcal{J}_2}^2 = \operatorname{tr}(\mathcal{R}_k^*\mathcal{R}_k\mathcal{R}_k^*\mathcal{R}_k)$.

In the case $n = 2$, one replaces the integral kernel $e^{i\omega|x-y|}/(4\pi|x-y|)$ in (14) by the Hankel function $(i/4)H_0^{(1)}(\omega|x-y|)$ of order zero and first kind (cf. [6]). We use the following result.

***Theorem* 11.** ([5], [7, p. 1449].) *Assume* $P^2, Q^2 \in L^2(\mathbb{R}^2) \cap L^{4/3}(\mathbb{R}^2)$. *Consider on* $L^2(\mathbb{R}^2)$ *the operator* $\mathcal{K}_\omega$ *with integral kernel*

$$K_\omega(x,y) = P(x)\frac{i}{4}H_0^{(1)}(\omega|x-y|)Q(y), \tag{16}$$

where $x, y \in \mathbb{R}^2$, $x \neq y$, $\operatorname{Im}(\omega) \geq 0$, $\omega \neq 0$. *Then*

$$\begin{aligned} \|\mathcal{K}_\omega\|_{\mathcal{J}_2(L^2(\mathbb{R}^2))} \quad \leq \quad & c_1|\omega|^{-1/2}\|P^2\|_{L^{4/3}(\mathbb{R}^2)}^{1/2}\|Q^2\|_{L^{4/3}(\mathbb{R}^2)}^{1/2} \\ & + c_2|\omega|^{-1}\|P^2\|_{L^2(\mathbb{R}^2)}^{1/2}\|Q^2\|_{L^2(\mathbb{R}^2)}^{1/2}, \end{aligned} \tag{17}$$

provided $\operatorname{Im}(\omega) \geq 0$ *and* $\omega \neq 0$.

Note that $P^2, Q^2 \in L^2(\mathbb{R}^2) \cap L^{4/3}(\mathbb{R}^2)$ for exponentially decaying continuous potentials P and Q.

In the case $n = 1$, one replaces the integral kernel $e^{i\omega|x-y|}/(4\pi|x-y|)$ in equation (14) by $(i/2\omega)e^{i\omega|x-y|}$ and introduces

$$K_\omega(x,y) = P(x)\frac{i}{2\omega}e^{i\omega|x-y|}Q(y), \quad x,\ y \in \mathbb{R},\ \operatorname{Im}(\omega) \geq 0,\ \omega \neq 0, \tag{18}$$

assuming $P^2, Q^2 \in L^1(\mathbb{R})$. The corresponding operator $\mathcal{K}_\omega$ in $L^2(\mathbb{R})$, with integral kernel $K_\omega(x,y)$, given by (18) then satisfies the elementary estimate

$$\|\mathcal{K}_\omega\|_{\mathcal{J}_2(L^2(\mathbb{R}))} \leq c|\omega|^{-1}\|P^2\|_{L^1(\mathbb{R})}^{1/2}\|Q^2\|_{L^1(\mathbb{R})}^{1/2}. \tag{19}$$

To establish the connection with the discussion in the current paper, we recall that $(-\Delta - \omega^2)^{-1}$ for $\operatorname{Im}(\omega) > 0$ is an integral operator with integral kernel $e^{i\omega|x-y|}/(4\pi|x-y|)$ in the case $n = 3$, integral kernel $(i/4)H_0^{(1)}(\omega|x-y|)$ for $n = 2$, and integral kernel $(i/2\omega)e^{i\omega|x-y|}$ for $n = 1$.

Second proof of Proposition 8 (for $1 \le n \le 3$)**.** In the cases $n = 1, 2$ we apply the estimate (19) and Theorem 11 to the operator $\mathcal{K}_\omega := P(-\Delta - \omega^2)^{-1}Q$ in $L^2(\mathbb{R}^n)$ for $n = 1, 2$ and conclude that $\|P(-\Delta - \omega^2)^{-1}Q\| \to 0$ as $|\mathrm{Re}\,(\omega^2)| \to \infty$. This proves Proposition 8 for $n = 1, 2$.

In the case $n = 3$ one needs one more calculation. If $\omega = k + im$ with $m > 0$, $k \in \mathbb{R}$, then the integral kernel $K_\omega(x, y)$ of the Hilbert-Schmidt operator $\mathcal{K}_\omega := P(-\Delta - \omega^2)^{-1}Q$ in $L^2(\mathbb{R}^3)$ is given by the formula

$$K_\omega(x, y) = R_k(x, y)e^{-m|x-y|}, \quad x,\ y \in \mathbb{R}^3,\ x \neq y.$$

where $R_k(x, y)$ is defined in (14) (with $\omega = k$, $\mathrm{Im}\,(\omega) = 0$ in (14)).

