# A sharp formula for the essential spectral radius of the Ruelle transfer operator on smooth and Hölder spaces*


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**Abstract**

We study Ruelle's transfer operator $\mathcal{L}$ induced by a $C^{\mathbf{r}+1}$–smooth expanding map $\varphi$ of a smooth manifold and a $C^{\mathbf{r}}$–smooth bundle automorphism $\Phi$ of a real vector bundle $\mathcal{E}$. We prove the following exact formula for the essential spectral radius of $\mathcal{L}$ on the space $C^{\mathbf{r},\alpha}$ of $\mathbf{r}$-times continuously differentiable sections of $\mathcal{E}$ with $\alpha$-Hölder $\mathbf{r}$-th derivative:

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) = \exp\left(\sup_{\nu\in \mathrm{Erg}} \{h_\nu + \lambda_\nu - (\mathbf{r}+\alpha)\chi_\nu\}\right),$$

where Erg is the set of $\varphi$-ergodic measures, $h_\nu$ the entropy of $\varphi$ with respect to $\nu$, $\lambda_\nu$ the largest Lyapunov exponent of the cocycle induced by $\Phi$, and $\chi_\nu$ the smallest Lyapunov exponent for the differential $D\varphi$.



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# 1 Introduction and Results

The transfer operator is important in many questions of dynamical systems and statistical mechanics. It is particularly helpful for studying the mixing and statistical properties of measures, investigations of zeta functions and Fredholm determinants, piecewise monotone transformations, etc. We refer to the books [B, KS, PP, R4] and recent papers [Ba, F, K, Rg], and the bibliography therein.

Estimates *from above* for the essential spectral radius of the transfer operator are obtained in [R2, R3], see also [CL, ChL]. Results on the *calculation* of the essential spectral radius in the scalar case are obtained in [CI, Ke] for dimension one and in [H], see also [BH], for higher dimensions. In the current paper we derive an estimate from below, and, as a result, an *exact formula* in terms of Lyapunov–Oseledets exponents for the essential spectral radius of the matrix coefficient transfer operator for the multidimensional case in the following setting, which is due to Ruelle [R2].

Let $\varphi$ be a $C^{\mathbf{r}+1}$–smooth expanding map (small distances are increased by a factor $\rho_{\rm ex} > 1$) of a smooth compact $\mathbf{d}$-dimensional connected manifold $\Theta$. We assume that $\varphi$ is not one-to-one, that is $\mathbf{N} = \mathrm{card}\{\varphi^{-1}\{\theta\}\} > 1$. Let $\mathcal{E}$ be a smooth real $\ell$-dimensional vector bundle over $\Theta$, $\mathcal{E}_\theta \simeq \mathbb{R}^\ell$, and $\Phi$ be a $C^{\mathbf{r}+1}$–smooth bundle automorphism over $\varphi$, that is $\Phi(\theta) : \mathcal{E}_\theta \to \mathcal{E}_{\varphi\theta}$ and $\det \Phi(\theta) \neq 0$. For $\mathbf{r} = 0, 1, \ldots$ and $\alpha \in [0, 1]$ let $C^{\mathbf{r},0} = C^{\mathbf{r}}$ denote the space of $\mathbf{r}$-times continuously differentiable sections $f$ of $\mathcal{E}$, and $C^{\mathbf{r},\alpha}$, $0 < \alpha \leq 1$, denote the space of $\mathbf{r}$-times continuously differentiable sections $f$ with $\mathbf{r}$-th derivative that satisfies a (global) Hölder condition with exponent $\alpha$. On the space $C^{\mathbf{r},\alpha}$, $\alpha \in [0, 1]$, consider the matrix coefficient transfer operator, $\mathcal{L}$, defined as follows:

$$(\mathcal{L}f)(\theta) = \sum_{\eta\in\varphi^{-1}\theta} \Phi(\eta)f(\eta), \quad \theta \in \Theta, \quad f \in C^{\mathbf{r},\alpha}.$$

Let Erg $=$ Erg$(\varphi, \Theta)$ denote the set of all $\varphi$-invariant ergodic Borel probability measures on $\Theta$, and $h_\nu$ denote the entropy of $\varphi$ with respect to $\nu \in$ Erg. For each $\nu \in$ Erg, let $\lambda_\nu$ denote the largest Lyapunov-Oseledets exponent of the cocycle $\Phi^k(\theta) = \Phi(\varphi^{k-1}\theta)\cdot\ldots\cdot\Phi(\theta)$, generated by $\Phi$ and $\varphi$, and let $\chi_\nu$ denote the smallest Lyapunov-Oseledets exponent of the differential $D\varphi^k(\theta)$, $\theta \in \Theta$, $k = 1, 2, \ldots$.

In the current paper we give the proof of the following result announced in [GL].

**Theorem 1.1.**

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) = \rho_\#(\mathbf{r}, \alpha), \quad \textit{where} \quad \rho_\#(\mathbf{r}, \alpha) := \exp\Big(\sup_{\nu\in\mathrm{Erg}} \{h_\nu + \lambda_\nu - (\mathbf{r} + \alpha)\chi_\nu\}\Big). \tag{1.1}$$

Theorem 1.1 gives the following formula for the scalar case $\ell = 1$, $\Phi : \Theta \to \mathbb{R} \setminus \{0\}$:

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) = \exp\Big(\sup_{\nu\in\mathrm{Erg}} \{h_\nu + \int_\Theta \log|\Phi(\theta)|\, d\nu - (\mathbf{r} + \alpha)\chi_\nu\}\Big), \tag{1.2}$$

and for the one-dimensional case $\mathbf{d} = 1$, $\Theta = S^1$:

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) = \exp\Big(\sup_{\nu\in\mathrm{Erg}} \{h_\nu + \int_\Theta \log|\Phi(\theta)|\, d\nu - (\mathbf{r} + \alpha)\int_\Theta \log|\varphi'(\theta)|\, d\nu\}\Big). \tag{1.3}$$

The transfer operator $\mathcal{L}$ shares many common properties with the *evolution operator* $T$ defined for $f \in C^{\mathbf{r},\alpha}$ by the formula $(Tf)(\theta) = \Phi(\varphi^{-1}_{\mathrm{diffeo}}\theta)f(\varphi^{-1}_{\mathrm{diffeo}}\theta)$, where $\varphi_{\mathrm{diffeo}}$ is a diffeomorphism on $\Theta$ (see [ChL] for a systematic discussion and many applications of evolution operators, and also related topics in [L]). In particular, using the techniques of the current paper, one can prove that if the set of the aperiodic points of $\varphi_{\mathrm{diffeo}}$ is dense in $\Theta$, then

$$\mathrm{r}_{\mathrm{ess}}(T; C^{\mathbf{r},\alpha}) = \exp\Big( \sup_{\nu\in\mathrm{Erg}} \{\lambda_\nu - (\mathbf{r}+\alpha)\chi_\nu\}\Big). \tag{1.4}$$

Formulas of the type (1.1)–(1.4) have a fairly long history that we will now briefly review. For the *scalar case* $\ell = 1$, $\Phi : \Theta \to \mathbb{R}$ and $\mathbf{r} = \alpha = 0$, and for the spectral radius $\mathrm{r}_{\mathrm{sp}}(\cdot)$ the following formula was known, probably, since [R1]:

$$\mathrm{r}_{\mathrm{sp}}(\mathcal{L}, C^0) = \exp\Big( \sup_{\nu\in\mathrm{Erg}} \{h_\nu + \int_\Theta \log|\Phi|\, d\nu\}\Big). \tag{1.5}$$

Its counterpart, $\mathrm{r}_{\mathrm{sp}}(T; C^0) = \exp\Big( \sup_{\nu\in\mathrm{Erg}}\{\int_\Theta \log|\Phi|\, d\nu\}\Big)$ for the evolution operator $T$ was obtained in [AL, CS, Ki1]. In the case $\mathbf{d} = 1$, $\Theta = S^1$, a formula for the essential spectral radius $\mathrm{r}_{\mathrm{ess}}(T; C^{\mathbf{r},\alpha})$, similar to (1.3), that is, with no entropy $h_\nu$, can be also obtained, cf., e.g., [An]. In the important paper [CI] in the case $\ell = \mathbf{d} = 1$ and $\alpha = 0$ it was proved that the essential spectrum of $\mathcal{L}$ on $C^{\mathbf{r},0}$ is a disk and a formula for $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},0})$ was given (cf. Lemma 2.1 below); this formula can be modified to the form (1.3).

For the *matrix case* $\ell > 1$ the transfer operator on $C^{\mathbf{r},\alpha}$ was first studied in [T] and [R2]. For $\mathbf{r} = \alpha = 0$ the formula

$$\mathrm{r}_{\mathrm{sp}}(\mathcal{L}, C^0) = \exp\Big( \sup_{\nu\in\mathrm{Erg}} \{h_\nu + \lambda_\nu\}\Big) \tag{1.6}$$

was obtained in [CL, Theorem 3] (see also [LS, Theorem 4.15], where another type of transfer operators was considered). Also, it was proved in [LS] that $\mathrm{r}_{\mathrm{sp}}(T, C^0) = \exp\Big( \sup_{\nu\in\mathrm{Erg}}\{\lambda_\nu\}\Big)$. We remark that a comparison of the right-hand side of (1.5) and (1.6) shows that the expression $\sup_{\nu\in\mathrm{Erg}}\{h_\nu + \lambda_\nu\}$ should play the role of topological pressure $P(\log|\Phi|)$ [W] in the case of matrix-valued $\Phi$'s. A simple proof of the inequality $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},0}) \le \rho_\#(\mathbf{r}, 0)$ was given in [CL, Theorem 2]; unfortunately, the technique of [CL] does not work for $\alpha \neq 0$. Note that the inequality $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) \le \rho_{\#}(\mathbf{r},\alpha)$ in Theorem 1.1 improves the following estimate due to Ruelle [R2]:

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) \le \exp\Big( \sup_{\nu\in \mathrm{Erg}} \{h_\nu + \int_\Theta \log \|\Phi(\theta)\|\, d\nu - (\mathbf{r}+\alpha)\log \rho_{\mathrm{ex}}\}\Big)$$

and hence, see Remark 3.2 for $\mathbf{r} = 0$ below, yields also an improvement of Ruelle's estimate for the radius of convergence of the power series for $\zeta$–function.

