Planar Systems of Differential Equations

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1 Introduction

A system of differential equations is a set of equations involving the derivatives of several functions of the same independent variable. Systems of differential equations are used to model many physical situations. For example, the following system of differential equations arises in the study of predator-prey interactions in ecology:

$$\begin{cases} x' = x\left(\alpha - \beta y\right), \\ y' = y\left(-\gamma + \delta x\right), \end{cases} \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta$ are positive constants and x and y are functions of the same independent variable t, and x', y' their derivatives with respect to t, i.e.

$$x'(t) = \frac{dx}{dt}, \qquad y'(t) = \frac{dy}{dt}.$$

In system (1.1), $x(t)$ and $y(t)$ represent the populations of the prey and predator species, respectively, at time t. Systems of differential equations are also used to model electrical circuits. For example, let L, C and R be the inductance, capacitance and resistance, respectively, in the parallel LCR circuit shown in Figure 1.1. Assume that these quantities L, C and R are held constant. Let V be the voltage drop across the capacitor and I be the current through the inductor. Then V and I satisfy the system of differential equations

$$\begin{cases} \dfrac{dI}{dt} = \dfrac{V}{L}, \\ \dfrac{dV}{dt} = -\dfrac{I}{C} - \dfrac{V}{RC}. \end{cases}$$

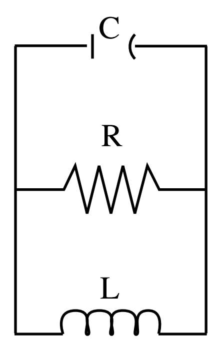

Figure 1.1: A parallel LCR circuit

In these notes we will focus on a special type of system of differential equations, namely the linear 2×2 systems. These are systems of differential equations that can be written in the form

$$\begin{cases} x' = ax + by + f(t), \\ y' = cx + dy + g(t), \end{cases} \tag{1.2}$$

where $x(t)$ and $y(t)$ are functions of the independent variable t, and a, b, c and d are constant real numbers and f and g are continuous functions on some open interval I of the real numbers. For example,

$$\begin{cases} x' = 2x + 7y + t^2, \\ y' = 3x + 12y + e^t, \end{cases}$$

is a 2×2 linear system of differential equations. We choose to focus on this type of system because (1) the theory is accessible to students who have taken only Math 1500, (2) this subject provides a good introduction to the theory of higher-dimensional systems, and (3) the linearizations of many nonlinear systems that involve only two functions of one independent variable are linear 2×2 systems, which provide good first-order approximations to the local behavior of the nonlinear system. Linear 2×2 systems also arise directly in applications as the following example shows.

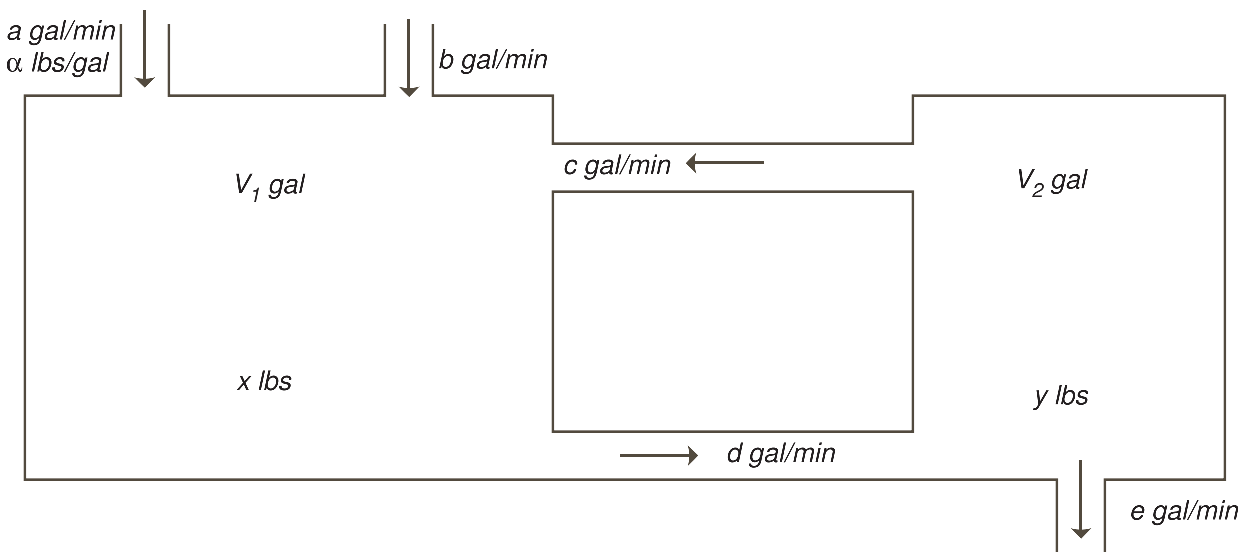

Figure 1.2: Tanks with pipes

Example. Many industrial processes involve "mixing." Consider, for instance, the arrangement of tanks and pipes depicted in Figure 1.2. In this case, liquid is pumped between the tanks at the rates shown. Also, liquid enters the first tank, which has liquid volume V_1, from two sources. The first source pumps in pure water at the rate of b gal/min; the second source enters at the rate of a gal/min and contains a certain chemical (solute) at the concentration of α lbs/gal. The solution enters and leaves the first tank at the rates c and d, and it leaves the second tank, which has liquid volume V_2, at the rate e. The liquid volume of each tank is assumed to remain constant. We are also given the initial amounts of the chemical in each tank. The problem is to determine the amount of the chemical in each of the tanks as a function of time.

The mathematical model is based on conservation of mass. Note first that the rates at which the solution enters and leaves the tanks cannot be arbitrary; the amount of solution coming in must equal the amount going out. Equivalently, the sum of the rates in must equal the sum of the rates out. For the first tank we must have

$$a + b + c = d$$

and for the second

$$d = c + e.$$

These relations are important in the analysis of the system. In particular, we must have $d > c$ to be in a physically realistic situation.

We will use conservation of mass again to set up our differential equations. Let x and y denote *the amount measured in lbs* of the chemical dissolved in the first and second tanks, respectively. The differential equations are simply a statement of conservation of mass:

$$\text{rate of change of amount with respect to time} = \text{rate in} - \text{rate out}. \tag{1.3}$$

Consider the first tank. The rate of change of the amount with respect to time is dx/dt. The "rate in" means the rate at which the chemical enters the tank. This must be expressed in units of lbs/min. The rate in from the outside source is simply

$$a \text{ gal/min times } \alpha \text{ lbs/gal} = a\alpha \text{ lbs/min}.$$

The rate of increase due to the pipe entering from the second tank must be

$$c \text{ gal/min times a factor measured in lbs/gal}.$$

We now come to an essential point: In the second tank y denotes the number of lbs of the chemical. By our assumptions, the volume in the second tank remains constant. Also, we will assume that all mixing in the tanks is instantaneous, so that the concentration is the same everywhere in the tank. Under these assumptions, the appropriate factor measured in lbs/gal is y/V_2. The "rate in" via the pipe from the second tank is yc/V_2 lbs/min. Likewise, the rate out is xd/V_1 lbs/min. These facts and the conservation law (1.3) lead to the differential equation

$$x' = (a\alpha + \frac{c}{V_2}y) - \frac{d}{V_1}x.$$

By a similar procedure (taking into account the relation $e = d - c$), we find that

$$y' = \frac{d}{V_1}x - \frac{d}{V_2}y.$$

This system together with the initial conditions takes the form

$$\begin{cases} x' = -\dfrac{d}{V_1}x + \dfrac{c}{V_2}y + a\alpha, \\ y' = \dfrac{d}{V_1}x - \dfrac{d}{V_2}y, \end{cases} \tag{1.4}$$

and

$$x(0) = x_0, \qquad y(0) = y_0. \qquad \square$$

You will soon learn how to solve such systems. A special case of our mixing problem will be solved in the example in Section 7.

In Sections 4–7 we will divide 2×2 systems into 4 classes, and for each of these 4 classes, we shall give one method of finding the general solution of systems of differential equations in the class. To accomplish this task, we shall first need some concepts from matrix theory and linear algebra, which we will describe in Section 2.

We have already encountered in disguise a special type of linear 2×2 systems in Chapter 3 of Boyce & DiPrima: second order linear differential equations with constant coefficients.

Example. Consider the second order differential equation

$$x'' - 3x' - 4x = 3e^{2t}. \tag{1.5}$$

We can rewrite this equation as a linear 2×2 system by introducing the function $y = x'$ so that (1.5) becomes

$$\begin{cases} x' = 0x + y + 0, \\ y' = 4x + 3y + 3e^{2t}, \end{cases} \tag{1.6}$$

which is in the form (1.2). By convention, we drop the terms with zero coefficients and write this system as

$$\begin{cases} x' = y, \\ y' = 4x + 3y + 3e^{2t}. \end{cases}$$

Note that the second equation in system (1.6) follows from equation (1.5) if we take into account that $y = x'$. □

In general, given a linear second order differential equation

$$\alpha x'' + \beta x' + \gamma x = f(t) \tag{1.7}$$

where α, β and γ are constant real numbers with $\alpha \neq 0$, we can, by putting $y = x'$, rewrite it as a linear 2×2 system in the form of (1.2):

$$\begin{cases} x' = y, \\ y' = -\dfrac{\gamma}{\alpha}x - \dfrac{\beta}{\alpha}y + \dfrac{1}{\alpha}f(t). \end{cases} \tag{1.8}$$

Again the second equation in system (1.8) follows from (1.7) by taking into account that $y = x'$.

Conversely, given a 2×2 system as in (1.2), it is always possible to replace one equation with a linear second order equation with constant coefficients involving only x or only y.

Example. Consider the system

$$\begin{cases} x' = 2x - y, \\ y' = -x + 2y + t. \end{cases} \tag{1.9}$$

From the first equation we derive $y = 2x - x'$, hence $y' = 2x' - x''$, so that the second equation becomes

$$2x' - x'' = -x + 2(2x - x') + t = 3x - 2x' + t,$$

which can be rearranged as

$$x'' - 4x' + 3x = -t.$$

In total, then, the system (1.9) is equivalent to

$$\begin{cases} y = 2x - x', \\ x'' - 4x' + 3x = -t. \end{cases} \tag{1.10}$$

The second equation can easily be solved in $x(t)$ using the methods we learned in Section 3. Once $x(t)$ is found, the first equation gives $y(t)$. □

Even though solutions of linear systems can be found using the previous techniques, we will present here a different way to treat such systems, which is more algebraic in nature, and at the same time more elegant and powerful. In the meantime, we will also learn how to master some tools in linear algebra, such as matrices, eigenvalues and eigenvectors, all of which are extremely useful in applications.