A direct calculation using $m > 0$ then shows

$$\begin{aligned}(20)\ \operatorname{tr}&(\mathcal{K}_\omega^*\mathcal{K}_\omega\mathcal{K}_\omega^*\mathcal{K}_\omega)\\ &= \iiiint_{\mathbb{R}^{12}} e^{-ik(|x-y|-|y-z|+|z-w|-|w-x|)}\frac{|Q(x)|^2|P(y)|^2|P(w)|^2|Q(z)|^2}{|x-y||y-z||z-w||w-x|}\\ &\qquad\qquad \times (4\pi)^{-4}e^{-m(|x-y|+|y-z|+|z-w|+|w-x|)}\,dx\,dy\,dz\,dw\\ &\le \operatorname{tr}(\mathcal{R}_k^*\mathcal{R}_k\mathcal{R}_k^*\mathcal{R}_k).\end{aligned}$$

But then relation (15) yields $\|\mathcal{K}_\omega\|^4 \le \operatorname{tr}(\mathcal{R}_k^*\mathcal{R}_k\mathcal{R}_k^*\mathcal{R}_k) \to 0$ as $|k| \to \infty$. Similarly, $\|\mathcal{K}_\omega\|^4 \le \operatorname{tr}(\mathcal{K}_\omega^*\mathcal{K}_\omega\mathcal{K}_\omega^*\mathcal{K}_\omega) \to 0$ as $m \to +\infty$, by the dominated convergence theorem, and hence $\|\mathcal{K}_\omega\| \to 0$ as $|\mathrm{Re}\,(\omega^2)| = |k^2 - m^2| \to \infty$. Thus, $\|P[-\Delta - \omega^2]^{-1}Q\| \to 0$ as $|\mathrm{Re}\,(\omega^2)| \to \infty$, and hence Proposition 8 is proved. ❐

Proof of Lemma 4. We recall that $\xi = a + i\tau$, $a \in \mathbb{R} \setminus \{0\}$, $\tau \in \mathbb{R}$. Since D^2 is self-adjoint,

$$\|\xi(\xi^2+D^2)^{-1}\| \le \frac{|\xi|}{|\mathrm{Im}\,(\xi^2)|} = \frac{(a^2+\tau^2)^{1/2}}{2|a\tau|} \le c \quad \text{uniformly with respect to } |\tau|.$$

On the other hand, $[\xi^2 + D^2]^{-1}D = h(D)$ with h defined as $h(p) = p(\xi^2 + p^2)^{-1}$. Therefore,

$$\begin{aligned}(21)\quad \|(\xi^2 + D^2)^{-1}D\| &= \max_{p\in\sigma(D)} |h(p)|\\ &= \max_{p\ge-\beta} |p|[(a^2 - \tau^2 + p^2)^2 + 4a^2\tau^2]^{-1/2}.\end{aligned}$$

To verify that the right-hand side of (21) is bounded as $|\tau| \to \infty$, one considers two cases. First, let $p^2 \geq 2\tau^2$. Since $p^2 - \tau^2 \geq \frac{1}{2}p^2$,

$$\max_{|p|\geq\sqrt{2}|\tau|} |p|[(a^2-\tau^2+p^2)^2+4a^2\tau^2]^{-1/2} \leq \max_{p\in\mathbb{R}} |p| \left[a^2+\frac{p^2}{2}\right]^{-1/2} \leq \sqrt{2}.$$

Second, for $p^2 \leq 2\tau^2$, one has

$$\max_{|p|\leq\sqrt{2}|\tau|} |p|[(a^2-\tau^2+p^2)^2+4a^2\tau^2]^{-1/2} \leq \max_{|p|\leq\sqrt{2}|\tau|} \frac{1}{2|a|}\frac{|p|}{|\tau|} \leq \frac{1}{\sqrt{2}|a|}.$$

This completes the proof of Lemma. ❒

4. Systems of nonlinear Schrödinger equations

Systems of nonlinear Schrødinger equations arise in many applications of nonlinear optics. From the perspective of the current paper, the fundamental issue is the same as for a single equation, namely the existence and stability of standing wave solutions. For an introduction to this subject see [32]. A particularly interesting example of multiple pulses was studied recently by Yew [29, 30, 31]. The problem of second harmonic generation occurs in a slab waveguide with a quadratically nonlinear response. Yew showed that multiple pulses could be generated from the base pulse through a resonant homoclinic bifurcation. The details of this process are not pertinent for the current work, only the outcome, which is the presence of multiple pulses that Yew has shown to be unstable due to real, positive eigenvalues. It can be shown easily that the essential spectrum for these pulses remains on the imaginary axis and hence the spectral configuration for the linearization at the multiple pulses is analogous to the cases considered above for scalar equations. Since the technology of this paper can be applied to this system, Theorem 2 holds in this case. In the following we show how the above considerations can be adapted to this case of systems.