For the scalar case $\ell = 1$ and $\Phi_{\mathrm{PF}}(\theta) = |\det D\varphi(\theta)|^{-1}$, the transfer operator $\mathcal{L}$ is the Perron-Frobenius operator $(\mathcal{L}_{\mathrm{PF}} f)(\theta) = \sum_{\eta\in\varphi^{-1}\{\theta\}} |\det D\varphi(\eta)|^{-1} f(\eta)$, $\theta \in \Theta$, $f : \Theta \to \mathbb{R}$. We remark that in the case of the isolated rate of decay [BY, p.358] the right-hand side of (1.2) with this $\Phi_{\mathrm{PF}}$ gives an estimate from below for the rate of decay of correlations $\tau_0(\mathbf{r},\alpha)$. The quantity $\tau_0(\mathbf{r},\alpha)$ is defined (see, e. g., [BY, p.357]) as the smallest number such that the following holds: for each $\tau > \tau_0(\mathbf{r},\alpha)$ and each pair of test functions $f_1, f_2 \in C^{\mathbf{r},\alpha}$ we have

$$\Big| \int_\Theta (f_1 \circ \varphi^k) \cdot f_2\, dm - (\int_\Theta f_1\, dm) \cdot (\int_\Theta f_2\, dm) \Big| \le c(\tau, \|f_1\|_{\mathbf{r},\alpha}, \|f_1\|_{\mathbf{r},\alpha})\, \tau^k, \quad k \in \mathbb{N}.$$

Here $m$ is the unique $\varphi$–invariant probability measure on $\Theta$ which is absolutely continuous with respect to Lebesgue measure on $\Theta$. Note that for the topological pressure we have $P(\log \Phi_{\mathrm{PF}}) = h_m - \int_\Theta \log |D\varphi(\theta)|\, dm = 0$. Also, in this case $1 = \mathrm{r}_{\mathrm{sp}}(\mathcal{L}_{\mathrm{PF}})$ is a simple eigenvalue for $\mathcal{L}_{\mathrm{PF}}$, and the spectrum $\sigma(\mathcal{L}_{\mathrm{PF}}; C^{\mathbf{r};\alpha}) = \{1\} \cup \sigma_0$, where $|\sigma_0| := \sup\{|\lambda| : \lambda \in \sigma_0\}$ is *strictly* smaller than 1, see [R2, Thm.3.6]. It is proved in [BY, p.358] that if $|\sigma_0| > \mathrm{r}_{\mathrm{ess}}(\mathcal{L}_{\mathrm{PF}}; C^{\mathbf{r},\alpha})$ (the case of the *isolated* rate of decay) then $\tau_0(\mathbf{r},\alpha) = |\sigma_0|$. Thus, in this case $\tau_0(\mathbf{r},\alpha)$ is bounded from below by the right-hand side of (1.2) as claimed.

Our strategy of the proof of Theorem 1.1 is as follows. We obtain the inequality $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) \le \rho_{\#}(\mathbf{r}, \alpha)$ by refining the Ruelle's technique from [R2]. The idea is to keep track of the "local" contraction rate of the inverses for the iteration $\varphi^k$, and use Nussbaum's formula. The actual implementation of this idea is rather technical (see Lemmas 2.1 and 3.1 below). For the inequality $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) \ge \rho_{\#}(\mathbf{r}, \alpha)$ the difference between the matrix $\ell > 1$ and the scalar case becomes more substantial, cf. [R3, Rem.1.2(b)]. To prove this inequality we develop further the operator-theoretical approach from [CL, ChL]. We study yet another transfer operator, $\mathcal{K}_{\mathbf{r},\alpha}$, acting on the space of continuous sections over an extended compact space $\Theta_{\mathbf{r},\alpha}$, and induced by an extension $\psi_{\mathbf{r},\alpha}$ of $\varphi$ and $\Psi_{\mathbf{r},\alpha}$ of $\Phi$. We give a generalization of [ChL, Thm. 8.56] for $\alpha > 0$, and prove that $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) \ge \mathrm{r}_{\mathrm{sp}}(\mathcal{K}_{\mathbf{r},\alpha}; C^0(\Theta_{\mathbf{r},\alpha}))$ (Lemma 3.3). The method behind this proof goes back to Mather [M], see also [ChL] for a detailed discussion of the "Mather localization", and papers [BJL, Ke, Ki2] where similar ideas have been used for the scalar $\ell = 1$ case. Note that the presence of the Hölder seminorm makes our proof somewhat different from [ChL, Thm.8.56]. Next, we show that $\mathrm{r}_{\mathrm{sp}}(\mathcal{K}_{\mathbf{r},\alpha}; C^0(\Theta_{\mathbf{r},\alpha})) \ge \rho_{\#}(\mathbf{r}, \alpha)$ (Lemma 3.4). This proof is based on the strategy of Bowen's proof of the variational principle [Bo] and the proof of [ChL, Thm. 8.5]. A critical new moment here is the use of our Lemma 2.1 for the estimate for $\rho_{\#}(\mathbf{r}, \alpha)$ from above.

**Notation:** $c(a, b)$ - generic constant depending on parameters $a, b$; $k \in \mathbb{N}$; $\mathrm{dist}(\cdot, \cdot)$ - distance in $\Theta$; $|\cdot| = \|\cdot\|_{\mathbb{R}^{\mathbf{d}}}$; $\|\cdot\| = \|\cdot\|_{\mathbb{R}^\ell}$; $\|\cdot\|_{\mathbf{r},\alpha} = \|\cdot\|_{C^{\mathbf{r},\alpha}}$; $\mathcal{B}_{\mathbf{r}}$ – the set of $\mathbf{r}$–multilinear operators; $\sigma_{\mathrm{ap}}(A)$ – the approximate point spectrum of $A$, that is, $\sigma_{\mathrm{ap}}(A) = \{z \in \mathbb{C} :$ for each $\epsilon > 0$ there exists $g_\epsilon$ such that $\|(z - A)g_\epsilon\| \le \mathrm{const} \cdot \epsilon \cdot \|g_\epsilon\|\}$. We use bold face to denote $\mathbf{r}$-tuples of unit vectors, e.g., $\mathbf{v} = (v_1, \ldots, v_{\mathbf{r}})$, and bars to denote $(\mathbf{r} + 1)$-tuples of unit vectors, e.g., $\overline{\mathbf{v}} = (v_0, \mathbf{v}) = (v_0, v_1, \ldots, v_{\mathbf{r}})$.

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## 2 Preliminaries

Fix $\omega > 0$ small and consider a Markov partition $\{\Theta_1, \ldots, \Theta_s\}$ of $\Theta$ with $\operatorname{diam}\Theta_j < \omega$ (see, e.g., [R2, Prop. 2.1, 2.2]). Define, for $i, j \in \{1, \ldots, s\}$, $\pi_{ij} = 1$ if $\varphi(\Theta_i) \supset \Theta_j$ and $\pi_{ij} = 0$ otherwise. Fix open sets $U_j \supset \Theta_j$ with $\operatorname{diam} U_j < \omega$ such that $\varphi(U_i)$ contains the closure of $U_j$ if and only if $\pi_{ij} = 1$ and $\Theta_{i_0} \cap \Theta_{i_1} \cap \ldots \cap \Theta_{i_k} = \emptyset$ implies $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_k} = \emptyset$ (see, e.g., [R2, Prop. 2.3]). For $\pi_{ij} = 1$ there is a unique local inverse $\varphi_{ij}^{-1} : U_j \to U_i$ of the map $\varphi$ and $\operatorname{dist}(\varphi_{ij}^{-1}\theta, \varphi_{ij}^{-1}\eta) \le \rho_{\text{ex}}^{-1} \operatorname{dist}(\theta, \eta)$ whenever $\theta, \eta \in U_j$. We will assume that $\omega$ is so small that each $U_i$ belongs to a single coordinate chart on the manifold $\Theta$, thus, we will view $U_i$ as a subset of $\mathbb{R}^n$. We will write $i\theta$ to denote the $\varphi$-preimage of $\theta \in \Theta$ that is in $U_i$. A $k$-tuple $i^k = i_1 \ldots i_k \in \{1, \ldots, s\}^k$ is called admissible if $\pi_{i_1 i_2} = \ldots = \pi_{i_{k-1} i_k} = 1$. For each admissible $k$-tuple, $k > 1$, we set $U_{i^k} = \varphi_{i_1 i_2}^{-1} \circ \varphi_{i_2 i_3}^{-1} \circ \ldots \circ \varphi_{i_{k-1} i_k}(U_{i_k})$ and let $i^k\theta = i_1 \ldots i_k\theta \in U_{i^k j}$ denote the corresponding preimage of $\theta \in U_j$ under $\varphi^k$. Since $\varphi$ is expanding, we have $\operatorname{diam} U_{i^k} \le \omega \rho_{\text{ex}}^{k-1}$.

We will define the extended compact space $\Theta_{\mathbf{r},\alpha}$ and the extensions $\psi_{\mathbf{r},\alpha}$ of the map $\varphi$ and $\{\Psi_{\mathbf{r},\alpha}^k\}$ of the cocycle $\{\Phi^k\}$ as follows. Let $\Theta_{\mathbf{r},\alpha} = \{(\theta, \overline{\mathbf{v}}) : \theta \in \Theta, v_j \in \mathcal{T}_\theta^1, j = 0, \ldots, \mathbf{r}\}$ and $\Theta_{\mathbf{r},0} = \{(\theta, \mathbf{v}) : \theta \in \Theta, v_j \in \mathcal{T}_\theta^1, j = 1, \ldots, \mathbf{r}\}$, where $\mathcal{T}_\theta^1$ is the fiber in the unit tangent bundle to $\Theta$. We define maps $\psi_{\mathbf{r},\alpha} : \Theta_{\mathbf{r},\alpha} \to \Theta_{\mathbf{r},\alpha}$ and $\psi_{\mathbf{r},0} : \Theta_{\mathbf{r},0} \to \Theta_{\mathbf{r},0}$ by

$$\psi_{\mathbf{r},\alpha} : (\theta, v_0, v_1, \dots v_{\mathbf{r}}) \mapsto \left(\varphi\theta, \frac{D\varphi(\theta)v_0}{|D\varphi(\theta)v_0|}, \frac{D\varphi(\theta)v_1}{|D\varphi(\theta)v_1|}, \dots, \frac{D\varphi(\theta)v_{\mathbf{r}}}{|D\varphi(\theta)v_{\mathbf{r}}|}\right),$$

$$\psi_{\mathbf{r},0} : (\theta, v_1, \dots v_{\mathbf{r}}) \mapsto \left(\varphi\theta, \frac{D\varphi(\theta)v_1}{|D\varphi(\theta)v_1|}, \dots, \frac{D\varphi(\theta)v_{\mathbf{r}}}{|D\varphi(\theta)v_{\mathbf{r}}|}\right).$$

Throughout, we use letter $\eta$ to denote $\varphi$–preimages of $\theta$, and letter $\overline{\mathbf{u}}$ to denote the corresponding component in $\psi$–preimages of $(\theta, \overline{\mathbf{v}})$. In particular, if $(\theta_0, \overline{\mathbf{v}}^0)$, $(\theta_1, \overline{\mathbf{v}}^1)$, and $(\theta_2, \overline{\mathbf{v}}^2)$ are any given points in $\Theta_{\mathbf{r},\alpha}$, and $i^k$ is admissible, then we denote

$$\eta_l = i^k\theta_l, \quad \overline{\mathbf{u}}^l = \left\{[D\varphi^k(i^k\theta_l)]^{-1}v_j^l/|[D\varphi^k(i^k\theta_l)]^{-1}v_j^l|\right\}_{j=0}^{\mathbf{r}}, \tag{2.1}$$

such that $\psi_{\mathbf{r},\alpha}^{-k}(\theta_l, \overline{\mathbf{v}}^l) = \{(\eta_l, \overline{\mathbf{u}}^l) : i^k\}$ and $\psi_{\mathbf{r},0}^{-k}(\theta_l, \mathbf{v}^l) = \{(\eta_l, \mathbf{u}^l) : i^k\}$ for $l = 0, 1, 2$.