Homework Assignments

1.1. Rewrite the second-order differential equation

$$2x'' + 4x' - 5x = te^{-3t}$$

as a 2×2 system of differential equations.

1.2. Rewrite the linear 2×2 system of differential equations

$$\begin{cases} x' = y \\ y' = 3x - y + 4e^t \end{cases}$$

as a linear second-order differential equation.

1.3. Using the change of variables

$$x = u + 2v, \qquad y = 3u + 4v$$

show that the linear 2×2 system of differential equations

$$\begin{cases} \dfrac{du}{dt} = 5u + 8v \\ \dfrac{dv}{dt} = -u - 2v \end{cases}$$

can be rewritten as a linear second-order differential equation.

2 Some Concepts from Matrix Theory and Linear Algebra

2.1 Matrices and column vectors

A 2×2 *matrix*, with real entries, is a rectangular array of real numbers arranged in two columns and two rows. So all 2×2 matrices can be written in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.1}$$

where a, b, c and d are real numbers. For example

$$\begin{bmatrix} 2 & 1.7 \\ 6.2 & -4.3 \end{bmatrix} \tag{2.2}$$

is a 2×2 matrix.

We define a 2×1 matrix, (2 rows, 1 column) or *column vector* as any array of the form

$$\begin{bmatrix} a \\ c \end{bmatrix},$$

and 1×2 matrix (1 row, 2 columns) as any array of the type

$$\begin{bmatrix} a & b \end{bmatrix}.$$

Two matrices or two column vectors are considered equal, if the corresponding entries are equal.

Throughout these notes, vectors in the plane will be denoted with boldface letters, $\mathbf{v}$,$\mathbf{u}$,$\mathbf{x}$, and so on. A vector $\mathbf{v}$ with components (v_1, v_2) will be identified with the column vector having entries v_1, v_2, that is, we will write

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

For example, the vector that starts at the origin $(0,0)$ and terminates at the point $(1,3)$ will be denoted by $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (see Figure 2.1).

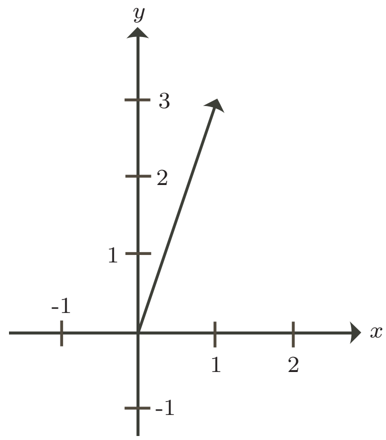

Figure 2.1: A vector in the plane

The *zero vector* is the column vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Given two vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

the matrix having **u** and **v** as columms will be denoted by

$$\left[\mathbf{u} \mid \mathbf{v}\right] = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}.$$

A common and convenient notation for matrix entries that we will sometimes use is "a_{jk}", meaning that the number a_{jk} is positioned in the j-th row and the k-th column. For example, if A is a 2×2 matrix, then its entries will be a_{11}, a_{12}, a_{21} and a_{22} arranged as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{2.3}$$

In general, an $m \times n$ *matrix* is an array of numbers arranged in m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ ... & & & \\ ... & & & \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix}.$$

In these notes we will only consider matrices and vectors up to two rows and two columns, however much of the theory extends to more general matrices.

2.2 Operations with matrices

Sum and difference of two matrices. Given the matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} \sigma & \omega \\ \tau & \rho \end{bmatrix},$$

then $A + B$ is the matrix

$$A + B = \begin{bmatrix} a+\sigma & b+\omega \\ c+\tau & d+\rho \end{bmatrix}.$$

and $A - B$ is the matrix

$$A - B = \begin{bmatrix} a-\sigma & b-\omega \\ c-\tau & d-\rho \end{bmatrix}.$$

Example.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}. \qquad \square$$

Multiplying a matrix by a number. If λ is a real number and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then λA is the matrix

$$\lambda A = \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix};$$

i.e., to multiply a matrix A by a real number λ, we just multiply every entry of A by λ.

Example.

$$7 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 21 & 28 \end{bmatrix}. \qquad \square$$

Multiplying a matrix by a column vector. Given a matrix A and a column vector $\mathbf{v}$ as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} \sigma \\ \tau \end{bmatrix}$$

define the product $A\mathbf{v}$ as the column vector

$$A\mathbf{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \sigma \\ \tau \end{bmatrix} = \begin{bmatrix} a\sigma + b\tau \\ c\sigma + d\tau \end{bmatrix}. \tag{2.4}$$

Example. We compute that

$$\begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \times 6 + 4 \times 7 \\ 1 \times 6 + 5 \times 7 \end{bmatrix} = \begin{bmatrix} 46 \\ 41 \end{bmatrix}. \qquad \square$$

Note that for any matrix A we have

$$A\mathbf{0} = \mathbf{0}.$$

Also note that for any matrix A, vector $\mathbf{v}$, and number α we have

$$\alpha(A\mathbf{v}) = (\alpha A)\mathbf{v} = A(\alpha\mathbf{v}). \tag{2.5}$$

As an exercise, check for example that for each real α

$$\alpha\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}3\\ 1\end{bmatrix}\right) = \begin{bmatrix}\alpha & 2\alpha\\ 3\alpha & 4\alpha\end{bmatrix}\begin{bmatrix}3\\ 1\end{bmatrix} = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}3\alpha\\ \alpha\end{bmatrix}.$$

Multiplying two matrices. Given the matrices

$$A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \qquad B = \begin{bmatrix}\sigma & \omega\\ \tau & \rho\end{bmatrix},$$

then AB is the matrix

$$AB = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}\sigma & \omega\\ \tau & \rho\end{bmatrix} = \begin{bmatrix}a\sigma + b\tau & a\omega + b\rho\\ c\sigma + d\tau & c\omega + d\rho\end{bmatrix}. \tag{2.6}$$

In other words, if we write $B = [\mathbf{u} \mid \mathbf{v}]$ with

$$\mathbf{u} = \begin{bmatrix}\sigma\\ \tau\end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix}\omega\\ \rho\end{bmatrix}$$

then AB is the matrix with columns $A\mathbf{u}$ and $A\mathbf{v}$.

Example. We compute that

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix} = \begin{bmatrix}1\times 5 + 2\times 7 & 1\times 6 + 2\times 8\\ 3\times 5 + 4\times 7 & 3\times 6 + 4\times 8\end{bmatrix} = \begin{bmatrix}19 & 22\\ 43 & 50\end{bmatrix}. \qquad \square$$

Matrix operations enjoy algebraic properties similar to those satisfied by the real numbers. For example, matrix multiplication is *associative* in the sense that if A, B, and C are 2×2 matrices, then

$$(AB)C = A(BC), \tag{2.7}$$

and the same is true for matrix addition. The *distributive property* also holds:

$$A(B + C) = AB + AC. \tag{2.8}$$

The *commutative property* is true for the sum

$$A + B = B + A$$

but in general it is NOT TRUE for the product, that is, in general we have

$$AB \neq BA.$$

This is a major difference between matrix multiplication and scalar multiplication!

Example. Consider the 2×2 matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

Then we see that

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix},$$

while

$$BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}.$$

Hence, in this case, $AB \neq BA$. □

The *identity matrix* is defined as

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and plays the same role for matrices as the number "1" does for real numbers, in the sense that for any 2×2 matrix A, we have

$$AI = IA = A.$$

and for any vector $\mathbf{v} \in \mathbb{R}^2$

$$I\mathbf{v} = \mathbf{v}.$$

The *zero matrix* is the matrix

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which has the property that for any matrix 2×2 matrix A,

$$A0 = 0A = 0, \qquad A + 0 = 0 + A = A.$$

Notice that we are making a very common abuse of notation by writing 0 to mean both the scalar $0 \in \mathbb{R}$ and the 2×2 zero matrix.

In the next section we will define the *inverse* of a matrix A, that is a matrix denoted by A^{-1} with the property that $AA^{-1} = A^{-1}A = I$.

2.3 Determinants, Inverses, Linear Dependence, Linear Systems, Eigenvalues and Eigenvectors

Determinant of a matrix. Consider the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The *determinant of* A, denoted by either $|A|$ or $\det(A)$, is the real number defined by

$$|A| = a_{11}a_{22} - a_{12}a_{21}.$$

We will use any of the following notations

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Example. We compute that

$$\det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = 2 \times 5 - 3 \times 4 = -2.$$

As a second example, let's find the determinant of another matrix:

$$\begin{vmatrix} 4 & -1 \\ 5 & 7 \end{vmatrix} = 4 \times 7 - (-1) \times 5 = 33. \qquad \square$$

Fact. Given two matrices A and B we have

$$|AB| = |A| \cdot |B|.$$

Also,

$$|\lambda A| = \lambda^2 |A|.$$

If, on the other hand, we multiply all entries in a single row of the matrix A by λ, then the resulting matrix has determinant $\lambda|A|$. The same is true if we multiply all the entries in any one column by λ.

For example by direct computation

$$\begin{vmatrix} \lambda a_{11} & \lambda a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \lambda a_{11}a_{22} - \lambda a_{12}a_{21} = \lambda|A|.$$

Inverse of a matrix. Consider a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Question: under which condition can we find a matrix B such that

$$AB = BA = I? \tag{2.9}$$

By applying the determinant on both sides of the equation $AB = I$ we get that $|A| \cdot |B| = |I| = 1$ which shows that $|A|$ cannot be zero. Conversely, the following fact shows that when $|A| \neq 0$ such B can be found.

Fact. Given a matrix
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
with $|A| \neq 0$, there exists one and only one matrix B satisfying (2.9). Such matrix will be denoted by A^{-1} and called the *inverse of* A, and it is given by the formula
$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \begin{bmatrix} \frac{a_{22}}{|A|} & -\frac{a_{12}}{|A|} \\ -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{bmatrix}. \tag{2.10}$$
With this notation we have
$$AA^{-1} = A^{-1}A = I.$$

Example. Let
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$
We see that the determinant $|A| = -2$, which is not zero. Thus A is invertible and from (2.10) the formula for A^{-1} is
$$A^{-1} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}. \qquad \square$$

Linear dependence of vectors. Two plane vectors $\mathbf{a}$ and $\mathbf{b}$ are *linearly dependent* if one is a multiple of the other, i.e. if either $\mathbf{a} = \mu\mathbf{b}$ for some μ real, or $\mathbf{b} = \mu\mathbf{a}$ for some μ real. For example,
$$\mathbf{a} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 10 \\ -15 \end{bmatrix}$$
are linearly dependent since $\mathbf{b} = 5\mathbf{a}$, or, equivalently, $\mathbf{a} = \frac{1}{5}\mathbf{b}$.
According to this definition, $\mathbf{a}$ and $\mathbf{0}$ are linearly dependent for any $\mathbf{a}$, since $\mathbf{0} = 0\mathbf{a}$.