The equations governing the second harmonic generation problem are a system of coupled nonlinear Schrødinger equations of the form

$$\begin{aligned} i\frac{\partial w}{\partial t} + \Delta w - \theta w + \bar{w}v &= 0, \\ i\sigma\frac{\partial v}{\partial t} + \Delta v - \alpha v + \frac{1}{2}w^2 &= 0, \end{aligned} \tag{22}$$

where $w = w(t,x)$ and $v = v(t,x)$, $x \in \mathbb{R}^n$, $n \geq 1$, are complex valued functions, and α, σ, and θ are positive parameters. For the one-dimensional case, $n = 1$, the questions of the existence of standing waves for (22) and, as mentioned above, the structure of the spectrum of the linearization around the standing waves are well-understood, see [30, 31].

The linearization at a standing wave $\boldsymbol{u} = (\varphi, \psi)$ is given by the operator

$$\mathcal{A} = \begin{bmatrix} 0 & -L_R \\ L_I & 0 \end{bmatrix}$$

that has the same structure as in (3), but L_R and L_I are now 2×2 operator matrices defined as follows,

$$(23) \quad L_R := \begin{bmatrix} -\Delta + \theta - \psi & -\varphi \\ -\dfrac{\varphi}{\sigma} & \dfrac{-\Delta + \alpha}{\sigma} \end{bmatrix}, \quad L_I := \begin{bmatrix} -\Delta + \theta + \psi & -\varphi \\ -\dfrac{\varphi}{\sigma} & \dfrac{-\Delta + \alpha}{\sigma} \end{bmatrix},$$

where the functions φ and ψ are assumed to be continuous and exponentially decaying at infinity.

***Theorem* 12.** *The Spectral Mapping Theorem holds for the group generated on the space* $[L^2(\mathbb{R}^n)]^4$ *by the operator* $\mathcal{A}$ *with* L_R *and* L_I *defined in* (23).

Proof. As above, one needs to show that $\|(a + i\tau - \mathcal{A})^{-1}\|$ remains bounded as $|\tau| \to \infty$. Denote

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \quad \text{where } D_1 = -\Delta + \theta,\ D_2 = -(1/\sigma)\Delta + \alpha/\sigma,$$

and consider the following matrix potentials exponentially decaying at infinity,

$$(24) \qquad Q_1 = \begin{bmatrix} -\psi & -\varphi \\ -\dfrac{\varphi}{\sigma} & 0 \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} \psi & -\varphi \\ -\dfrac{\varphi}{\sigma} & 0 \end{bmatrix}.$$

With this new notation and $\xi = a + i\tau$ formula (4) still holds.

By transposing the second and third rows and columns in the 4×4 matrix

$$\begin{bmatrix} \xi & D \\ -D & \xi \end{bmatrix},$$

we remark that this matrix is similar to the block-diagonal matrix, with the blocks

$$\begin{bmatrix} \xi & D_1 \\ -D_1 & \xi \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \xi & D_2 \\ -D_2 & \xi \end{bmatrix}$$

on the main diagonal and zero remaining entries. Lemma 4, applied to each of the diagonal blocks, shows that the norm of the operator (4a) in the matrix case remains bounded as $|\tau| \to \infty$.

Formula (5) is also valid in the new notation. For $j = 1, 2$ and the matrices Q_j defined in (24), consider the polar decomposition $Q_j(x) = |Q_j(x)|U_j(x)$, where $|Q_j(x)| = [Q_j^*(x)Q_j(x)]^{1/2}$ and $U_j(x)$ is a partial isometry. Note that $U_j(x)$ and $|Q_j(x)|$ commute. To be consistent with our previous notations in (7), we denote $|Q_j|^{1/2}(x) = |Q_j(x)|^{1/2}$ and $Q_j^{1/2}(x) = |Q_j(x)|^{1/2}U_j(x)$, so that $Q_j = Q_j^{1/2}|Q_j|^{1/2} = |Q_j|^{1/2}Q_j^{1/2}$, $j = 1, 2$. We again have $T(\xi) = A(\xi)B$, where $A(\xi)$ and B for the matrix case are defined as in (8). The proof of Theorem 1 remains the same as in the scalar case, as soon as we can show that Lemma 6 holds for matrix-valued potentials $P(x) = [p_{jk}(x)]_{j,k=1}^2$ and $Q(x) = [q_{jk}(x)]_{j,k=1}^2$, with exponentially decaying continuous entries. We remark that

$$P[\xi^2 + D^2]^{-1}DQ = [p_{j1}[\xi^2 + D_1^2]^{-1}D_1q_{1k} + p_{j2}[\xi^2 + D_2^2]^{-1}D_2q_{2k}]_{j,k=1}^2.$$

The norm of each summand in these entries tends to zero as $|\tau| \to \infty$, by the scalarversion of Lemma 6. Thus, $\|P[\xi^2 + D^2]^{-1}DQ\| \to 0$ and, similarly, $\|P\xi[\xi^2 + D^2]^{-1}Q\| \to 0$ as $|\tau| \to \infty$. ❒

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