We define cocycles $\{\Psi_{\mathbf{r},\alpha}^k\}_{k\in\mathbb{N}}$ over $\psi_{\mathbf{r},\alpha}$ by

$$\Psi_{\mathbf{r},\alpha}^k(\theta, v_0, v_1, \dots v_{\mathbf{r}}) = \Phi^k(\theta)|D\varphi^k(\theta)v_0|^{-\alpha}\prod_{j=1}^{\mathbf{r}}|D\varphi^k(\theta)v_j|^{-1},$$

and, similarly, cocycles $\{\Psi_{\mathbf{r},0}^k\}_{k\in\mathbb{N}}$ over $\psi_{\mathbf{r},0}$ setting $\alpha = 0$. On the space $C^0(\Theta_{\mathbf{r},\alpha})$ of *continuous* sections over $\Theta_{\mathbf{r},\alpha}$ we define the extended transfer operator, $\mathcal{K}_{\mathbf{r},\alpha}$, by the rule

$$(\mathcal{K}_{\mathbf{r},\alpha}F)(\theta, \overline{\mathbf{v}}) = \sum_{(\eta,\overline{\mathbf{u}})\in\psi_{\mathbf{r},\alpha}^{-1}(\theta,\overline{\mathbf{v}})} \Psi_{\mathbf{r},\alpha}(\eta, \overline{\mathbf{u}})F(\eta, \overline{\mathbf{u}}),$$

where $\overline{\mathbf{v}} = (v_0, v_1, \dots, v_{\mathbf{r}})$. Similarly, $\mathcal{K}_{\mathbf{r},0}$ is defined on $C^0(\Theta_{\mathbf{r},0})$ by $\psi_{\mathbf{r},0}$ and $\Psi_{\mathbf{r},0}$ with $\mathbf{v} = (v_1, \dots, v_{\mathbf{r}})$. The operator $\mathcal{K}_{\mathbf{r},0}$ appears naturally when we differentiate $\mathcal{L}f$ (see [CI] and [ChL, Lemma 8.57], and (2.12) below). Using (2.1), it is easy to see that

$$(\mathcal{K}_{\mathbf{r},\alpha}^k F)(\theta, \overline{\mathbf{v}}) = \sum_{i^k} \Phi^k(i^k\theta)|[D\varphi^k(i^k\theta)]^{-1}v_0|^\alpha \prod_{j=1}^{\mathbf{r}} |[D\varphi^k(i^k\theta)]^{-1}v_j|F(i^k\theta, \overline{\mathbf{u}}). \tag{2.2}$$

We will need the growth rates $R(\mathbf{r},\alpha)$, $\rho(\mathbf{r},\alpha)$ and $s(\mathbf{r},\alpha)$, defined as follows:

$$R_k(\mathbf{r},\alpha) = \sup_{(\theta,\overline{\mathbf{v}})\in\Theta_{\mathbf{r},\alpha}} \sum_{\eta\in\varphi^{-k}\theta} \|\Phi^k(\eta)\| \, |[D\varphi(\eta)]^{-1}v_0|^\alpha \prod_{j=1}^{\mathbf{r}} |[D\varphi(\eta)]^{-1}v_j|, \tag{2.3}$$

$$R(\mathbf{r},\alpha) = \lim_{k\to\infty} R_k(\mathbf{r},\alpha)^{1/k};$$

$$\rho_k(\mathbf{r},\alpha) = \sum_{i^k} \sup_\theta \|\Phi^k(i^k\theta)\| \cdot \|[D\varphi^k(i^k\theta)]^{-1}\|^{\mathbf{r}+\alpha},$$

$$\rho(\mathbf{r},\alpha) = \lim_{k\to\infty} \rho_k(\mathbf{r},\alpha)^{1/k};$$

$$s_k(\mathbf{r},\alpha) = \max_{q=1,\dots,\mathbf{r}} \sum_{i^k} \sup_{\theta_1,\theta_2} \|\Phi^k(i^k\theta_1)\| \|[D\varphi^k(i^k\theta_1)]^{-1}\|^q \|[D\varphi^k(i^k\theta_2)]^{-1}\|^{\mathbf{r}-q} \tag{2.4}$$

$$\times \Big[\mathrm{dist}(i^k\theta_1, i^k\theta_2)/\mathrm{dist}(\theta_1,\theta_2)\Big]^\alpha, \quad s(\mathbf{r},\alpha) = \lim_{k\to\infty} s_k(\mathbf{r},\alpha)^{1/k}.$$

Here the supremum in (2.4) is taken over all $\theta_1,\theta_2\in\Theta$ if $\alpha=0$ and $\theta_1\neq\theta_2$ if $\alpha>0$. We note that $\psi_{\mathbf{r},\alpha}$ is a covering, but, generally, is *not* expanding. A general formula for the spectral radius of a transfer operator induced by a covering map ([ChL, Prop. 8.50]) gives:

$$\mathrm{r_{sp}}(\mathcal{K}_{\mathbf{r},\alpha}; C^0(\Theta_{\mathbf{r},\alpha})) = \lim_{k\to\infty} \Bigg( \sup_{(\theta,\overline{\mathbf{v}})\in\Theta_{\mathbf{r},\alpha}} \sum_{(\eta,\overline{\mathbf{u}})\in\psi^{-k}_{\mathbf{r},\alpha}(\theta,\overline{\mathbf{v}})} \|\Psi^k_{\mathbf{r},\alpha}(\eta,\overline{\mathbf{u}})\| \Bigg)^{1/k}. \tag{2.5}$$

If $(\theta,\overline{\mathbf{v}})\in\Theta_{\mathbf{r},\alpha}$ and $\overline{\mathbf{v}}=(v_0,v_1,\dots,v_{\mathbf{r}})$, then for each $(\eta,\overline{\mathbf{u}})\in\psi^{-k}_{\mathbf{r},\alpha}(\theta,\overline{\mathbf{v}})$ we have $\eta\in\varphi^{-k}(\theta)$ and $\overline{\mathbf{u}} = \{[D\varphi^k(\eta)]^{-1}v_j/|[D\varphi^k(\eta)]^{-1}v_j|\}_{j=0}^{\mathbf{r}}$ as in (2.1). By the definition of $\Psi^k_{\mathbf{r},\alpha}$ we then have:

$$\Psi^k_{\mathbf{r},\alpha}(\eta,\overline{\mathbf{u}}) = \Phi^k(\eta) \Big| D\varphi^k(\eta) \frac{[D\varphi^k(\eta)]^{-1}v_0}{|[D\varphi^k(\eta)]^{-1}v_0|} \Big|^{-\alpha} \prod_{j=1}^{\mathbf{r}} \Big| D\varphi^k(\eta) \frac{[D\varphi^k(\eta)]^{-1}v_j}{|[D\varphi^k(\eta)]^{-1}v_j|} \Big|^{-1}$$

$$= \Phi^k(\eta) \Big| [D\varphi^k(\eta)]^{-1}v_0 \Big|^\alpha \prod_{j=1}^{\mathbf{r}} \Big| [D\varphi^k(\eta)]^{-1}v_j \Big|.$$

We substitute this expression in (2.5), and observe that the summation over all $(\eta,\overline{\mathbf{u}})\in\psi^{-k}_{\mathbf{r},\alpha}(\theta,\overline{\mathbf{v}})$ in (2.5) is the same as the summation over all $\eta\in\varphi^{-k}(\theta)$. Using (2.3), we see that the right-hand side of (2.5) is equal to $\lim_{k\to\infty}(R_k(\mathbf{r},\alpha))^{1/k}$. Therefore, we have proved that

$$\mathrm{r}_{\mathrm{sp}}(\mathcal{K}_{\mathbf{r},\alpha}; C^0(\Theta_{\mathbf{r},\alpha})) = R(\mathbf{r},\alpha). \tag{2.6}$$

We will prove below that, in fact, $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) = R(\mathbf{r},\alpha)$.

**Lemma 2.1.** *$R(\mathbf{r},\alpha) = \rho(\mathbf{r},\alpha) = s(\mathbf{r},\alpha)$.*

*Proof.* By standard calculus, $R_k(\mathbf{r},\alpha) \le \rho_k(\mathbf{r},\alpha) \le s_k(\mathbf{r},\alpha)$. To see that $s(\mathbf{r},\alpha) \le R(\mathbf{r},\alpha)$, for each $\epsilon > 0$ we choose a $\delta = \delta(\epsilon) > 0$ such that if $\mathrm{dist}(\eta_1,\eta_2) \le \delta$ then

$$\|\Phi(\eta_2)\| \le \|\Phi(\eta_1)\|(1+\epsilon) \quad \text{and} \quad \|[D\varphi(\eta_2)]^{-1}\| \le \|[D\varphi(\eta_1)]^{-1}\|(1+\epsilon). \tag{2.7}$$

Consider $\theta_1$ and $\theta_2$ with $\mathrm{dist}(\theta_1,\theta_2) \le \delta$ as vectors in $\mathbb{R}^n$ (up to a constant factor), and let $\gamma_i(t) = i\gamma(t)$, $t \in [0,1]$, be the $i$-th preimage under $\varphi$ of the point $\gamma(t) = (\theta_1 - \theta_2)t + \theta_2$ that belongs to the segment connecting $\theta_1$ and $\theta_2$. Taking the inf over all smooth curves $\chi$ that connect $i\theta_1 = \gamma_i(1)$ and $i\theta_2 = \gamma_i(0)$, we have $\mathrm{dist}(i\theta_1, i\theta_2) \le \int_0^1 |[D\varphi(\gamma_i(t))]^{-1}\dot\gamma(t)|\, dt \le \max_{0\le t\le 1} \|[D\varphi(\gamma_i(t))]^{-1}\| \cdot |\theta_1 - \theta_2|$. Since $\varphi$ is an expanding map, we have $\mathrm{dist}(\gamma_i(t),\gamma_i(1)) \le \delta$. By (2.7), we have the inequality $\max_{0\le t\le 1}\|[D\varphi(\gamma_i(t))]^{-1}\| \le \|[D\varphi(i\theta_1)]^{-1}\|(1+\epsilon)$. Therefore,

$$\big[\mathrm{dist}(i^k\theta_1, i^k\theta_2)/\mathrm{dist}(\theta_1,\theta_2)\big]^\alpha \le \|[D\varphi(i^k\theta_1)]^{-1}\|^\alpha \ldots \|[D\varphi(i_k\theta_1)]^{-1}\|^\alpha (1+\epsilon)^{\alpha k}.$$

Fix $m$ so large that $\mathrm{diam}\, U_{i^m} \le \delta$. Using the last inequality, we have

$$s_{k+m}(\mathbf{r},\alpha) \le \max_{q=1,\ldots,\mathbf{r}} \sum_{i^m} c(i^m) \sum_{i^k} \sup_{\theta_1,\theta_2\in U_{i^m}} \|\Phi(i_k\theta_1)\| \ldots \|\Phi(i^k\theta_1)\|(1+\epsilon)^{\alpha k} \times \tag{2.8}$$
$$\|[D\varphi(i_k\theta_1)]^{-1}\|^{q+\alpha} \ldots \|[D\varphi(i^k\theta_1)]^{-1}\|^{q+\alpha} \|[D\varphi(i_k\theta_2)]^{-1}\|^{\mathbf{r}-q} \ldots \|[D\varphi(i^k\theta_2)]^{-1}\|^{\mathbf{r}-q}.$$

For each $i^m$ fix $\theta = \theta(i^m) \in U_{i^m}$ independent of $i^k$. Since $\varphi$ expands, we have the inequality $\mathrm{dist}(i_p \dots i_k\theta, i_p \dots i_k\theta_l) \le \delta$ for $p = 1, \dots, k$, $l = 1, 2$. Now (2.7) and (2.8) imply

$$s_{k+m}(\mathbf{r},\alpha) \le \sum_{i^m} c(i^m) \sum_{i_k} \dots \sum_{i_1} (1+\epsilon)^{k+\alpha k+\mathbf{r}k} \|\Phi(i_k\theta)\| \dots \|\Phi(i^k\theta)\|$$
$$\times \|[D\varphi(i_k\theta)]^{-1}\|^{\mathbf{r}+\alpha} \dots \|[D\varphi(i^k\theta)]^{-1}\|^{\mathbf{r}+\alpha}.$$

Since $\theta$ does not depend on $i^k$, we have $s_{k+m}(\mathbf{r},\alpha) \le c(m)(1+\epsilon)^{k(1+\alpha+\mathbf{r})}(\overline{R}_1)^k$, where we denote $\overline{R} := \lim_{n\to\infty} \overline{R}_n^{1/n}$ and $\overline{R}_n = \sup_{\theta\in\Theta} \sum_{i^n} \|\Phi^n(i^n\theta)\| \|[D\varphi^n(i^n\theta)]^{-1}\|^{\mathbf{r}+\alpha}$, $n = 1, 2, \dots$. The same argument applied for $\varphi^n$ and $\Phi^n$ gives $s_{kn+m}(\mathbf{r},\alpha) \le c(m)(1+\epsilon)^{k(1+\alpha+\mathbf{r})}(\overline{R}_n)^k$, which implies $s(\mathbf{r},\alpha) \le \overline{R}$.