Another, more common, way to state this definition is the following:
Two vectors $\mathbf{a}$ and $\mathbf{b}$ are *linearly dependent* if there exist real numbers α and β, <u>not both 0</u>, such that
$$\alpha\mathbf{a} + \beta\mathbf{b} = \mathbf{0}. \tag{2.11}$$

If this is the case, assuming for example $\alpha \neq 0$ we have $\mathbf{a} = -(\beta/\alpha)\mathbf{b}$, i.e. $\mathbf{a}$ is a multiple of $\mathbf{b}$. The advantage of the above definition is that it is easily extendable to more than two vectors in more than two dimensions.

Geometrically, the plane vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent if and only if they lie on the same line when the initial points of these vectors are placed at the origin $(0, 0)$. For example, referring to Figure 2.2, the vectors
$$\mathbf{a} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix}$$

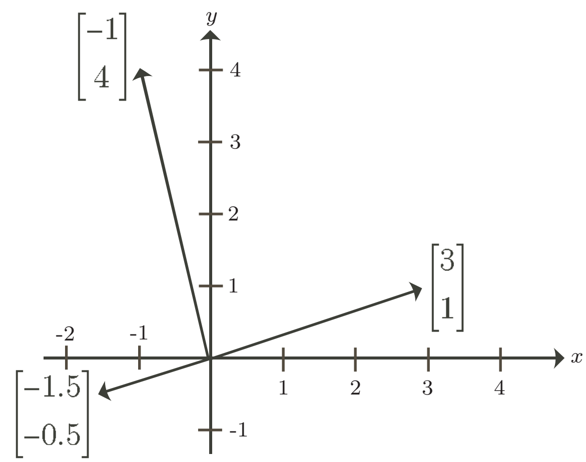

Figure 2.2: Three vectors in the plane

are linearly dependent because when their starting points are placed at the origin $(0, 0)$, these two vectors lie on the same line. We can also see this in terms of (2.11) by taking $\alpha = 1$ and $\beta = 2$. On the other hand, the vectors

$$\mathbf{a} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{c} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

are linearly independent, because when we place their starting points at $(0, 0)$, these two vectors lie on different lines.

Example. The vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

are linearly dependent. To see this, take $\alpha = 1$ and $\beta = -\frac{1}{3}$ in (2.11):

$$\mathbf{a} - \frac{1}{3}\mathbf{b} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \left(-\frac{1}{3}\right) \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Alternatively, observe that

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{3}, \qquad \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \frac{1}{3}\mathbf{b},$$

or, more simply, $3\mathbf{a} = \mathbf{b}$. This clearly shows that both vectors lie on the same line if their initial points are both placed at the origin. □

Linear dependence and determinants. There is a relationship between the concept of linear dependence and determinants. The main point is the following fact about determinants:

Fact. Given a matrix A, then $\det(A) = 0$ if and only if one column is a multiple of the other, and one row is a multiple of the other

This is fairly easy to verify, in one direction. Suppose that the second row of A, $[c \;\; d]$ is a multiple of the first row, $[a \;\; b]$, that is $c = \mu a$, and $d = \mu b$ for some real number μ. Then

$$\begin{vmatrix} a & b \\ \mu a & \mu b \end{vmatrix} = \mu \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0.$$

Notice here that we have used the previously discussed fact about how multiplying a row of a matrix by a scalar affects the determinant. Conversely, if $|A| = ad - bc = 0$ then, it is possible to check that one row or column is a multiple of the other.

As a consequence of this result we obtain the following characterization of linear independence:

Fact. Two plane vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent if and only if

$$\det[\mathbf{u} \mid \mathbf{v}] \neq 0.$$

Explicitly, $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are linearly independent if and only if

$$\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \neq 0.$$

Linear independence of vector functions. Recall from multivariate calculus that a (two-dimensional) *vector function* or *vector-valued function* takes the form

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$

where $u_1(t)$ and $u_2(t)$ are two scalar-valued functions of t lying in some interval I.

In analogy with vectors, we say that two vector functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are *linearly dependent* on the interval I if, for each $t \in I$, the vectors $\mathbf{u}(t), \mathbf{v}(t)$ are linearly dependent. In view of what has been said earlier, we then have that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly dependent if there exist two numbers α, β not both zero such that

$$\alpha\mathbf{u}(t) + \beta\mathbf{v}(t) = \mathbf{0} \qquad \text{for all } t \in I.$$

This condition is the same as saying that there exists a scalar μ so that $\mathbf{u}(t) = \mu\mathbf{v}(t)$ for every t or $\mathbf{v}(t) = \mu\mathbf{u}(t)$ for every t.

Definition. The *Wronskian* of two vector functions

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \qquad \mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix},$$

is defined as

$$W(t) = \det[\mathbf{u}(t) \mid \mathbf{v}(t)] = \begin{vmatrix} u_1(t) & v_1(t) \\ u_2(t) & v_2(t) \end{vmatrix} = u_1(t)v_2(t) - v_1(t)u_2(t).$$

Linear systems in matrix notation. Matrix theory is very useful to treat linear systems in a compact and efficient way. The easiest example is that of 2×2 linear systems of type

$$\begin{cases} ax + by = \sigma \\ cx + dy = \tau. \end{cases} \tag{2.12}$$

If we let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} \sigma \\ \tau \end{bmatrix}$$

then the system in (2.14) can be written in compact form as

$$A\mathbf{x} = \mathbf{u}.$$

Indeed,

$$A\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} \sigma \\ \tau \end{bmatrix} = \mathbf{u}.$$

Moreover, the algebra of matrices allows us to solve the system, multiplying the equation $A\mathbf{x} = \mathbf{u}$ by A^{-1}:

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{u}$$

which is the same as (since $A^{-1}A = I$)

$$\mathbf{x} = A^{-1}\mathbf{u}.$$

This operation is possible only when $|A| \neq 0$, since the inverse only exists under this hypothesis.

This operation is analogous to the one used to solve the equation $ax = b$, when a, x, b are real numbers and $a \neq 0$, namely $x = a^{-1}b$.

Fact. Given a 2×2 matrix A, if $|A| \neq 0$ then for any vector $\mathbf{u}$ the linear system $A\mathbf{x} = \mathbf{u}$ has a unique solution given as

$$\mathbf{x} = A^{-1}\mathbf{u}.$$

In particular, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$.

What happens when $|A| = 0$? Consider for example the system

$$\begin{cases} 2x - y = 0 \\ 4x - 2y = 0 \end{cases} \tag{2.13}$$

which can be written $A\mathbf{x} = \mathbf{0}$ for the matrix A defined by

$$A = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}.$$

Clearly $|A| = 0$ since the second row is a multiple of the first one. In fact the second equation of the system is equivalent to the first one (divide it by 2). So all solutions of the system satisfy the equation

$$2x = y.$$

If $x = \alpha$ for any α, then $y = 2\alpha$ and all solutions can be written as

$$\mathbf{x} = \begin{bmatrix} \alpha \\ 4\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

This fact is true in general, and will be used later:

Fact. Given a nonzero 2×2 matrix A, if $|A| = 0$ then the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, has infinitely many solutions. Every solution can be written as $\mathbf{x} = \alpha\mathbf{u}$ for some nonzero vector $\mathbf{u}$, any $\alpha \in \mathbb{R}$. Such solutions can be found by using one of the two equations of the system.

For example if the system is given as

$$\begin{cases} ax + by = 0 \\ cx + dy = 0, \end{cases} \tag{2.14}$$

when $ad = bc$ the second equation is automatically a multiple of the first and we can use $ax + by = 0$ to find the solutions. In fact, if $a \neq 0$, then we have $x = -(b/a)y$, so all solutions are of type

$$\mathbf{x} = \alpha \begin{bmatrix} -b/a \\ 1 \end{bmatrix}.$$

Clearly one can also use $y = -(a/b)x$ and write the solutions as

$$\mathbf{x} = \alpha \begin{bmatrix} 1 \\ -a/b \end{bmatrix}.$$

On the other hand, if $a = 0$, then one of b or c must also be zero (because $bc = ad = 0$). If $b = 0$, then the first equation in (2.14) is completely annihilated, so we are left with only the second, which corresponds to an entire line of points in $\mathbb{R}^2$. Finally, if $c = 0$, then we are free to choose x to be anything we like, so again there are infinitely many solutions.

Matrices and vectors with complex entries. So far we have considered the entries of our matrices and vectors to be real numbers. We also multiplied matrices and vectors by real numbers. The theory presented above, however, remains true verbatim if instead of real numbers we use complex numbers.

For example, consider the matrix

$$A = \begin{bmatrix} -1 & 2i \\ i & 2 \end{bmatrix}.$$

Then $|A| = (-1) \cdot 2 - (2i)i = -2 + 2 = 0$. Indeed the two complex valued column vectors

$$\mathbf{u} = \begin{bmatrix} -1 \\ i \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 2i \\ 2 \end{bmatrix}$$

are (complex) scalar multiples of each other: $\mathbf{v} = (-2i)\mathbf{u}$. Likewise, the two rows are multiples of each other: $\begin{bmatrix} -1 & 2i \end{bmatrix} = i \begin{bmatrix} i & 2 \end{bmatrix}$. In particular, solving $A\mathbf{x} = \mathbf{0}$, that is,

$$\begin{cases} -x + 2iy = 0 \\ ix + 2y = 0, \end{cases}$$

is equivalent to solving either just the first equation or just the second equation. Looking at the first, say, we find that $x = 2iy$, where y is any *complex* number. Hence, setting $y = \alpha$ all solutions of the above system are of type

$$\mathbf{x} = \alpha \begin{bmatrix} 2i \\ 1 \end{bmatrix},$$

where α can be any complex number.

Eigenvalues and eigenvectors of a matrix. Let A be a 2×2 matrix in the form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

In general we cannot expect that given a nonzero vector $\mathbf{v}$, the vector $A\mathbf{v}$ be a multiple of $\mathbf{v}$, i.e. we cannot expect that $A\mathbf{v}$ and $\mathbf{v}$ are linearly dependent. When this happens however, we assign to such $\mathbf{v}$ a special name.