It remains to show that $R(\mathbf{r},\alpha) \ge \overline{R}$. Fix $\epsilon \in (0,1)$ and for each $\theta \in \Theta$ let $G_\theta = \{v_h\}_{h=1}^N$ denote an $\epsilon$-net in $\mathcal{T}_\theta^1$. For each $\theta \in \Theta$ and $i^k$ choose $w = w(i^k,\theta) \in \mathcal{T}_\theta^1$ such that $\|[D\varphi^k(i^k\theta)]^{-1}\| = |[D\varphi^k(i^k\theta)]^{-1}w|$, and find $v = v_h \in G_\theta$, $h = h(i^k,\theta)$, such that $|v - w| < \epsilon$. Then

$$\overline{R}_k \le \sup_\theta \sum_{i^k} \|\Phi^k(i^k\theta)\| \left| [D\varphi^k(i^k\theta)]^{-1} \frac{w-v}{|w-v|} \right|^{\mathbf{r}+\alpha} |v-w|^{\mathbf{r}+\alpha} +$$
$$+ \sup_\theta \sum_{i^k} \|\Phi^k(i^k\theta)\| |[D\varphi^k(i^k\theta)]^{-1}v|^{\mathbf{r}+\alpha} \le \epsilon^{\mathbf{r}+\alpha}\overline{R}_k + R_k(\mathbf{r},\alpha).$$

Thus, $(1-\epsilon^{\mathbf{r}+\alpha})\overline{R}_k \le R_k(\mathbf{r},\alpha)$, and the lemma is proved. □

For $F \in C^{0,\alpha}(\Theta_{\mathbf{r},0})$ we denote $\|F\|_\alpha = \sup_{\theta_1\ne\theta_2} \sup_{\mathbf{v}} \|F(\theta_1,\mathbf{v}) - F(\theta_2,\mathbf{v})\| \,\mathrm{dist}(\theta_1,\theta_2)^{-\alpha}$, such that $\|f\|_{\mathbf{r},\alpha} = \|D^{\mathbf{r}}f\|_{C^0(\Theta_{\mathbf{r},0})} + \|D^{\mathbf{r}}f\|_\alpha$.

**Lemma 2.2.** *If $f \in C^{\mathbf{r},\alpha}(\Theta)$, then*

$$\|\mathcal{L}^k f\|_{\mathbf{r},\alpha} \leq \|\mathcal{K}^k_{\mathbf{r},0} D^{\mathbf{r}} f\|_\alpha + c(k)\|f\|_{\mathbf{r},0}, \tag{2.9}$$

$$\|\mathcal{L}^k f\|_{\mathbf{r},\alpha} \geq \|\mathcal{K}^k_{\mathbf{r},0} D^{\mathbf{r}} f\|_\alpha - c(k)\|f\|_{\mathbf{r},0}, \tag{2.10}$$

$$\|\mathcal{L}^k f\|_{\mathbf{r},\alpha} \leq s_k(\mathbf{r},\alpha)\|D^{\mathbf{r}} f\|_\alpha + c(k)\|f\|_{\mathbf{r},0}. \tag{2.11}$$

*Proof.* Similarly to [ChL, Lemma 8.57], the chain rule implies

$$(D^{\mathbf{r}}\mathcal{L}^k f)(\theta,\mathbf{v}) = (\mathcal{K}^k_{\mathbf{r},0} D^{\mathbf{r}} f)(\theta,\mathbf{v}) + (L_k f)(\theta,\mathbf{v}), \tag{2.12}$$

where $L_k f$ contains only the derivatives of $f$ up to the order $\mathbf{r}-1$. Thus, $\|\mathcal{L}^k f\|_{\mathbf{r},0} \leq c(k)\|f\|_{\mathbf{r},0}$ and $\|L_k f\|_\alpha \leq c(k)\|f\|_{\mathbf{r},0}$. Since $\|\mathcal{L}^k f\|_{\mathbf{r},\alpha} \geq \|D^{\mathbf{r}}\mathcal{L}^k f\|_\alpha$, we have (2.9) and (2.10). To finish the proof of (2.11), for $\theta_1 \neq \theta_2$ and any $\mathbf{v}$ we estimate

$$\|(\mathcal{K}^k_{\mathbf{r},0} D^{\mathbf{r}} f)(\theta_1,\mathbf{v}) - (\mathcal{K}^k_{\mathbf{r},0} D^{\mathbf{r}} f)(\theta_2,\mathbf{v})\| \leq a(\theta_1,\theta_2) + b(\theta_1,\theta_2),$$

where, recalling (2.1), we denote

$$a(\theta_1,\theta_2) = \sum_{i^k} \|\Phi^k(\eta_1)\| \|[D\varphi^k(\eta_1)]^{-1}\|^{\mathbf{r}} \|D^{\mathbf{r}} f(\eta_1)(\mathbf{u}^1) - D^{\mathbf{r}} f(\eta_2)(\mathbf{u}^2)\|,$$
$$b(\theta_1,\theta_2) = \sum_{i^k} \left\|\Phi^k(\eta_1)\prod_{j=1}^{\mathbf{r}} |[D\varphi^k(\eta_1)]^{-1} v_j| - \Phi^k(\eta_2)\prod_{j=1}^{\mathbf{r}} |[D\varphi^k(\eta_2)]^{-1} v_j|\right\| \cdot \|f\|_{\mathbf{r},0}.$$

Since $\Phi^k$ and $D\varphi^k$ are $\alpha$-Hölder, we have that $b(\theta_1,\theta_2) \leq c(k)\,\mathrm{dist}(\theta_1,\theta_2)^\alpha \|f\|_{\mathbf{r},0}$. Since $D^{\mathbf{r}} f(\eta_l)$, $l = 1,2$, are multilinear operators, $\|D^{\mathbf{r}} f(\eta_1)(\mathbf{u}^1) - D^{\mathbf{r}} f(\eta_2)(\mathbf{u}^2)\|$ is dominated by

$$\|D^{\mathbf{r}} f(\eta_1) - D^{\mathbf{r}} f(\eta_2)\|_{\mathcal{B}_{\mathbf{r}}} + \|D^{\mathbf{r}} f(\eta_2)\|_{\mathcal{B}_{\mathbf{r}}} \sum_{j=1}^{\mathbf{r}} (\prod_{p<j} |u^1_p|)|u^1_j - u^2_j|(\prod_{p>j} |u^2_p|).$$

Since $D\varphi^k$ is $\alpha$-Hölder, we have the inequality $|u^1_j - u^2_j| \leq c(k)\,\mathrm{dist}(\eta_1,\eta_2)^\alpha$. Also, $\|D^{\mathbf{r}} f(\eta_1) -$

$D^{\mathbf{r}}f(\eta_2)\|_{\mathcal{B}_{\mathbf{r}}} \leq \|D^{\mathbf{r}}f\|_\alpha \operatorname{dist}(\eta_1,\eta_2)^\alpha$. Using the definition of $s_k(\mathbf{r},\alpha)$, we have that $\|\mathcal{K}^k_{\mathbf{r},0}D^{\mathbf{r}}f\|_\alpha$ is dominated by the right-hand side of (2.11). □

**Lemma 2.3.** *There exists a constant $a = a(\mathbf{d},\mathbf{r},\alpha)$, $a \in (0,1)$, such that for each $\eta_0 \in \Theta$, each neighborhood $\mathcal{U} \ni \eta_0$, and each $\overline{\mathbf{u}} = (u_0, u_1, \dots, u_{\mathbf{r}})$, $|u_j| = 1$, $j = 0, 1, \dots \mathbf{r}$, one can find a scalar $C^{\mathbf{r},\alpha}$-smooth function $\beta$ such that the following holds:* $\operatorname{supp}\beta \subset \mathcal{U}$, *$\|\beta\|_{\mathbf{r},0} \leq \delta$, $\|\beta\|_{\mathbf{r},\alpha} \leq 1$, and*

$$\lim_{\tau\to 0} \tau^{-\alpha} D^{\mathbf{r}}\beta(\eta_0 + \tau u_0)(u_1, \dots, u_{\mathbf{r}}) = a. \tag{2.13}$$

*Proof.* Without loss of generality we will assume that $\mathcal{U}$ is so small that $\mathcal{U} \subset \mathbb{R}^{\mathbf{d}}$, and that $\eta_0 = 0$. We will choose $\beta(\theta) = \beta_0(\theta)\beta_1(|\theta|)$, $\theta \in \mathbb{R}^{\mathbf{d}}$. Here $\beta_1 : \mathbb{R}_+ \to [0,1]$ is a "flat" $C^{\mathbf{r}+1}$-cutoff function such that $\beta_1(\tau) = 1$ for $\tau \in [0,b]$ and $\beta_1(\tau) = 0$ for $\tau \geq 2b$, and $\sup_{0\leq\tau\leq 2b}|D^k\beta(\tau)| = O(b^{-k})$ as $b \to 0$, $k = 0, \dots, \mathbf{r}+1$. Parameter $b$ will be selected small to make sure that $\operatorname{supp}\beta \subset \mathcal{U}$, $\|\beta\|_{\mathbf{r},0} \leq \delta$, and $\|\beta\|_{\mathbf{r},\alpha} \leq 1$. The $C^{\mathbf{r},\alpha}$-smooth function $\beta_0 : \mathbb{R}^{\mathbf{d}} \to \mathbb{R}$ will be selected as follows to satisfy (2.13).

First, we claim that for a sufficiently small $a_1 = a_1(\mathbf{d},\mathbf{r},\alpha) \in (0,1)$ and each choice of $\overline{\mathbf{u}}$ there exists a vector $w = w(\overline{\mathbf{u}}) \in \mathbb{R}^{\mathbf{d}}$, $|w| = 1$, such that $|u_0 \cdot w|^\alpha \prod_{j=1}^{\mathbf{r}} |u_j \cdot w| \geq a_1$. Here $\cdot$ denotes the scalar product in $\mathbb{R}^{\mathbf{d}}$. For instance, if $\mathbf{r} = 0$ then $w = u_0$; if $\mathbf{r} = 1$ then $w$ bisects the smallest angle between $u_0$ and $\pm u_1$. To prove the claim for any $\mathbf{r}$, let us suppose that for each $k \in \mathbb{N}$ there exists $\overline{\mathbf{u}}(k) = (u_0(k), \dots, u_{\mathbf{r}}(k))$, $|u_j(k)| = 1$, $j = 0, \dots, \mathbf{r}$, such that for all $w, |w| = 1$, we have $|u_0(k) \cdot w|^\alpha \prod_{j=1}^{\mathbf{r}} |u_j(k) \cdot w| \leq k^{-1}$. Using compactness of the unit sphere in $\mathbb{R}^{\mathbf{d}}$, we may assume that $\{\overline{\mathbf{u}}(k)\}_{k=1}^\infty$ converges to some $\overline{\mathbf{u}} = (u_0, \dots, u_{\mathbf{r}})$, $|u_j| = 1$. Passing to the limit, we have that for each $w$, $|w| = 1$, at least one of the $\mathbf{r}+1$ unit vectors

$u_j$ is perpendicular to $w$. Since each of the sets $\{w : |w| = 1, w \perp u_j\}$, $j = 0, \ldots, \mathbf{r}$, has zero surface measure on the unit sphere in $\mathbb{R}^{\mathbf{d}}$, we have a contradiction, and the claim is proved.

Next, take any $a \in (0, 1)$, and for a given $\overline{\mathbf{u}}$ pick $w = w(\overline{\mathbf{u}})$ as indicated in the claim above. Define

$$\beta_0(\theta) = a\Big[(\mathbf{r} + \alpha) \cdots (1 + \alpha)|u_0 \cdot w|^\alpha \prod_{j=1}^{\mathbf{r}} |u_j \cdot w|\Big]^{-1} |\theta \cdot w|^{\mathbf{r}+\alpha}, \quad \theta \in \mathbb{R}^{\mathbf{d}}.$$

We will show that for sufficiently small $a$ and $b$, independent of $\overline{\mathbf{u}}$, the function $\beta(\theta) = \beta_0(\theta)\beta_1(|\theta|)$ is as required.

Note, that $\beta_0$ is constant along the hyperplanes orthogonal to $w$, and $\nabla \beta_0$ is parallel to $w$. Also, we have $|\beta_0(\theta)| \le c|\theta|^{\mathbf{r}+\alpha}$, where $c$ does not depend on $\overline{\mathbf{u}}$ by the claim above. For each $\mathbf{v} = (v_1, \ldots, v_{\mathbf{r}})$, $|v_j| = 1$, we have:

$$D^{\mathbf{r}}\beta_0(\theta)(v_1, \ldots, v_{\mathbf{r}}) = a\left[|u_0 \cdot w|^\alpha \prod_{j=1}^{\mathbf{r}} |u_j \cdot w|\right]^{-1} \prod_{j=1}^{\mathbf{r}} |v_j \cdot w||\theta \cdot w|^\alpha.$$

In particular, $D^{\mathbf{r}}\beta_0(\tau u_0)(u_1, \ldots, u_{\mathbf{r}}) = a\tau^\alpha$. Since $\beta_1$ is identically one in the ball $|\theta| \le b$, we have (2.13). Using the claim above, we have $\sup_{|\theta| \le 2b} \|D^k \beta_0(\theta)\|_{\mathcal{B}_k} = O(b^{\mathbf{r}-k+\alpha})$ as $b \to 0$ for $k = 0, 1, \ldots, \mathbf{r}$. Then, using the choice of $\beta_1$ and product rule, we have:

$$\|\beta\|_{\mathbf{r},0} = \sum_{k=0}^{\mathbf{r}} \sup_{|\theta| \le 2b} \|D^k(\beta_0(\theta)\beta_1(|\theta|))\|_{\mathcal{B}_k} = O(b^\alpha) \text{ as } b \to 0.$$

Thus, $\|\beta\|_{\mathbf{r},0} \le \delta$ for sufficiently small $b$. Since $\|D^{\mathbf{r}}(\beta_0\beta_1)\|_\alpha \le \|\beta_1 D^{\mathbf{r}}\beta_0\|_\alpha + \|\beta_2\|_\alpha$, where $\beta_2$ contains derivatives of $\beta_0$ up to the order $\mathbf{r} - 1$, we can choose sufficiently small $a$ and $b$, independent of $\overline{\mathbf{u}}$, to satisfy $\|\beta\|_{\mathbf{r},\alpha} \le 1$. $\square$

# 3 Proof of Theorem 1.1

The inequality $\mathrm{r_{sp}}(\mathcal{K}_{\mathbf{r},\alpha}; C^0(\Theta_{\mathbf{r},\alpha})) \le \rho_\#(\mathbf{r},\alpha)$ can be obtained as in [ChL, Thm 8.60]. Thus, by (2.6) and Lemma 2.1, the inequality $\mathrm{r_{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) \le \rho_\#(\mathbf{r},\alpha)$, yielding one direction of the proof of Theorem 1.1, is implied by the following lemma, whose proof is an adaptation of Ruelle's argument in [R2, p.248-249].

**Lemma 3.1.** *$\mathrm{r_{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) \le s(\mathbf{r},\alpha)$.*

*Proof.* By identifying $C^{\mathbf{r},\alpha}(\Theta)$ with $\bigoplus_i C^{\mathbf{r},\alpha}(U_i)$ one can define operators $\mathcal{M}$ and $\mathcal{M}^{(k)}$, which are equivalent, respectively, to $\mathcal{L}$ and $\mathcal{L}^k$ (see [R2, p. 246–247]):

$$(\mathcal{M}f)_j(\theta) = \sum_i \Phi(ij\theta) f(ij\theta),\ (\mathcal{M}^{(k)}f)_{i_k}(\theta) = \sum_{i_0\dots i_{k-1}} \Phi^k(i_0\dots i_{k-1}i_k\theta) f(i_0\dots i_{k-1}i_k\theta), \quad (3.1)$$

where $f_j = f|_{U_j}$. We have $\mathrm{r_{ess}}(\mathcal{L}) \le \mathrm{r_{ess}}(\mathcal{M})$. By Nussbaum's formula, it suffices to show that

$$\liminf_{k\to\infty} \|\mathcal{M}^{(k)} - K_{(k)}\|^{1/k} \le s(\mathbf{r},\alpha), \quad (3.2)$$

where $K_{(k)}$ has finite rank for each $k$. Following the proof of [R2, Thm. 3.2, p.249], choose $\bar\theta = \bar\theta(i_0,\dots,i_k) \in U_{i_0\dots i_k}$ and define $K_{(k)}f = \mathcal{M}^{(k)}\mathcal{F}_{\mathbf{r}}f$, where $\mathcal{F}_{\mathbf{r}}f$ is the Taylor expansion of $f$ of order $\mathbf{r}$ at $\bar\theta$. Let $h$ denote the remainder of the Taylor expansion. To see (3.2), we need to estimate the $C^{\mathbf{r},\alpha}$-norm of the function

$$(\mathcal{M}^{(k)} - K_{(k)})f(\theta) = \sum_{i_0\dots i_{k-1}} \Phi^k(i_0\dots i_{k-1}\theta) h(i_0\dots i_{k-1}\theta), \qquad \theta \in U_{i_k}.$$

As in [R2, p. 249] we have

$$\|D^{\mathbf{r}-q}h(\theta)\| \le c\|f\|_{\mathbf{r},\alpha}\,\mathrm{dist}(\theta,\bar\theta)^{q+\alpha} \qquad \text{for } q=0,\dots,\mathbf{r} \qquad \text{and } \theta\in U_{i_k}. \tag{3.3}$$

Since $\lim_{k\to\infty} s_k(0,\mathbf{r}+\alpha)^{1/k} \le s(\mathbf{r},\alpha)$, and $h(\bar\theta)=D^q h(\bar\theta)=0$, $q=1,2,\dots,\mathbf{r}$, for each $\epsilon>0$ we have

$$\begin{aligned}\|(\mathcal{M}^{(k)}-K_{(k)})f\|_{C^0} &\le c\|f\|_{\mathbf{r},\alpha}\sup_{i_k}\sup_{\theta\in U_{i_k}}\sum_{i_0\dots i_{k-1}}\|\Phi^k(i_0\dots i_{k-1}\theta)\|\left[\frac{\mathrm{dist}(i_0\dots i_{k-1}\theta,\bar\theta)}{\mathrm{dist}(\theta,\varphi^k\bar\theta)}\right]^{\mathbf{r}+\alpha}\\ &\le c(\epsilon)(s(\mathbf{r},\alpha)+\epsilon)^k\|f\|_{\mathbf{r},\alpha}.\end{aligned}$$

In the remainder of the proof we show that similar estimates (up to a polynomial in $k$) hold for the $C^0$-norms of the derivatives and Hölder seminorm of $\mathcal{L}^k h=(\mathcal{M}^{(k)}-K_{(k)})h$. We will work with the $\mathbf{r}$-th derivative; lower order derivatives can be considered similarly. By (2.12) we have $D^{\mathbf{r}}\mathcal{L}^k h=\mathcal{K}^k_{\mathbf{r},0}D^{\mathbf{r}}h+h_{\mathbf{r}}$, where $h_{\mathbf{r}}$ contains a variety of terms with derivatives of order $\mathbf{r}-1$, but no derivatives of order $\mathbf{r}$ of $h$.

To estimate $\|\mathcal{K}^k_{\mathbf{r},0}D^{\mathbf{r}}h\|_{C^0(\Theta_{\mathbf{r},0})}$, we let $\theta_1=\theta$, $\theta_2=\varphi^k(\bar\theta)$, $i^k=i_0\dots i_{k-1}$. By the definition of $\mathcal{K}_{\mathbf{r},0}$ and the bound (3.3) on $\|(D^{\mathbf{r}}h)(\theta)\|$, we have that $\|\mathcal{K}^k_{\mathbf{r},0}D^{\mathbf{r}}h\|_{C^0(\Theta_{\mathbf{r},0})}$ is dominated by

$$\max_{(\theta,\mathbf{v})}\sum_{i^k}\|\Phi^k(i^k\theta)\|\prod_{j=1}^{\mathbf{r}}|[D\varphi^k(i^k\theta)]^{-1}v_j|\,\mathrm{dist}(i^k\theta,\bar\theta)^\alpha\cdot\|f\|_{\mathbf{r},\alpha}\le c(\epsilon)(s(\mathbf{r},\alpha)+\epsilon)^k\|f\|_{\mathbf{r},\alpha}.$$

Similarly for the other derivatives and, hence, $\|h_{\mathbf{r}}\|_{C^0(\Theta_{\mathbf{r},0})}$ and $\|D^{\mathbf{r}}\mathcal{L}^k h\|_{C^0(\Theta_{\mathbf{r},0})}$ are dominated by the expression $c(\epsilon)p(k)(s(\mathbf{r},\alpha)+\epsilon)^k\|f\|_{C^{\mathbf{r},\alpha}}$ for a polynomial $p(\cdot)$. The same estimate holds for the Hölder seminorm of $h_{\mathbf{r}}$. Thus, to finish the proof, we need to show that

$$\sup_{\theta_1\ne\theta_2}\max_{\mathbf{v}}\|(\mathcal{K}^k_{\mathbf{r},0}D^{\mathbf{r}}h)(\theta_1,\mathbf{v})-(\mathcal{K}^k_{\mathbf{r},0}D^{\mathbf{r}}h)(\theta_2,\mathbf{v})\|\,\mathrm{dist}(\theta_1,\theta_2)^{-\alpha}\le c(\epsilon)p(k)\|f\|_{\mathbf{r},\alpha}(s(\mathbf{r},\alpha)+\epsilon)^k.$$