> **Definition.** We say that a real or complex number λ is an *eigenvalue of A* if there is a nonzero vector $\mathbf{v}$ such that
>
> $$A\mathbf{v} = \lambda\mathbf{v},$$
>
> and we call such $\mathbf{v}$ an *eigenvector corresponding to λ*, or an *eigenvector for λ*.

The reason why we ask that $\mathbf{v} \neq \mathbf{0}$ is because $A\mathbf{0} = \mathbf{0}$, so any complex number would be an eigenvalue ($\lambda\mathbf{0} = \mathbf{0}$).

There is a simple way to find eigenvalues. Note first that the equation $A\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $A\mathbf{v} = \lambda I\mathbf{v}$, and also equivalent to

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

This means that the eigenvector $\mathbf{v}$ is a nonzero solution of the homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$. According to what has been said earlier, such system has nonzero solution when, and only when,

$$\det(A - \lambda I) = 0. \tag{2.15}$$

Such equation can be written out more explicitly, since

$$\begin{aligned} |A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}. \end{aligned} \tag{2.16}$$

Hence, a real or complex number λ is an eigenvalue of A if it's a solution of the quadratic equation

$$|A - \lambda I| = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0. \tag{2.17}$$

we will call the above equation the *characteristic equation of A.*

An easy way to remember the characteristic equation is to notice that it has the form

$$\lambda^2 - \text{trace}(A)\lambda + \det(A) = 0, \tag{2.18}$$

where the trace of a matrix is defined to be the sum of its diagonal elements.

Since the eigenvalues of a matrix A are roots of the quadratic equation (2.17) with real coefficients, there are three possibilities:

(i) A has two distinct real eigenvalues;

(ii) A has only one real eigenvalue, which is a double root of the characteristic equation (i.e., (2.17) can be written in the form $(\lambda - E)^2 = 0$ where E is a real number);

(iii) A has two complex eigenvalues that are complex conjugates of each other.

Example. Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}. \tag{2.19}$$

The eigenvalues of A are the roots of the equation

$$\begin{aligned} 0 = |A - \lambda I| &= \begin{vmatrix} 4 - \lambda & -2 \\ 3 & -3 - \lambda \end{vmatrix} \\ &= (4 - \lambda)(-3 - \lambda) + 6 \\ &= \lambda^2 - \lambda - 6. \end{aligned} \tag{2.20}$$

By solving the characteristic equation

$$0 = \lambda^2 - \lambda - 6, \tag{2.21}$$

we see that the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -2$. To find the eigenvectors for $\lambda_1 = 3$ we need to find a nonzero solution $\mathbf{v}$ of the system $(A - 3I)\mathbf{v} = \mathbf{0}$, i.e.

$$\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is the same as

$$\begin{cases} v_1 - 2v_2 = 0 \\ 3v_1 - 6v_2 = 0. \end{cases}$$

Since the second row is a multiple of the first, we can discard it, so that all solutions of the above systems must satisfy $v_1 = 2v_2$. Hence for any α real, if $v_2 = \alpha \neq 0$ and $v_1 = 2\alpha$ we obtain that all eigenvectors for $\lambda_1 = 3$ are of type

$$\mathbf{v} = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

as α ranges over the nonzero reals.

Similarly, for $\lambda_2 = -2$ to find the corresponding eigenvectors we solve the system $(A + 2I)\mathbf{v} = \mathbf{0}$, which gives

$$\begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is the same as

$$\begin{cases} 6v_1 - 2v_2 = 0 \\ 3v_1 - v_2 = 0. \end{cases}$$

This time it is more convenient to use the second equation, which gives $3v_1 = v_2$. If $v_1 = \alpha \neq 0$ then $v_2 = 3\alpha$ and all eigenvectors of $\lambda_2 = -2$ are of type

$$\mathbf{v} = \begin{bmatrix} \alpha \\ 3\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

as α ranges over the nonzero reals. □

Example. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}. \tag{2.22}$$

The eigenvalues of A are the roots of the equation

$$\begin{aligned} 0 = |A - \lambda I| &= \begin{vmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} \\ &= (\lambda+1)^2 + 4 \\ &= \lambda^2 + 2\lambda + 5. \end{aligned} \tag{2.23}$$

Using the quadratic formula to solve $\lambda^2 + 2\lambda + 5 = 0$, we see that the eigenvalues of A are the complex conjugates $\lambda_1 = -1 + 2i$ and $\lambda_2 = -1 - 2i$.

The eigenvectors

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

associated to $\lambda_1 = -1 + 2i$ satisfy the equation

$$(A - (-1 + 2i)I)\,\mathbf{v} = \mathbf{0},$$

that is

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$\begin{cases} -2iv_1 + 2v_2 = 0 \\ -2v_1 - 2iv_2 = 0. \end{cases} \tag{2.24}$$

Solving (2.24) using the first equation we have $v_2 = iv_1$, so that all eigenvectors associated to the eigenvalue $-1 + 2i$ must be of the form

$$\mathbf{v} = \begin{bmatrix} \alpha \\ i\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ i \end{bmatrix}$$

for some non-zero comlpex number α.

Similarly, the eigenvectors $\mathbf{v}$ associated to $-1 - 2i$ satisfy the equation

$$(A - (-1 - 2i)I)\,\mathbf{v} = \mathbf{0},$$

that is

$$\begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$\begin{cases} 2iv_1 + 2v_2 = 0 \\ -2v_1 + 2iv_2 = 0. \end{cases} \tag{2.25}$$

Solving (2.25) using the second equation (for example) we have $v_1 = iv_2$, so that all eigenvectors associated to the eigenvalue $-1 - 2i$ must be of the form

$$\mathbf{v} = \begin{bmatrix} \alpha i \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}$$

for some non-zero number α. Thus all the eigenvectors associated to the eigenvalue $-1 - 2i$ must be of the form

$$\overline{\mathbf{v}} = \begin{bmatrix} \alpha \\ -i\alpha \end{bmatrix},$$

for some non-zero complex number α. □

The last two examples show that every eigenvalue of the matrix A is associated with infinitely many eigenvectors, in particular multiples of a single eigenvector. They also show that two eigenvectors associated with different eigenvalues are linearly independent. These facts are true in general and summarized below.

Facts. Let A be a 2×2 matrix.

1. If λ is an eigenvalue of A and $\mathbf{v}$ a corresponding eigenvector, then $\alpha\mathbf{v}$ is also an eigenvector for λ, for any $\alpha \neq 0$, real or complex.
2. If $\mathbf{u}$ is another eigenvector for λ, then $\mathbf{u}+\mathbf{v}$ is also an eigenvector for λ, provided that $\mathbf{u}+\mathbf{v} \neq \mathbf{0}$.
3. If λ_1 and λ_2 are two different eigenvalues of A and $\mathbf{u}$, $\mathbf{v}$ are eigenvectors associated to λ_1 and λ_2, respectively, then the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

The justification of the first fact, for example, is as follows. Suppose that $A\mathbf{v} = \lambda\mathbf{v}$ then, using (2.5),

$$A(\alpha\mathbf{v}) = (\alpha A)\mathbf{v} = \alpha(A\mathbf{v}) = \alpha(\lambda\mathbf{v}) = \lambda(\alpha\mathbf{v}),$$

which shows that $\alpha\mathbf{v}$ is also an eigenvector corresponding to λ.

Homework Assignments

2.1. Draw the vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -6 \\ -5 \end{bmatrix}$ on the same diagram.

2.2. Evaluate the following

1. $\begin{bmatrix} 1 & 3 \\ -4 & 6 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$
2. $\begin{bmatrix} 2.5 & -6 \\ 5 & -3 \end{bmatrix} - \begin{bmatrix} -6 & 0 \\ 2 & -7 \end{bmatrix}$
3. $(-4)\begin{bmatrix} -6 & 20 \\ 6 & -3 \end{bmatrix}$
4. $e^{2t}\begin{bmatrix} t^2e^{3t} & 6 \\ -7 & -6t \end{bmatrix}$, where t is a real number
5. $\begin{bmatrix} 6 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ -2 \end{bmatrix}$
6. $e^{5t}\begin{bmatrix} -t^2 & e^{6t} \\ 2 & 6t \end{bmatrix}\begin{bmatrix} t^4 \\ -t \end{bmatrix}$, where t is a real number

7. $\begin{bmatrix} 6 & -5 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -5 & 1 \\ 3 & -4 \end{bmatrix}$

8. $\begin{bmatrix} -4 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & -7 \end{bmatrix}$

9. $\begin{bmatrix} e^{2t}t^2 & 6t \\ 5e^{4t} & 3t \end{bmatrix} \begin{bmatrix} e^{-4t} & e^{4t} \\ te^{3t} & t^{-1} \end{bmatrix}$, where t is a real number, $t \neq 0$

2.3. Calculate the determinants of the following matrices.

1. $\begin{bmatrix} 6 & 7 \\ -4 & 3 \end{bmatrix}$

2. $\begin{bmatrix} -5 & 4 \\ -2 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -3 \\ -4 & 0 \end{bmatrix}$

2.4. Determine whether the following pairs of vectors are linearly dependent. If they are linearly dependent, find real numbers α and β, not both zero, such that

$$\alpha \mathbf{u} + \beta \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

1. $\mathbf{u} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} 2c^2 \\ c \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2c^3 \\ c \end{bmatrix}$, where c is a non-zero real number

3. $\mathbf{u} = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} \sqrt{8} \\ 2 \end{bmatrix}$

4. $\mathbf{u} = \begin{bmatrix} e^{5c} \\ e^{2c} \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} e^{3c} \\ 1 \end{bmatrix}$, where c is a real number

2.5. For each of the following matrices, find all eigenvalues and associated eigenvectors.

1. $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -3 \\ \frac{2}{3} & 3 \end{bmatrix}$

3. $\begin{bmatrix} 8 & 9 \\ -4 & -4 \end{bmatrix}$

2.6. For each of the following matrices, determine whether it is invertible. If it is invertible, find its inverse.

1. $\begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}$

2. $\begin{bmatrix} -7 & \sqrt{2} \\ 6 & -4 \end{bmatrix}$

3. $\begin{bmatrix} e^{2t} & t \\ t^2 e^t & \sqrt{t} \end{bmatrix}$, where t is a non-zero real number.