Consider $\mathbf{r}$-linear operators $A=(D^{\mathbf{r}}h)(i^k\theta_1)$, $B=(D^{\mathbf{r}}h)(i^k\theta_2)$ and split (for fixed $\theta_1,\theta_2\in\Theta$,

$\mathbf{v} \in (\mathcal{T}^1)^{\mathbf{r}}$) the difference $(\mathcal{K}^k_{\mathbf{r},0} D^{\mathbf{r}} h)(\theta_1, \mathbf{v}) - (\mathcal{K}^k_{\mathbf{r},0} D^{\mathbf{r}} h)(\theta_2, \mathbf{v}) = S_1 + S_2 + S_3$, where

$$\begin{aligned} S_1 &= \sum_{i^k} [\Phi^k(i^k\theta_1) - \Phi^k(i^k\theta_2)] A(\{[D\varphi^k(i^k\theta_1)]^{-1} v_j\}_{j=1}^{\mathbf{r}}), \\ S_2 &= \sum_{i^k} \Phi^k(i^k\theta_2)[A(\{[D\varphi^k(i^k\theta_1)]^{-1} v_j\}_{j=1}^{\mathbf{r}}) - B(\{[D\varphi^k(i^k\theta_1)]^{-1} v_j\}_{j=1}^{\mathbf{r}})], \\ S_3 &= \sum_{i^k} \Phi^k(i^k\theta_2)[B(\{[D\varphi^k(i^k\theta_1)]^{-1} v_j\}_{j=1}^{\mathbf{r}}) - B(\{[D\varphi^k(i^k\theta_2)]^{-1} v_j\}_{j=1}^{\mathbf{r}})]. \end{aligned}$$

Since $\Phi^k(i^k\theta_2) - \Phi^k(i^k\theta_1)$ can be written as

$$\sum_{p=0}^{k-1} \Phi^p(i_{k-p+1} \dots i_k\theta_1)[\Phi(i_{k-p} \dots i_k\theta_2) - \Phi(i_{k-p} \dots i_k\theta_1)]\Phi^{k-p-1}(i_1 \dots i_k\theta_2)$$

and $\|A(\{[D\varphi^k(i^k\theta_1)]^{-1} v_j\}_{j=1}^{\mathbf{r}})\| \le \|f\|_{\mathbf{r},\alpha} \prod_{j=1}^{\mathbf{r}} |[D\varphi^k(i^k\theta_1)]^{-1} v_j|$, we have

$$\begin{aligned} \|S_1\| \;\le\;& c\|f\|_{\mathbf{r},\alpha} \sum_{p=0}^{k-1} \sum_{i_{k-p}\dots i_k} \|\Phi^p(i_{k-p+1} \dots i_k\theta_1)\| \prod_{j=1}^{\mathbf{r}} |[D\varphi^{p+1}(i_{k-p} \dots i_k\theta_1)]^{-1} v_j| \times \\ & \times \operatorname{dist}(i_{k-p} \dots i_k\theta_1, i_{k-p} \dots i_k\theta_2)^\alpha \sum_{i_1 \dots i_{k-p-1}} \|\Phi^{k-p-1}(i^k\theta_2)\| \prod_{j=1}^{\mathbf{r}} |[D\varphi^{k-p-1}(i^k\theta_1)]^{-1} v_j|. \end{aligned}$$

Thus $\|S_1\| \le c(\epsilon)p(k)(s(\mathbf{r},\alpha) + \epsilon)^k \|f\|_{\mathbf{r},\alpha}$. A similar, but longer argument works for $\|S_2\|$ and $\|S_3\|$. □

*Remark* 3.2. For $\mathbf{r} = 0$ let $d(\cdot)$ be the generalized $\zeta$–function, see [R2, Thm. A1], defined as the formal power series $d(z) = \exp\Big( - \sum_{k=1}^\infty z^k k^{-1} \sum_{\theta \in \mathrm{Fix}\, \varphi^k} \mathrm{Tr}\ \Phi^k(\theta)\Big)$, where Tr denotes the trace. Combining the proof of [R2, Thm. A1] and Lemma 3.1 we have that the power series converges in $\{z \in \mathbb{C} : |z| s(0,\alpha) < 1\}$ (that is, in fact, for $|z| < 1/\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{0,\alpha})$). Indeed, arguing as in the proof of Lemma 3.1, we can replace in the estimates (A.3) and (A.6) of [R2] the quantity $\theta^\alpha e^{P+\epsilon}$ by the quantity $s(0,\alpha) + \epsilon$; the rest of the proof of [R2, Thm. A1] remains unchanged. We suspect that for $\mathbf{r} > 0$ similar changes could be made in (1.5) and Proposition 3.2 of [R3].

We split the proof of the inequality $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) \geq \rho_{\#}(\mathbf{r}, \alpha)$ in two lemmas.

**Lemma 3.3.** $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}; C^{\mathbf{r},\alpha}) \geq \mathrm{r}_{\mathrm{sp}}(\mathcal{K}^{\mathbf{r},\alpha}; C^0(\Theta_{\mathbf{r},\alpha}))$.

*Proof.* For $\alpha = 0$ the proof is given in [ChL, Thm. 8.56, p. 328]. We will describe how to extend this proof for $\alpha > 0$. As in [ChL], passing to the operator $z^{-1}\mathcal{K}_{\mathbf{r},\alpha}$ for some $z$ with $|z| = \mathrm{r}_{\mathrm{sp}}(\mathcal{K}_{\mathbf{r},\alpha})$, we will assume without loss of generality that $\mathrm{r}_{\mathrm{sp}}(\mathcal{K}_{\mathbf{r},\alpha}; C^0(\Theta_{\mathbf{r},\alpha})) = 1$ and $1 \in \sigma_{\mathrm{ap}}(\mathcal{K}_{\mathbf{r},\alpha}; C^0)$. We will prove that $1 \in \sigma_{\mathrm{ap}}(\mathcal{K}_{\mathbf{r},\alpha}; C^0(\Theta_{\mathbf{r},\alpha}))$ implies $1 \in \sigma_{\mathrm{ap}}(\mathcal{L}; C^{\mathbf{r},\alpha})$. This will be done by an explicit construction for any $\epsilon > 0$ of a $g = g_\epsilon \in C^{\mathbf{r},\alpha}$ satisfying

$$\|g - \mathcal{L}g\|_{\mathbf{r},\alpha} \leq c(\mathbf{d}, \mathbf{r}, \alpha) \cdot \epsilon \cdot \|g\|_{\mathbf{r},\alpha}. \tag{3.4}$$

Fix $\epsilon < 1/8$, and choose a large $N = N(\epsilon) \in \mathbb{N}$, and small $\delta = \delta(N, \epsilon)$, which will be specified later. Since $1 \in \sigma_{\mathrm{ap}}(\mathcal{K}_{\mathbf{r},\alpha}; C^0(\Theta_{\mathbf{r},\alpha}))$, we can, as in [ChL, p.328], pick an $F = F_{\epsilon,N} \in C^0(\Theta_{\mathbf{r},\alpha})$ and $(\theta_0, \overline{\mathbf{v}}^0) \in \Theta_{\mathbf{r},\alpha}$ such that

$$\|F\|_{C^0(\Theta_{\mathbf{r},\alpha})} = 1, \quad \|\mathcal{K}^N_{\mathbf{r},\alpha} F - F\|_{C^0(\Theta_{\mathbf{r},\alpha})} \leq \epsilon, \quad \|F(\theta_0, \overline{\mathbf{v}}^0)\| \geq 1/2. \tag{3.5}$$

Using (3.5) we have that for each $N$ there exist an aperiodic point $(\theta_0, \overline{\mathbf{v}}^0)$ and vectors $F(\eta_0, \overline{\mathbf{u}}^0)$, for $(\eta_0, \overline{\mathbf{u}}^0) \in \psi^{-N}_{\mathbf{r},\alpha}(\theta_0, \overline{\mathbf{v}}^0)$, such that

$$\|F(\eta_0, \overline{\mathbf{u}}^0)\| \leq 1 \quad \text{and} \quad \Big\| \sum_{\eta_0 \in \varphi^{-N}\theta_0} \Psi^N_{\mathbf{r},\alpha}(\eta_0, \overline{\mathbf{u}}^0) F(\eta_0, \overline{\mathbf{u}}^0) \Big\| \geq 3/4. \tag{3.6}$$

We also recall that $\mathrm{r}_{\mathrm{sp}}(\mathcal{K}_{\mathbf{r},\alpha}; C^0(\Theta_{\mathbf{r},\alpha})) = R(\mathbf{r}, \alpha) = 1$ by (2.6). Using (3.6) we will construct $g$ as in (3.4), as follows.

Choose a small open set $B_0$ in $\Theta$ such that $\theta_0 \in B_0$ and the components of $\varphi^{k-N}(B_0)$, $k = 0, \dots, 2N+1$ are disjoint. Take a smaller ball $B \subset B_0$ such that $\theta_0 \in B$. Let $\chi$ :

$\Theta \to [-2,2]$ be any $(\mathbf{r},\alpha)$-smooth bump-function such that: $\chi(\theta)=1$ for $\theta \in \bigcup_{k=0}^{N} \varphi^{k-N}(B)$, $\chi(\theta)=(\epsilon-1)^{-1}$ for $\theta \in \bigcup_{k=N+1}^{2N} \varphi^{k-N}(B)$, and $\chi(\theta)=0$ for $\theta \notin \bigcup_{k=0}^{2N} \varphi^{k-N}(B_0)$. Note that $\|\chi\|_{\mathbf{r},\alpha}$ grows as $\epsilon \to 0$ and $N \to \infty$.

For a sufficiently small $\delta>0$, use Lemma 2.3 to choose a function $\beta : \Theta \to [0,1]$ such that $\operatorname{supp}\beta \subset \varphi^{-N}(B)$, $\|\beta\|_{\mathbf{r},0} \le \delta$, $\|\beta\|_{\mathbf{r},\alpha} \le 1$. In addition, we select $\beta$ such that $\lim_{\tau\to 0} \tau^{-\alpha} D^{\mathbf{r}}\beta(\eta_0+\tau u_0^0)(\mathbf{u}^0) = a$ for each $\eta_0 = i^N\theta_0$. Here $\overline{\mathbf{v}}^0 = (v_0^0, \mathbf{v}^0)$ is as in (3.6), notation (2.1) is used, and $a = a(\mathbf{d},\mathbf{r},\alpha) \in (0,1)$ is the constant from Lemma 2.3. Then $D^{\mathbf{r}}\beta(\eta_0)(\mathbf{u}^0)=0$ and

$$\lim_{t\to 0} t^{-\alpha} D^{\mathbf{r}}\beta(\varphi_{i^N}^{-N}(\theta_0+tv_0^0))(\mathbf{u}^0) = a|[D\varphi^N(\eta_0)]^{-1}v_0^0|^{\alpha}. \tag{3.7}$$

Define $h(\eta)=\beta(\eta)F(\eta_0,\overline{\mathbf{u}}^0)$ for $\eta \in \varphi_{i^N}^{-N}(B)$ that belong to the same component of $\varphi_{i^N}^{-N}(B)$ that contains $\eta_0$ (for each $i^N$), and let $h(\theta)=0$ for $\theta \notin \varphi^{-N}(B)$. Let

$$g(\theta) = (1-\epsilon)^{|N-k|}(\mathcal{L}^k h)(\theta) \quad \text{for} \quad \theta \in \varphi^{k-N}(B),\ k=0,\dots,2N,$$

and $g(\theta)=0$ for $\theta \notin \bigcup_{k=0}^{2N}\varphi^{k-N}(B)$, cf. [ChL, (8.140),(8.141)]. Note that if $k=0,\dots,2N+1$, then $\operatorname{supp}\mathcal{L}^k h \subset \varphi^{k-N}(B) \subset \varphi^{k-N}(B_0)$.