3 Homogeneous 2×2 Systems

In the following 4 sections we give a complete treatment of homogeneous 2×2 systems of differential equations, namely those written in the form

$$\begin{cases} x' = ax + by \\ y' = cx + dy, \end{cases}$$

where we assume that a, b, c and d are real numbers, and x, y are differentiable functions of t, in some interval I. Using matrix notation, the above system can be written in the compact form as

$$\mathbf{x}' = A\mathbf{x},$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

We emphasize that $\mathbf{x}$ and $\mathbf{x}'$ are both functions of the variable t. We will also use the notation $\mathbf{x}(t)$, $\mathbf{x}'(t)$, to make the dependence on t more explicit when needed. In this section, unless otherwise indicated, functions of t will be denoted $\mathbf{x}$, $\mathbf{x}_1$, $\mathbf{x}_2$, and so on, whereas $\mathbf{u}$, $\mathbf{v}$, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ will be used for fixed vectors.

An *initial value problem* on an interval I, is a system $\mathbf{x}' = A\mathbf{x}$ together with an initial condition of type

$$\mathbf{x}(t_0) = \mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

where $\mathbf{x}_0$ is a given (constant) vector and $t_0 \in I$ is an fixed initial time. Written in components, this initial value problem takes the form

$$\begin{cases} x' = ax + by \\ y' = cx + dy, \end{cases}$$

with

$$x(t_0) = x_0, \qquad y(t_0) = y_0.$$

Independent solutions, general solution. In complete analogy with the treatment of linear homogeneous equations of second order, we have the following general result:

Fact. If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two linearly independent solutions of a homogeneous system $\mathbf{x}' = A\mathbf{x}$, then all solutions $\mathbf{x}(t)$ of the system can be written as

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t) \tag{3.1}$$

where c_1, c_2 are any real or complex numbers. For any given A two independent solutions can always be found, and they are defined on $\mathbb{R}$.

An expression such as (3.1), which gives all solutions of $\mathbf{x}' = A\mathbf{x}$, is called *general solution* of the system. Just as in the scalar case, one can use the Wronskian to efficiently check whether two solutions are indeed linearly independent. This is recorded in the following fact:

Fact. Suppose that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are both solutions of the system $\mathbf{x}' = A\mathbf{x}$ for $t \in I$. They are linearly independent on I if and only if $W(t) \neq 0$ for every $t \in I$.

Example. Consider the 2×2 system

$$\begin{cases} x' = -\dfrac{1}{2}x + y, \\ y' = -x - \dfrac{1}{2}y. \end{cases} \tag{3.2}$$

One can check that

$$\mathbf{x}_1(t) = \begin{bmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2(t) = \begin{bmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{bmatrix}$$

are both solutions of the system (3.2). (We will learn how to obtain these solutions later, for now let us focus on determining whether together they give a general solution or not.) To find out if they are linearly independent, we form their Wronskian function defined in Section 2:

$$W(t) = \begin{vmatrix} e^{-t/2}\cos t & e^{-t/2}\sin t \\ -e^{-t/2}\sin t & e^{-t/2}\cos t \end{vmatrix} = e^{-t}(\cos^2 t + \sin^2 t) = e^{-t}.$$

This shows that $W(t) \neq 0$ for any t, and so the fact above ensures that the solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent. Thus, *every* solution $\mathbf{x}(t)$ of (3.2) has the form

$$\mathbf{x}(t) = c_1 \begin{bmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{bmatrix},$$

for some $c_1, c_2 \in \mathbb{R}$. Expressed in terms of components this becomes

$$\begin{cases} x(t) = c_1 e^{-t/2} \cos t + c_2 e^{-t/2} \sin t \\ y(t) = -c_1 e^{-t/2} \sin t + c_2 e^{-t/2} \cos t \end{cases}$$

where c_1 and c_2 are real numbers. □

Generating solutions using eigenvalues and eigenvectors. Linear systems $A\mathbf{x} = \mathbf{b}$ are formally similar to linear equations $ax = b$. In the same way planar system $\mathbf{x}' = A\mathbf{x}$ are formally similar to first order equations $x' = ax$. The solutions of latter, as we know, are given by $x(t) = ce^{at}$. Therefore, it is not too surprising that a similar formula can generate some solutions of $\mathbf{x}' = A\mathbf{x}$:

Fact. Given a matrix A, if λ is an eigenvalue of A and $\mathbf{u}$ is any eigenvector for λ, then

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{u} \tag{3.3}$$

is a solution of the system $\mathbf{x}' = A\mathbf{x}$, defined on the entire real line.

To verify this statement, suppose that $A\mathbf{u} = \lambda\mathbf{u}$. Then, if $\mathbf{x}(t) = e^{\lambda t}\mathbf{u}$,

$$\mathbf{x}'(t) = \lambda e^{\lambda t}\mathbf{u} = e^{\lambda t}(\lambda\mathbf{u}) = e^{\lambda t}A\mathbf{u} = A(e^{\lambda t}\mathbf{u}) = A\mathbf{x}$$

(here we used again (2.5), with $\alpha = e^{\lambda t}$), hence $\mathbf{x}$ is a solution of the system.

The above result implies that if we come up with an eigenvalue λ and a fixed eigenvector $\mathbf{u}$ then any function of type

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{u}$$

is a solution of the system $\mathbf{x}' = A\mathbf{x}$ for any constant c_1. We cannot expect however that all solutions have this form, since, as stated in formula (3.1), we need two independent solutions to generate all solutions. In the next three sections we will describe how to produce such independent solutions, depending on the types of eigenvalues of A.

Homework Assignments

3.1. Let A be a 2×2 matrix with distinct real eigenvalues λ_1, λ_2. Suppose that $\mathbf{u}$ is an eigenvector for λ_1 and $\mathbf{v}$ is an eigenvector for λ_2. Compute the Wronskian of the vector-valued functions $e^{\lambda_1 t}\mathbf{u}$ and $e^{\lambda_2 t}\mathbf{v}$. What does this tell you?

3.2. Use the definition of linear dependence to determine whether the vector-valued functions

$$\mathbf{u}(t) = \begin{bmatrix} t^4 \\ t^2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}(t) = \begin{bmatrix} t^2 \\ 1 \end{bmatrix}$$

are linearly dependent or linearly independent on the interval $(-10, 10)$. Then compute the determinant of

$$\begin{bmatrix} t^4 & t^2 \\ t^2 & 1 \end{bmatrix}.$$

The determinant looks a bit like a Wronskian. Have you found a contradiction? Why or why not?

3.3. Given that

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{4t} \begin{bmatrix} -3 \\ 3 \end{bmatrix} \tag{3.4}$$

and

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{4t} \begin{bmatrix} -3t+1 \\ 3t \end{bmatrix} \tag{3.5}$$

are solutions of the system

$$\begin{cases} x' = x - 3y, \\ y' = 3x + 7y, \end{cases}$$

determine whether these solutions are linearly independent.

3.4. Check that the functions given in (3.4) and (3.5) really are solutions of the system in exercise 3.3.

4 Case 1: A has two real distinct eigenvalues

If the matrix A has two real distinct eigenvalues, $\lambda_1 \neq \lambda_2$, then the corresponding eigenfunctions $\mathbf{u}, \mathbf{v}$ are linearly independent, and so are the functions

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}, \qquad \mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}$$

since for each t they are multiples of eigenvectors of the corresponding eigenvalues, hence still eigenvectors. Thus, we have the following:

Fact. If the eigenvalues λ_1, λ_2 of A are real and distinct, with respective eigenvectors $\mathbf{u}, \mathbf{v}$, then the general real-valued solution of $\mathbf{x}' = A\mathbf{x}$ is defined on $\mathbb{R}$ and given as

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{u} + c_2 e^{\lambda_2 t}\mathbf{v} \tag{4.1}$$

where c_1, c_2 are any real numbers. For any given vector $\mathbf{x}_0$ and $t_0 \in \mathbb{R}$ there is a unique solution satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$.

Example. Find the general solution of the system of differential equations

$$\begin{cases} x' = 3x - y, \\ y' = 4x - 2y. \end{cases} \tag{4.2}$$

Also, find the solution that satisfies the initial condition

$$x(0) = 4, \quad y(0) = 3. \tag{4.3}$$

The coefficient matrix

$$A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \tag{4.4}$$

has eigenvalues 2 and -1. The real eigenvectors associated to the eigenvalue 2 all have the form

$$\mathbf{u} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix},$$

and the eigenvectors associated to the eigenvalue -1 all have the form

$$\mathbf{v} = \begin{bmatrix} \alpha \\ 4\alpha \end{bmatrix},$$

where α is a non-zero real number. Taking, for example, $\alpha = 1$, we see that

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

are eigenvectors associated to the eigenvalues 2 and -1, respectively. Thus, by (4.1), the general solution of (4.2) is

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \tag{4.5}$$

where c_1 and c_2 are arbitrary constants.

We next choose the unique c_1 and c_2 that ensure the initial condition (4.3) is satisfied. At $t = 0$, equation (4.5) becomes

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

This vector equation is equivalent to the system of equations

$$\begin{cases} 4 = c_1 + c_2, \\ 3 = c_1 + 4c_2. \end{cases} \tag{4.6}$$

By solving (4.6), we obtain $c_1 = \frac{13}{3}$ and $c_2 = \frac{-1}{3}$. Therefore, the unique solution of the initial value problem (4.2)–(4.3) is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{13}{3} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{3} e^{-t} \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

or, equivalently,

$$\begin{cases} x(t) = \dfrac{13}{3} e^{2t} - \dfrac{1}{3} e^{-t}, \\[2ex] y(t) = \dfrac{13}{3} e^{2t} - \dfrac{4}{3} e^{-t}. \end{cases} \qquad \square$$

Homework Assignments

4.1. Find the general solution of the system

$$\begin{cases} x' = 4x + y \\ y' = 3x + 2y. \end{cases}$$

4.2. Find the solution of the system

$$\begin{cases} x' = 3x + 2y \\ y' = x + 2y \end{cases}$$

with $x(2) = 3$ and $y(2) = 1$.

4.3. Find the general solution of the system

$$\begin{cases} x' = 2x + y \\ y' = x + y. \end{cases}$$

What are the possible behaviors of a solution of the system as $t \to \infty$?

4.4. (i) Rewrite the linear second order equation

$$2x'' - x' - 6x = 0 \tag{4.7}$$

as a linear 2×2 system.

(ii) Show that the roots of the characteristic equation of (4.7) are the same as the eigenvalues of the corresponding linear 2×2 system.

(iii) Find the general solution of (4.7) by the method of Chapter 3 in Boyce and DiPrima.

(iv) Find the general solution of the linear 2×2 system corresponding to (4.7) and reconcile it to your answer to (iii).

4.5. For the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

describe the behavior of a solution as $t \to \infty$ without finding the general solution.

5 Case 2: A has two complex conjugate eigenvalues

In this case A has eigenvalues $\lambda_1 = \sigma + i\mu$ and $\lambda_2 = \sigma - i\mu$, for some real numbers σ, μ, with $\mu \neq 0$. Recall that given a complex number $z = a + ib$ its conjugate is the number $\overline{z} = a - ib$. Hence , $\lambda_2 = \overline{\lambda_1}$.

An eigenvector $\mathbf{u}$ for λ_1 will have the form

$$\mathbf{u} = \begin{bmatrix} \alpha_1 + i\beta_1 \\ \alpha_2 + i\beta_2 \end{bmatrix}$$

for some real numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$.

There is an easy relation between the eigenvectors of λ_1 and those of λ_2 :

Fact. If $\mathbf{u} = \begin{bmatrix} \alpha_1 + i\beta_1 \\ \alpha_2 + i\beta_2 \end{bmatrix}$ is an eigenvector for λ_1 then the vector

$$\overline{\mathbf{u}} = \begin{bmatrix} \alpha_1 - i\beta_1 \\ \alpha_2 - i\beta_2 \end{bmatrix}$$

is an eigenvector for $\lambda_2 = \sigma - i\mu$.

In other words, to obtain the eigenvector of the conjugate of an eigenvalue having eigenvector $\mathbf{u}$, we just conjugate the components of $\mathbf{u}$ (this is only true if the entries of A are real, as in our case.)

Knowing some algebraic properties of complex numbers can be very helpful for these types of computations. In particular, we are already familiar with the concept of the real and imaginary parts of a complex number. Consider now the complex vector $\mathbf{u}$ above. If we break each of its components into their real and imaginary parts, we get

$$\mathbf{u} = \begin{bmatrix} \alpha_1 + i\beta_1 \\ \alpha_2 + i\beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} i\beta_1 \\ i\beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + i \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

Notice that the two vectors on the right-hand side above are both in $\mathbb{R}^2$. In analogy to the scalar case, we say that the real part of $\mathbf{u}$ is $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ and the imaginary part of $\mathbf{u}$ is $\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$.

By the general theory, we know that the vectors $\mathbf{u}$ and $\overline{\mathbf{u}}$ are linearly independent, and so are the complex-valued functions

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}, \qquad \mathbf{x}_2(t) = e^{\lambda_2 t}\overline{\mathbf{u}}.$$

Thus, we obtain

Fact. If the eigenvalues λ_1, λ_2 of A are complex conjugate with respective eigenvectors $\mathbf{u}, \overline{\mathbf{u}}$, then the general complex-valued solution of $\mathbf{x}' = A\mathbf{x}$ is defined on $\mathbb{R}$ and given as

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{u} + c_2 e^{\lambda_2 t}\overline{\mathbf{u}} \tag{5.1}$$

where c_1, c_2 are any complex numbers. For any given complex vector $\mathbf{x}_0$ and $t_0 \in \mathbb{R}$ there is a unique solution satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$.

If we are only interested in real-valued solutions then the following result is true:

Fact. If the eigenvalues λ_1, λ_2 of A are complex conjugate and u is an eigenvector of λ_1, then the general real-valued solution of $\mathbf{x}' = A\mathbf{x}$ is defined on $\mathbb{R}$ and given as

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \tag{5.2}$$

where $\mathbf{x}_1(t)$ is the real part of $e^{\lambda_1 t}\mathbf{u}$, and $x_2(t)$ is the imaginary part of $e^{\lambda_1 t}\mathbf{u}$, and where c_1, c_2 are any real numbers. For any given real vector $\mathbf{x}_0$ and $t_0 \in \mathbb{R}$ there is a unique solution satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$.

If

$$\lambda_1 = \sigma + i\mu, \qquad \mathbf{u} = \begin{bmatrix} \alpha_1 + i\beta_1 \\ \alpha_2 + i\beta_2 \end{bmatrix},$$

are the eigenvalue and a corresponding eigenvector, then the solution in (5.2) can be computed explicitly using Euler's formula. Note that

$$\begin{aligned} e^{\lambda_1 t}\mathbf{u} = e^{(\sigma+i\mu)t} \begin{bmatrix} \alpha_1 + i\beta_1 \\ \alpha_2 + i\beta_2 \end{bmatrix} &= e^{\sigma t}(\cos(\mu t) + i\sin(\mu t)) \left(\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + i \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right) \\ &= e^{\sigma t}\left(\cos(\mu t) \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} - \sin(\mu t) \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right) + ie^{\sigma t}\left(\sin(\mu t) \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \cos(\mu t) \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right), \end{aligned}$$

which gives the real and imaginary parts of $e^{\lambda_1 t}\mathbf{u}$. It is possible to show that such real and imaginary parts of the complex-valued solution $e^{\lambda_1 t}\mathbf{u}$ are still solutions, and that they are also linearly independent. Therefore, the general *real-valued* solution in this case is given by

$$\begin{aligned} \mathbf{x}(t) = c_1 e^{\sigma t}&\left(\cos(\mu t) \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} - \sin(\mu t) \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right) \\ &+ c_2 e^{\sigma t}\left(\sin(\mu t) \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \cos(\mu t) \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right), \end{aligned} \tag{5.3}$$

where the constants c_1 and c_2 are *real numbers* that are determined by initial conditions.

It can be somewhat difficult to remember (5.3) as it's quite long. But, keep in mind that it comes from taking the real and imaginary parts of the complex solution $e^{\lambda_1 t}\mathbf{u}$. Rather than memorize (5.3), an alternative strategy is to first compute this complex-valued solution, then evaluate its real and imaginary parts directly.

To see how one does this, let $z = z_1 + iz_2$ be any complex number. Writing out z and $\mathbf{u}$ in terms of their real and imaginary parts, the product $z\mathbf{u}$ becomes

$$z\mathbf{u} = (z_1 + iz_2) \begin{bmatrix} \alpha_1 + i\beta_1 \\ \alpha_2 + i\beta_2 \end{bmatrix} = (z_1 + iz_2) \left(\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + i \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right).$$

If we now distribute and group together the terms with an i coefficient and those without, we find that

$$z\mathbf{u} = z_1 \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} - z_2 \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + i\left(z_2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + z_1 \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right).$$

So, in total,

$$\text{real part } z\mathbf{u} = z_1 \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} - z_2 \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}, \qquad \text{imaginary part } z\mathbf{u} = z_2 \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} + z_1 \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}.$$

Setting $z = e^{\lambda_1 t}$, this leads to exactly the general solution formula found in (5.3).

Example. Let's compute the real and imaginary parts of the product

$$z\mathbf{u} = (2+3i)\begin{bmatrix}1-i\\ 2+i\end{bmatrix}.$$

Breaking up the complex vector into its real and imaginary parts, this becomes

$$z\mathbf{u} = (2+3i)\left(\begin{bmatrix}1\\ 2\end{bmatrix} + i\begin{bmatrix}-1\\ 1\end{bmatrix}\right),$$

and so distributing and regrouping yields

$$z\mathbf{u} = 2\begin{bmatrix}1\\ 2\end{bmatrix} - 3\begin{bmatrix}-1\\ 1\end{bmatrix} + i\left(3\begin{bmatrix}1\\ 2\end{bmatrix} + 2\begin{bmatrix}-1\\ 1\end{bmatrix}\right) = \begin{bmatrix}5\\ 1\end{bmatrix} + i\begin{bmatrix}1\\ 8\end{bmatrix}.$$

Thus, the real part of $z\mathbf{u}$ is $\begin{bmatrix}5\\ 1\end{bmatrix}$ and its imaginary part is $\begin{bmatrix}1\\ 8\end{bmatrix}$. □

Example. Find the general real-valued solution of the system

$$\begin{cases} x' = -x + 2y, \\ y' = -2x - y. \end{cases} \tag{5.4}$$

The coefficient matrix is

$$A = \begin{bmatrix}-1 & 2\\ -2 & -1\end{bmatrix}$$

and so the characteristic equation is given by

$$0 = |A - \lambda I| = \lambda^2 + 2\lambda + 5.$$

Using the quadratic formula to find the roots, we see that the eigenvalues of A are

$$\lambda_1 = -1 + 2i, \qquad \lambda_2 = -1 - 2i.$$

To find an eigenvector associated to the eigenvalue $\lambda_1 = -1 + 2i$, we must solve $(A - \lambda_1 I)\mathbf{u} = \mathbf{0}$ i.e.

$$\begin{bmatrix}-2i & 2\\ -2 & -2i\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix},$$

or equivalently, the system

$$\begin{cases} -2iu_1 + 2u_2 = 0, \\ -2u_1 - 2iu_2 = 0. \end{cases}$$

Both equations are equivalent to the single equation $u_2 = iu_1$, (multiply the first one by $-i$ to obtain the second one) which has infinitely many solutions. A convenient choice for a solution is $u_1 = 1$ and $u_2 = i$; it corresponds to the complex solution

$$e^{(-1+2i)t}\begin{bmatrix}1\\ i\end{bmatrix} = e^{-t}(\cos(2t) + i\sin(2t))\begin{bmatrix}1\\ i\end{bmatrix} = e^{-t}\begin{bmatrix}\cos 2t\\ -\sin 2t\end{bmatrix} + ie^{-t}\begin{bmatrix}\sin 2t\\ \cos 2t\end{bmatrix}.$$

of the homogeneous system. A fundamental set of real solutions is given by the real and imaginary parts of this complex solution. In fact, these solutions are

$$e^{-t}\begin{bmatrix}\cos 2t\\ -\sin 2t\end{bmatrix}, \qquad e^{-t}\begin{bmatrix}\sin 2t\\ \cos 2t\end{bmatrix}.$$

Thus, the general solution of (5.4) is

$$\begin{bmatrix}x(t)\\ y(t)\end{bmatrix} = c_1e^{-t}\begin{bmatrix}\cos 2t\\ -\sin 2t\end{bmatrix} + c_2e^{-t}\begin{bmatrix}\sin 2t\\ \cos 2t\end{bmatrix}.$$ □

Homework Assignments

5.1. Find the real and imaginary parts of the following complex vectors:

(i) $(1+i)\begin{bmatrix}2+i\\ 3-2i\end{bmatrix}$.

(ii) $3i\begin{bmatrix}4i\\ -1+i\end{bmatrix}$.