Since supports of $\mathcal{L}^k h$, $k=0,\dots,2N$, are disjoint, we can re-write $g(\theta)$ for all $\theta\in\Theta$ as the following sum: $g(\theta)=\sum_{k=0}^{2N}(1-\epsilon)^{|N-k|}(\mathcal{L}^k h)(\theta)$. Then

$$(\mathcal{L}g)(\theta) = \sum_{k=1}^{2N}(1-\epsilon)^{|N-k+1|}(\mathcal{L}^k h)(\theta) + (1-\epsilon)^N(\mathcal{L}^{2N+1}h)(\theta) \quad \text{for} \quad \theta\in\Theta.$$

For $k=0,\dots,2N$ we denote, for brevity, $\epsilon_k = (1-\epsilon)^{|N-k|} - (1-\epsilon)^{|N-k+1|}$. Again using the fact that supports of $\mathcal{L}^k h$ are disjoint, we conclude that if $\theta\in\varphi^{-N}(B)$ then $g(\theta)-(\mathcal{L}g)(\theta) = (1-\epsilon)^N h(\theta)$; if $k=1,\dots,2N$ and $\theta \in \varphi^{k-N}(B)$ then $g(\theta)-(\mathcal{L}g)(\theta) = \epsilon_k(\mathcal{L}^k h)(\theta)$; if

$\theta \in \varphi^{N+1}(B)$ then $g(\theta) - (\mathcal{L}g)(\theta) = -(1-\epsilon)^N(\mathcal{L}^{2N+1}h)(\theta)$; and if $\theta \notin \bigcup_{k=0}^{2N+1} \varphi^{k-N}(B)$ then $g(\theta) - (\mathcal{L}g)(\theta) = 0$. For $\theta \in \varphi^{-N}(B)$ we rewrite $(1-\epsilon)^N h(\theta) = \epsilon_0(\mathcal{L}^0 h)(\theta) + (1-\epsilon)^{N+1}h(\theta)$. If $k = 0, \dots, N$ then $\epsilon_k = \epsilon(1-\epsilon)^{|N-k|}$. If $k = N+1, \dots, 2N$ then $\epsilon_k = \epsilon(\epsilon-1)^{-1}(1-\epsilon)^{|N-k|}$. Thus, if $k = 0, \dots, 2N$ and $\theta \in \varphi^{k-N}(B)$, then $\epsilon_k(\mathcal{L}^k h)(\theta) = \epsilon\chi(\theta)g(\theta)$. We stress, that $\mathrm{supp}(\chi g)$ belongs to the set $\bigcup_{k=0}^{2N} \varphi^{k-N}(B)$ and, therefore, $\chi g$ does not depend on the choice of $\chi$ outside of this set. Finally, we conclude that for all $\theta \in \Theta$ the following identity holds:

$$g(\theta) - (\mathcal{L}g)(\theta) = (1-\epsilon)^{N+1}h(\theta) + \epsilon\chi(\theta)g(\theta) - (1-\epsilon)^N(\mathcal{L}^{2N+1}h)(\theta). \tag{3.8}$$

First, we estimate $\|\mathcal{L}^{2N+1}h\|_{\mathbf{r},\alpha}$ from above. Since $R(\mathbf{r},\alpha) = 1$, by Lemma 2.1 there exists $c(\epsilon)$ such that $s_{2N+1}(\mathbf{r},\alpha) \le c(\epsilon)(1+\epsilon^2)^{2N+1}$. Apply (2.11) from Lemma 2.2 for $f = h$ and $k = 2N+1$. Since $\|h\|_{\mathbf{r},0} \le \delta$ and $\|D^{\mathbf{r}}h\|_\alpha \le 1$ by the choice of $\beta$, we have $\|\mathcal{L}^{2N+1}h\|_{\mathbf{r},\alpha} \le c(\epsilon)(1+\epsilon^2)^{2N+1} + c(N)\delta$. For the middle term in (3.8), using the product rule, we have that $\|\chi g\|_{\mathbf{r},\alpha} \le \|\chi\|_{C^0}\|g\|_{\mathbf{r},\alpha} + c(\mathbf{r})\|\chi\|_{\mathbf{r},\alpha}\|g\|_{\mathbf{r},0} \le 2\|g\|_{\mathbf{r},\alpha} + c(\epsilon,N)\delta$. Using these estimates and the inequality $\|h\|_{\mathbf{r},\alpha} \le 1$ in (3.8), we have:

$$\|g - \mathcal{L}g\|_{\mathbf{r},\alpha} \le 2\epsilon\|g\|_{\mathbf{r},\alpha} + (1-\epsilon)^{N+1}\Big[1 + c(\epsilon)(1+\epsilon^2)^{2N+1}\Big] + c(\epsilon,N)\delta. \tag{3.9}$$

We claim, that for a sufficiently small $\delta = \delta(\epsilon, N)$ we have

$$1 \le 2a^{-1}\|g\|_{\mathbf{r},\alpha}. \tag{3.10}$$

Recall that $a = a(\mathbf{d},\mathbf{r},\alpha)$ does not depend on $N$ nor $\epsilon$. As soon as the claim is proved, we use it in (3.9) and, first, choose $N$ sufficiently large, and, next, choose $\delta = \delta(\epsilon, N)$ sufficiently small to see (3.4).

To prove the claim, we use the inequality $\|h\|_{\mathbf{r},0} \le \delta$ and (2.10) from Lemma 2.2 for

$f = h$ and $k = N$ to obtain the inequality $\|g\|_{\mathbf{r},\alpha} \geq \|\mathcal{L}^N h\|_{\mathbf{r},\alpha} \geq \|\mathcal{K}^N_{\mathbf{r},0} D^{\mathbf{r}} h\|_\alpha - c(N)\delta$. Note that $(\mathcal{K}^N_{\mathbf{r},0} D^{\mathbf{r}} h)(\theta_0, \mathbf{v}^0) = 0$ by the choice of $\beta$. Thus,

$$\|\mathcal{K}^N_{\mathbf{r},0} D^{\mathbf{r}} h\|_\alpha \geq \lim_{t\to 0} t^{-\alpha} \|(\mathcal{K}^N_{\mathbf{r},0} D^{\mathbf{r}} h)(\theta_0 + t v^0_0, \mathbf{v}^0)\|$$
$$= \left\| \sum_{i^N} \Phi^N(i^N\theta_0) \prod_{j=1}^{\mathbf{r}} |[D\varphi^N(i^N\theta_0)]^{-1} v^0_j| F(i^N\theta_0, \overline{\mathbf{u}}^0) \lim_{t\to 0} t^{-\alpha} D^{\mathbf{r}}\beta(i^N(\theta_0 + t v^0_0))(\mathbf{u}^0) \right\|,$$

see notations (2.1). By (3.7) the last expression is equal to

$$a \left\| \sum_{i^N} \Phi^N(i^N\theta_0) \prod_{j=1}^{N} |[D\varphi^N(i^N\theta_0)]^{-1} v^0_j| \cdot |[D\varphi^N(i^N\theta_0)]^{-1} v^0_0|^\alpha F(i^N\theta_0, \overline{\mathbf{u}}^0) \right\|$$
$$= a\|(\mathcal{K}^N_{\mathbf{r},\alpha} F)(\theta_0, \overline{\mathbf{v}}^0)\| \geq 3a/4,$$

where we have used the choice of $F$ as indicated in (3.6). Thus, $\|g\|_{\mathbf{r},\alpha} \geq 3a/4 - c(N)\delta > a/2$ for all sufficiently small $\delta = \delta(N, \epsilon)$. □

By Lemma 3.3, (2.6) and Lemma 2.1 we have $\mathrm{r}_{\mathrm{ess}}(\mathcal{K}_{\mathbf{r},\alpha}; C^{\mathbf{r},\alpha}) \geq \rho(\mathbf{r}, \alpha)$. Thus, the inequality $\mathrm{r}_{\mathrm{ess}}(\mathcal{K}_{\mathbf{r},\alpha}; C^{\mathbf{r},\alpha}) \geq \rho_{\#}(\mathbf{r}, \alpha)$ is implied by the following lemma.

**Lemma 3.4.** $\rho(\mathbf{r}, \alpha) \geq \rho_{\#}(\mathbf{r}, \alpha)$.

*Proof.* We will make use of the Oseledec Multiplicative Ergodic Theorem (see, e.g., [Ar, Thm. 3.4.1]). This theorem gives the existence of exact Oseledec-Lyapunov exponents and corresponding subbundles almost everywhere whith respect to ergodic measures. We will apply the multiplicative ergodic theorem for three cocycles: $\{D\varphi^k\}$, $\{\Phi^k\}$, and $\{\Psi^k_{\mathbf{r},\alpha}\}$. First, we will select sets of full measure such that for each point from these sets the conclusions of the multiplicative ergodic theorem hold. This will allow us to recalculate the largest Lyapunov exponent, $\Lambda_\mu(\psi, \Psi)$, for the cocycle $\{\Psi^k_{r,\alpha}\}$ via the largest Lyapunov exponent for the cocycle $\{\Phi^k\}$ and the smallest Lyapunov exponent for the cocycle $\{D\varphi^k\}$. Next, we will use Furstenberg-Kesten formula for $\Lambda_\mu(\psi, \Psi)$. To finish the proof of the lemma, we will estimate $\Lambda_\mu(\psi, \Psi)$ following the plan of the Bowen's proof of the variational principle [Bo], thus generalizing the proof of [CL, Thm. 8.53].

Let $\nu \in \mathrm{Erg}(\varphi, \Theta)$. Recall that the conclusions of the multiplicative ergodic theorem hold for both cocycles $\{D\varphi^k\}$ and $\{\Phi^k\}$ on a subset of $\Theta$ of full $\nu$-measure. Let $\widetilde{\Theta}^{(\nu)}$ denote this subset. The conclusions that we will need are as follows. For each point $\theta \in \widetilde{\Theta}^{(\nu)}$ and each vector $v \in \mathcal{T}_\theta$ there exists exact limit $\chi_\nu(\theta, v) = \lim_{k\to\infty} k^{-1} \log |D\varphi^k(\theta)v|$, the Oseledec-Lyapunov exponent. There are only $p$, $p \le \mathbf{d}$, distinct values $\chi_\nu^{(1)}, \ldots \chi_\nu^{(p)}$ of the limits. They depend on the choice of $v \in \mathcal{T}_\theta$, but independent of $\theta \in \widetilde{\Theta}^{(\nu)}$. We order the Oseledec-Lyapunov exponents as follows: $\chi_\nu^{(1)} > \ldots > \chi_\nu^{(p)}$. For the smallest Oseledec-Lyapunov exponent we abbreviate $\chi_\nu = \chi_\nu^{(p)}$. Also, there are $p$ subbundles $V_1, \ldots, V_p$ such that if $v$ belongs to the fiber $V_i(\theta)$ over $\theta \in \widetilde{\Theta}^{(\nu)}$ of a subbundle $V_i$, then $\chi_\nu(\theta, v) = \chi_\nu^{(i)}, i = 1, \ldots, p$. We stress, that the value of $\chi_\nu(\theta, v)$ does not depend on $v$ as long as $v$ belongs to $V_i(\theta)$ with the same number $i$. That is, for any choice of vectors $v_j \in V_p(\theta)$ we have $\chi_\nu = \lim_{k\to\infty} k^{-1} \log |D\varphi^k(\theta)v_j|$, $j = 0, \ldots, \mathbf{r}$. Since the conclusions of the multiplicative ergodic theorem for the cocycle $\{\Phi^k\}$ hold on $\widetilde{\Theta}^{(\nu)}$, we also have that for each $\theta \in \widetilde{\Theta}^{(\nu)}$ there exists exact limit $\lim_{k\to\infty} k^{-1} \log \|\Phi^k(\theta)\|$. Here $\|\cdot\|$ denotes the norm of $(\ell \times \ell)$ matrices. This limit coincides with $\lambda_\nu = \lambda_\nu^{(1)}$, the largest Oseledec-Lyapunov exponent for the cocycle $\{\Phi^k\}$ over $\varphi$.

Let $\overline{V}_p(\theta)$, $\theta \in \widetilde{\Theta}^{(\nu)}$, $p \le \mathbf{d}$, denote the trace in $\mathcal{T}_\theta^1$ of the Oseledets subbundle $V_p$ that corresponds to the smallest Oseledets–Lyapunov exponent $\chi_\nu$ for the cocycle $\{D\varphi^k\}$ over $\varphi$. Let $\widetilde{\Theta}_{\mathbf{r},\alpha} \subset \Theta_{\mathbf{r},\alpha}$ denote the bundle with the base $\widetilde{\Theta}^{(\nu)}$ and fibers $\overline{V}_p(\theta) \times \ldots \times \overline{V}_p(\theta)$. Take any measure $\mu \in \mathrm{Erg}(\psi_{\mathbf{r},\alpha}, \Theta_{\mathbf{r},\alpha})$ such that $\nu = \mathrm{pr}\mu$ and that $\mu$ is supported in $\widetilde{\Theta}_{\mathbf{r},\alpha}$

(cf. [Ar, Thm.6.2.3]). Select a full $\mu$-measure subset $\widetilde{\Theta}^{(\mu)}_{\mathbf{r},\alpha} \subset \widetilde{\Theta}_{\mathbf{r},\alpha}$ such that for each $(\theta, \overline{\mathbf{v}}) \in \widetilde{\Theta}^{(\mu)}_{\mathbf{r},\alpha}$ the conclusions of the multiplicative ergodic theorem hold for the cocycle $\{\Psi^k_{\mathbf{r},\alpha}\}$ over $\psi_{\mathbf{r},\alpha}$. In particular, for each point $(\theta, \overline{\mathbf{v}}) \in \widetilde{\Theta}^{(\mu)}_{\mathbf{r},\alpha}$ there exists the exact limit $\Lambda_\mu(\psi, \Psi) = \lim_{k\to\infty} k^{-1} \log \|\Psi^k_{\mathbf{r},\alpha}(\theta, \overline{\mathbf{v}})\|$, the largest Oseledec-Lyapunov exponent for the cocycle $\{\Psi^k_{\mathbf{r},\alpha}\}$ over $\psi_{\mathbf{r},\alpha}$. Again, by the multiplicative ergodic theorem the limit is independent of $(\theta, \overline{\mathbf{v}}) \in \widetilde{\Theta}^{(\mu)}_{\mathbf{r},\alpha}$. Fix a point $(\theta, \overline{\mathbf{v}}) \in \widetilde{\Theta}^{(\mu)}_{\mathbf{r},\alpha}$ with $\overline{\mathbf{v}} = (v_0, \ldots, v_{\mathbf{r}})$. Since $\theta \in \widetilde{\Theta}^{(\nu)}$ and each $v_j \in \overline{V}_p(\theta)$, we have that $\chi_\nu = \lim_{k\to\infty} k^{-1} \log |D\varphi^k(\theta) v_j|$, $j = 0, \ldots, \mathbf{r}$ and $\lambda_\nu = \lim_{k\to\infty} k^{-1} \log \|\Phi^k(\theta)\|$ by the multiplicative ergodic theorem for the cocycles $\{D\varphi^k\}$ and $\{\Phi^k\}$, respectively. Using the definition of $\Psi^k_{\mathbf{r},\alpha}$, we have that $\Lambda_\mu(\psi, \Psi) = \lambda_\nu - (\mathbf{r} + \alpha)\chi_\nu$. Indeed,

$$
\begin{aligned}
\Lambda_\mu(\psi, \Psi) &= \lim_{k\to\infty} k^{-1} \log \|\Psi^k_{\mathbf{r},\alpha}(\theta, \overline{\mathbf{v}})\| \\
&= \lim_{k\to\infty} k^{-1} \log \|\Phi^k(\theta)\| - \alpha \lim_{k\to\infty} k^{-1} \log |D\varphi^k(\theta) v_0| - \sum_{j=1}^{\mathbf{r}} \lim_{k\to\infty} k^{-1} \log |D\varphi^k(\theta) v_j| \\
&= \lambda_\nu - (\mathbf{r} + \alpha)\chi_\nu.
\end{aligned}
$$

On the other hand, using the Furstenberg-Kesten Theorem ([Ar, Thm.3.3.3] or [W, Cor.10.1.2]) for $\mu$ and $\{\Psi^k_{\mathbf{r},\alpha}\}$, we can also compute $\Lambda_\mu(\psi, \Psi)$ as follows:

$$
\Lambda_\mu(\psi, \Psi) = \lim_{k\to\infty} \frac{1}{k} \int_{\Theta_{\mathbf{r},\alpha}} \log \|\Psi^k_{\mathbf{r},\alpha}(\theta, \overline{\mathbf{v}})\| \, d\mu.
$$

Using the disintegration $\mu_\theta(\cdot)$ of $\mu$ with respect to $\nu$ and the definition of $\Psi^k_{\mathbf{r},\alpha}$, we have:

$$
\begin{aligned}
\Lambda_\mu(\psi, \Psi) &= \lim_{k\to\infty} \frac{1}{k} \int_\Theta \int_{(\mathcal{T}^1_\theta)^{\mathbf{r}+1}} \log \frac{\|\Phi^k(\theta)\|}{|D\varphi^k(\theta) v_0|^\alpha \prod_{j=1}^{\mathbf{r}} |D\varphi^k(\theta) v_j|} \, d\mu_\theta(\overline{\mathbf{v}}) \, d\nu(\theta) \\
&\le \lim_{k\to\infty} \frac{1}{k} \int_\Theta \log \left( \|\Phi^k(\theta)\| \cdot \|[D\varphi^k(\theta)]^{-1}\|^{\mathbf{r}+\alpha} \right) d\nu(\theta).
\end{aligned}
$$

Let $\mathcal{A} = \{A_1, \ldots, A_{|\mathcal{A}|}\}$ denote a finite Borel partition of $\Theta$ such that each $\theta \in \Theta$ belongs to the closure of at most $M$ elements of $\mathcal{A}$. As in Step 1 of the proof of Theorem 8.53 in [ChL], it follows from the definition of entropy and a standard variational lemma [W, Lem.9.9] that

$$\begin{aligned} &H_\nu(\bigvee_{i=0}^{k-1}\varphi^{-i}\mathcal{A}) + \int_{\Theta_\nu}\log\Big(\|\Phi^k(\theta)\|\cdot\|[D\varphi^k(\theta)]^{-1}\|^{\mathbf{r}+\alpha}\Big)\,d\nu \qquad (3.11)\\ &\le \log\sum_{B\in\bigvee_{i=0}^{k-1}\varphi^{-i}\mathcal{A}}\sup_{\theta\in B}\Big(\|\Phi^k(\theta)\|\cdot\|[D\varphi^k(\theta)]^{-1}\|^{\mathbf{r}+\alpha}\Big)\\ &= \log\sum_{B\in\bigvee_{i=0}^{k-1}\varphi^{-i}\mathcal{A}}\Big(\|\Phi^k(\theta_B)\|\cdot\|[D\varphi^k(\theta_B)]^{-1}\|^{\mathbf{r}+\alpha}\Big)\end{aligned}$$

for suitable $\theta_B$ in the closure of $B$.

First, we remark that there is an $n = n(\mathcal{A})$ such that each $U_{i^n}$ intersects no more than $M$ sets $\overline{A}$ for $A\in\mathcal{A}$. Indeed, if not, then there is a nested sequence $U_{i^n}^{(m)}$ and $\theta = \lim_{m\to\infty}\theta^{(m)}$, $\theta^{(m)}\in U_{i^n}^{(m)}$, such that $\theta$ belongs to at least $M+1$ sets $\overline{A}$ for $A\in\mathcal{A}$. The contradiction to the assumptions on $\mathcal{A}$ proves the remark. Next, for each $\theta_B\in\overline{B}$, with $B = A_{j_0}\cap\varphi^{-1}(A_{j_1})\cap\ldots\cap\varphi^{-(k-1)}(A_{j_{k-1}})$, choose $i^{k+n} = i_1\ldots i_k i_{k+1}\ldots i_{k+n}$ such that $\theta_B\in U_{i^{k+n}}$. Define $f(B) := U_{i^{k+n}}$ for $B\in\bigvee_{i=0}^{k-1}\varphi^{-i}\mathcal{A}$. We claim that the multiplicity of the function $B\mapsto f(B)$ does not exceed $M^k$. Indeed, if $f(B) = f(B')$ then, for each $m = 0,\ldots,k-1$, we have that $\varphi^m\theta_B\in\overline{A}_{j_m}$ and $\varphi^m\theta_{B'}\in\overline{A}_{j'_m}$, but also that $\varphi^m\theta_B$ and $\varphi^m\theta_{B'}$ belong to $U_{i_{1+m}\ldots i_k\ldots i_{k+n}}$. By the remark above, there are no more than $M$ choices for $j_m\ne j'_m$ for each $m=0,\ldots,k-1$, and the claim is proved.

Therefore, (3.11) is dominated by

$$\begin{aligned}&\log\Big(M^k\sum_{i^{n+k}}\sup_{\theta\in U_{i^{k+n}}}\Big(\|\Phi^k(\theta)\|\cdot\|[D\varphi^k(\theta)]^{-1}\|^{\mathbf{r}+\alpha}\Big)\Big)\\ &\le\log\Big(M^k\sum_{i^n}\sum_{i^k}\sup_{\eta\in U_{i^n}}\Big(\|\Phi^k(i^k\eta)\|\cdot\|[D\varphi^k(i^k\eta)]^{-1}\|^{\mathbf{r}+\alpha}\Big)\Big)\le\log\Big(M^k c(n)\rho_k(\mathbf{r},\alpha)\Big).\end{aligned}$$

Since $\Lambda_\mu(\psi,\Psi) = \lambda_\nu - (\mathbf{r}+\alpha)\chi_\nu$, we conclude that $h_\nu(\varphi,\mathcal{A}) + \lambda_\nu - (\mathbf{r}+\alpha)\chi_\nu \le \log M +$

$\log \rho(\mathbf{r}, \alpha)$. Now, passing to the higher iterates of $\varphi$, the proof can be completed along the lines of [ChL], see Steps 2 and 3 on p. 320. $\square$

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