5.2. Find the general solution of the system

$$\begin{cases} x' = -x + y\\ y' = -4x - y. \end{cases}$$

5.3. Solve the initial value problem

$$\begin{cases} x' = x + 4y\\ y' = -25x + y \end{cases}$$

with the initial condition

$$x(0) = 2000, \quad y(0) = 5.$$

5.4. Find the general solution of the system

$$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix}-4 & 10\\ -5 & 6\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}.$$

Describe all possible behaviors of solutions as $t \to \infty$.

5.5. The motion of a spring-mass system is described by the initial value problem

$$\begin{cases} mu'' + \gamma u' + ku = 0 \\ u(t_0) = u_0, \quad u'(t_0) = v_0 \end{cases} \tag{5.5}$$

where m, γ, k are the mass, damping constant and spring constant, respectively, of the system (see Boyce and DiPrima, Section 3.8). We assume that these quantities m, γ and k are all positive and that

$$\gamma < 2\sqrt{mk}.$$

(i) Rewrite (5.5) as an initial value problem of an equivalent linear 2×2 system of differential equations.

(ii) Find the general solution of the 2×2 system in (i) (i.e. ignore the initial conditions).

(iii) Describe the behavior of the solution of the initial value problem of (i) as $t \to \infty$.

5.6. Find the general solution of the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and describe the behaviors of its solutions as $t \to \infty$.

6 Case 3: A has only one real eigenvalue

Suppose that λ is the unique eigenvalue of A. In this case there is a possibility that every vector is an eigenvector, but this can only happen if

$$A = aI = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

for some real number a. Indeed we have $|A - \lambda I| = (a - \lambda)^2 = 0$ only when $\lambda = a$. This means that the equation $(A - aI)\mathbf{u} = \mathbf{0}$ is satisfied for all $\mathbf{u}$, since $A - aI = 0$, the zero matrix. Each vector in the plane is then an eigenvector for A. So, if we pick

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which are clearly linearly independent, we get the general solution

$$\mathbf{x}(t) = c_1 e^{at} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{at} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^{at} \\ c_2 e^{at} \end{bmatrix}.$$

That is

$$x(t) = c_1 e^{at}, \qquad y(t) = c_2 e^{at}.$$

This is not surprising since the original system is

$$\begin{cases} x' = ax \\ y' = ay \end{cases} \tag{6.1}$$

both equations of which can be solved separately in the usual way (this is not really a genuine system, since the equations are not related to one another).

Aside from this trivial case, no pair of linearly independent eigenvectors can exist, and we must use a different method:

Fact. Suppose that A is not of type aI for some a. If λ is the only real eigenvalue of A, and $\mathbf{u}$ an eigenvector, then the general real-valued solution of $\mathbf{x}' = A\mathbf{x}$ is given as

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{u} + c_2 e^{\lambda t}(t\mathbf{u} + \mathbf{v}) \tag{6.2}$$

where $\mathbf{v}$ is a solution of the linear system $(A - \lambda I)\mathbf{v} = \mathbf{u}$, and c_1, c_2 any real numbers.

The main idea here is this one: we know that $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{u}$ is a solution of $\mathbf{x}' = A\mathbf{x}$. If we assume that $(A - \lambda I)\mathbf{v} = \mathbf{u}$, then one can show that the function $\mathbf{x}_2(t) = e^{\lambda t}(t\mathbf{u} + \mathbf{v})$ is also a solution and it is linearly independent from $\mathbf{x}_1(t)$. We can check that indeed $\mathbf{x}_2$ is a solution: Assuming $(A - \lambda I)\mathbf{v} = \mathbf{u}$, that is $A\mathbf{v} - \lambda\mathbf{v} = \mathbf{u}$ we get

$$\mathbf{x}_2'(t) = \lambda e^{\lambda t}(t\mathbf{u} + \mathbf{v}) + e^{\lambda t}\mathbf{u}$$

and

$$A\big(e^{\lambda t}(t\mathbf{u} + \mathbf{v})\big) = e^{\lambda t}tA\mathbf{u} + e^{\lambda t}A\mathbf{v} = e^{\lambda t}t\lambda\mathbf{u} + e^{\lambda t}(\mathbf{u} + \lambda\mathbf{v}) = \mathbf{x}_2'(t).$$

Example. Find the general solution of the system

$$\begin{cases} x' = -4x - y, \\ y' = 4x - 8y. \end{cases} \tag{6.3}$$

First find the eigenvalues of the coefficient matrix

$$A = \begin{bmatrix} -4 & -1 \\ 4 & -8 \end{bmatrix} \tag{6.4}$$

by solving the quadratic equation

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -4-\lambda & -1 \\ 4 & -8-\lambda \end{vmatrix} \\ &= \lambda^2 + 12\lambda + 36 \\ &= (\lambda + 6)^2 \\ &= 0. \end{aligned} \tag{6.5}$$

In this case $\lambda = -6$ is the only eigenvalue of A. To find the associated eigenvectors, solve the equation

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (A - (-6)I) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{6.6}$$

or equivalently, the single equation

$$2u_1 - u_2 = 0. \tag{6.7}$$

All the real eigenvectors associated to -6 are of the form $\begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix}$, where α can be any non-zero real number. In particular, with $\alpha = 1$, the vector

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is an eigenvector associated to -6. To find a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ so that $A(+6I)\mathbf{v} = \mathbf{u}$ we solve

$$(A + 6I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \tag{6.8}$$

or equivalently,

$$2v_1 - v_2 = 1. \tag{6.9}$$

We can set for example $v_1 = 1$ and find $v_2 = 2v_1 - 1 = 1$, so that $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, by (6.2), the general solution of (6.3) is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t} \left(t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right). \qquad \square$$

Homework Assignments

Find the general solution of the following systems:

6.1.

$$\begin{cases} x' = 4x - y \\ y' = x + 2y. \end{cases}$$

6.2.

$$\begin{cases} x' = -3x - 8y \\ y' = 2x + 5y. \end{cases}$$

6.3.

$$\begin{cases} x' = -\dfrac{1}{2}x + \dfrac{1}{2}y \\ y' = -\dfrac{9}{2}x - \dfrac{7}{2}y. \end{cases}$$

6.4.

$$\begin{cases} x' = 3x \\ y' = 3y. \end{cases}$$

Find the solution to the initial value problems:

6.5.

$$\begin{cases} x' = \dfrac{1}{2}x + \dfrac{1}{2}y \\ y' = -2x - \dfrac{3}{2}y, \end{cases}$$

with

$$x(0) = -5, \qquad y(0) = 6.$$

6.6.

$$\begin{cases} x' = 5x + y \\ y' = -4x + y, \end{cases}$$

with

$$x(0) = 4, \qquad y(0) = 2.$$

6.7. Find all the (real) values of s so that every solution of the system

$$\begin{cases} x' = sx - y \\ y' = x + (2+s)y \end{cases}$$

satisfies

$$\lim_{t\to\infty} x(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} y(t) = 0.$$

7 Solutions of Nonhomogeneous Systems

In this section we consider nonhomogeneous linear 2×2 systems of the form

$$\begin{cases} x' = ax + by + f(t), \\ y' = cx + dy + g(t), \end{cases} \tag{7.1}$$

where the functions f and g are continuous on an open interval I. In our matrix notation such systems can be written compactly in the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t), \tag{7.2}$$

where as usual

$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and now

$$\mathbf{F}(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

is a vector-valued nonhomogeneous term. Note that, to keep notation concise, the dependence of x, y, x', and y' on t has been suppressed.

Fact (Existence and uniqueness of solutions of initial value problems).
Given $t_0 \in I$ and any vector $\mathbf{x}_0 \in \mathbb{R}^2$, there is one and only one solution to the system (7.1) that satisfies the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

The procedure by which one finds a solution of (7.1) is the same as the one we used for second order linear equations: first one finds the general solution of the associated homogeneous system, then a particular solution of (7.1). The precise formulation of the method is contained in the following statement:

Fact. Suppose that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two linearly independent solutions of the homogeneous system associated with (7.1), that is,

$$\mathbf{x}' = A\mathbf{x}, \tag{7.3}$$

and suppose that $\mathbf{Y}(t)$ is any particular solution of the nonhomogeneous problem (7.1). Then *every* solution of (7.1) on the interval I can be written in the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \mathbf{Y}(t) \tag{7.4}$$

for some real constants c_1 and c_2.

Notice that the first two terms of (7.4) form the general solution of the corresponding homogeneous 2×2 system (7.3) associated with (7.1). Hence (7.4) tells us that the general solution of the nonhomogeneous system (7.1) is the sum of a particular solution and the general solution of the corresponding homogeneous system. Since we already know how to find the general solution of (7.3), if we can find a particular solution of (7.1), then we can construct the general solution of (7.1). In this section we will learn a method, called *variation of parameters* or *variation of constants* that gives an explicit formula for such a particular solution $\mathbf{Y}$ to (7.1). This is of course a generalization of the similarly titled method we have already encountered in our earlier studied of linear second order ODEs.

Description of the Method. Using the methods in Sections 4, 5 and 6, find two linearly independent solutions

$$\mathbf{x}_1(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

of the homogeneous system (7.3) corresponding to (7.1), and form the matrix

$$\Phi(t) = \big[\mathbf{x}_1(t) \mid \mathbf{x}_2(t)\big] = \begin{bmatrix} u_1(t) & v_1(t) \\ u_2(t) & v_2(t) \end{bmatrix}. \tag{7.5}$$

In general, a matrix of functions is called a *fundamental matrix* for the homogeneous system corresponding to (7.3) if its columns are linearly independent solutions of (7.3). (There are infinitely many fundamental matrices, depending on which two linearly independent solutions we use.) Because $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, so are the columns of $\Phi(t)$, which is therefore an invertible matrix.

Variation of parameters formula. If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent solutions of (7.3) and $\Phi(t)$ is the fundamental matrix defined in (7.5) then a particular solution $\mathbf{Y}(t)$ of (7.1) is given by

$$\mathbf{Y}(t) = \Phi(t) \int \Phi(t)^{-1}\mathbf{F}(t)\, dt. \tag{7.6}$$

The notation on the right-hand side of equation (7.6) is explained as follows.

1. Compute the vector $\Phi(t)^{-1}\mathbf{F}(t)$ expliciltly, and call its components $\psi_1(t)$, $\psi_2(t)$, that is

$$\begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) & v_1(t) \\ u_2(t) & v_2(t) \end{bmatrix}^{-1} \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

2. Find an antiderivative of the vector function $\Phi(t)^{-1}\mathbf{F}(t)$ by computing

$$\int \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} dt = \begin{bmatrix} \displaystyle\int \psi_1(t)\, dt \\ \displaystyle\int \psi_2(t)\, dt \end{bmatrix}. \tag{7.7}$$

   For example, if we find that $\psi_1(t) = 2t$ and $\psi_2(t) = \cos t$ then

$$\int \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} dt = \int \begin{bmatrix} 2t \\ \cos t \end{bmatrix} dt = \begin{bmatrix} \int 2t\, dt \\ \int \cos t\, dt \end{bmatrix} = \begin{bmatrix} t^2 \\ \sin t \end{bmatrix}.$$

3. Multiply our choice of antiderivative by the fundamental matrix $\Phi(t)$.

Explanation. We have

$$\Phi'(t) = A\Phi(t), \tag{7.8}$$

where A is the coefficient matrix. Let us look for a particular solution of (7.1) in the form $\mathbf{Y}(t) = \Phi(t)\mathbf{w}(t)$ where

$$\mathbf{w}(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

and the functions $w_1(t)$ and $w_2(t)$ are to be determined. Choose t_0 in the interval I. Let us determine the solution of (7.1) that vanishes at t_0. Using the product rule for differentiation, we have

$$\mathbf{Y}'(t) = \Phi'(t)\mathbf{w}(t) + \Phi(t)\mathbf{w}'(t) = A\Phi(t)\mathbf{w}(t) + \Phi(t)\mathbf{w}'(t). \tag{7.9}$$

Because we wish for $\mathbf{Y}$ to be a particular solution of (7.1), we should then have

$$\mathbf{Y}'(t) = A\mathbf{Y}(t) + \mathbf{F}(t) = A\Phi(t)\mathbf{w}(t) + \mathbf{F}(t).$$

Substituting this into (7.9) then yields

$$\Phi(t)\mathbf{w}'(t) = \mathbf{F}(t)$$

and multiplying both sides by $\Phi(t)^{-1}$ on the left gives

$$\mathbf{w}'(t) = \Phi(t)^{-1}\mathbf{F}(t).$$

Finally, we antidifferentiate both sides to get:

$$\mathbf{w}(t) = \int \Phi(t)^{-1}\mathbf{F}(t)\,dt.$$

But recall that we started with $\mathbf{Y} = \Phi(t)\mathbf{w}(t)$, and so using the expression for $\mathbf{w}$, we arrive at the variation of constants formula in (7.6).

Example. Solve the initial value problem

$$\begin{cases} x' = 3x - y + t, \\ y' = 4x - 2y - 2, \\ x(0) = 0, \quad y(0) = 1. \end{cases} \tag{7.10}$$

The matrix associated to the above system is

$$A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \tag{7.11}$$

and the forcing function

$$\mathbf{F}(t) = \begin{bmatrix} t \\ -2 \end{bmatrix}.$$

The homogeneous system associated to A was already solved in an example of Section 2: the matrix A has eigenvalues 2 and -1, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Hence

$$\mathbf{x}_1(t) = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

are linearly independent solutions of the homogeneous system corresponding to (7.10). Thus, by (7.5), a fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^{2t} & e^{-t} \\ e^{2t} & 4e^{-t} \end{bmatrix}, \tag{7.12}$$

and its inverse is computed using formula (2.10):

$$\Phi(t)^{-1} = \frac{1}{3e^t}\begin{bmatrix} 4e^{-t} & -e^{-t} \\ -e^{2t} & e^{2t} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4e^{-2t} & -e^{-2t} \\ -e^{t} & e^{t} \end{bmatrix}. \tag{7.13}$$

Thus,

$$\begin{aligned} \int \Phi(t)^{-1}\begin{bmatrix} t \\ -2 \end{bmatrix} dt &= \frac{1}{3}\int \begin{bmatrix} 4e^{-2t} & -e^{-2t} \\ -e^{t} & e^{t} \end{bmatrix}\begin{bmatrix} t \\ -2 \end{bmatrix} dt \\ &= \frac{1}{3}\int \begin{bmatrix} 4te^{-2t} + 2e^{-2t} \\ -te^{t} - 2e^{t} \end{bmatrix} dt \\ &= \frac{1}{3}\begin{bmatrix} \int (4t+2)e^{2t}\,dt \\ \int -(t+2)e^{t}\,dt \end{bmatrix}. \end{aligned} \tag{7.14}$$

To compute one antiderivative of each integral use integration by parts:

$$\begin{aligned} \int (4t+2)e^{-2t}\,dt &= -\frac{1}{2}(4t+2)e^{-2t} + \frac{1}{2}\int 4e^{-2t}\,dt = -(2t+1)e^{2t} - e^{2t} \\ &= -(2t+2)e^{-2t}, \end{aligned}$$

and

$$\begin{aligned} \int -(t+2)e^{t}\,dt &= -(t+2)e^{-t} + \int e^{-t} = -(t+2)e^{-t} - e^{-t} \\ &= -(t+1)e^{t}. \end{aligned}$$

This means that

$$\int \Phi(t)^{-1}\begin{bmatrix} t \\ -2 \end{bmatrix} dt = \frac{1}{3}\begin{bmatrix} -(2t+2)e^{-2t} \\ -(t+1)e^{t} \end{bmatrix}.$$

Using the variation of constants formula (7.6), we see that a particular solution of the inhomogeneous system (7.10) is given by

$$\begin{aligned} \mathbf{Y}(t) = \Phi(t)\int \Phi(t)^{-1}\begin{bmatrix} t \\ -2 \end{bmatrix} dt &= \begin{bmatrix} e^{2t} & e^{-t} \\ e^{2t} & 4e^{-t} \end{bmatrix}\frac{1}{3}\begin{bmatrix} -(2t+2)e^{-2t} \\ -(t+1)e^{t} \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} -(2t+2) - (t+1) \\ -(2t+2) - 4(t+1) \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} -3t-3 \\ -6t-6 \end{bmatrix} = \begin{bmatrix} -t-1 \\ -2t-2 \end{bmatrix}. \end{aligned} \tag{7.15}$$

By (7.4), the general solution of (7.10) is

$$\begin{aligned} \mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \mathbf{Y}(t) &= c_1e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} -t-1 \\ -2t-2 \end{bmatrix} \\ &= \begin{bmatrix} c_1e^{2t} + c_2e^{-t} - t - 1 \\ c_1e^{2t} + 4c_2e^{-t} - 2t - 2 \end{bmatrix}. \end{aligned} \tag{7.16}$$

To solve the initial value problem, set $t = 0$ and $x(0) = 0$ and $y(0) = 1$. Hence

$$\mathbf{x}(0) = \begin{bmatrix} c_1 + c_2 - 1 \\ c_1 + 4c_2 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{7.17}$$

or, equivalently,

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 + 4c_2 = 3. \end{cases} \tag{7.18}$$

It follows that $c_1 = \frac{1}{3}$ and $c_2 = \frac{2}{3}$. Substituting these into (7.16) we obtain the solution of the initial value problem:

$$\mathbf{x}(t) = \begin{bmatrix} \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} - t - 1 \\ \frac{1}{3}e^{2t} + \frac{8}{3}e^{-t} - 2t - 2 \end{bmatrix}. \qquad \square$$

Example. Let us solve the special case of the tank mixing problem (1.4) given by

$$\begin{cases} \dot{u} = -u + \dfrac{2}{9}v + \dfrac{1}{100}, \\ \dot{v} = 2u - 2v, \end{cases} \tag{7.19}$$

with initial conditions

$$u(0) = 1, \qquad v(0) = 0.$$

The original tank mixing problem (1.4) has several parameters. You might ask if the special form of (7.19) represents a reasonable example. You can skip this paragraph if you are interested only in the solution method. We will introduce a new (but easy) technique that might convince you that our special case is reasonable. We will also see that this technique tames the zoo of parameters in the general case. The idea here is to make the system *dimensionless* by a change of variables. This should always be done when solving real physical problems. The change of variables is of the form

$$u = \frac{x}{A}, \qquad v = \frac{y}{B}, \qquad s = \frac{t}{\lambda}$$

where A and B are measured in pounds and λ is measured in minutes. We start (using (1.4)) as follows:

$$\frac{du}{ds} = \frac{du}{dt}\frac{dt}{ds} = \frac{\lambda}{A}x' = \frac{\lambda}{A}\left(-\frac{Ad}{V_1}u + \frac{cB}{V_2}v + a\alpha\right).$$

After cleaning up the algebra and doing the same procedure for the variable v, we arrive at the system

$$\begin{cases} \dfrac{du}{ds} = -\dfrac{\lambda d}{V_1}u + \dfrac{cB\lambda}{AV_2}v + \dfrac{a\alpha\lambda}{A}, \\ \dfrac{dv}{ds} = \dfrac{A\lambda d}{BV_1}u - \dfrac{\lambda d}{V_2}v. \end{cases}$$

We now make some choices for the variables A, B and λ. To make the coefficient of u in the first equation -1, we choose $\lambda = V_1/d$. It is also convenient to choose

$A = \alpha V_1$ and $B = \alpha V_2$. Note that these quantities have the correct dimensions. After substitution into the system, we obtain the dimensionless system

$$\begin{cases} \dfrac{du}{ds} = -u + \beta v + \epsilon, \\ \dfrac{dv}{ds} = \gamma u - \gamma v, \end{cases} \tag{7.20}$$

where

$$\beta = \frac{c}{d}, \qquad \epsilon = \frac{a}{d}, \qquad \gamma = \frac{V_1}{V_2}.$$

It is now clear that the important parameters are the dimensionless ratios given in the last display, which makes perfect sense! In a real physical application, we must remember also to change the initial conditions to dimensionless form. Note that we can recover the solution in the original variables from a solution of the dimensionless system by a change of variables. After this explanation, it should be obvious that system (7.19) is a reasonable special case. Of course, system (7.20) is also in the best form for a more complete analysis of the original system. You are invited to find the general solution of the dimensionless system.

We will solve system (7.19) using variation of parameters. The coefficient matrix for the homogeneous system is

$$\begin{bmatrix} -1 & \frac{2}{9} \\ 2 & -2 \end{bmatrix}.$$

Its eigenvalues are $-7/3$ and $-2/3$, and its corresponding eigenvectors are

$$\begin{bmatrix} -\frac{1}{6} \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}.$$

Using these results, we have—by choosing the columns to be independent solutions—the fundamental matrix given by

$$\Phi(t) = \begin{bmatrix} -\frac{1}{6}e^{-7t/3} & \frac{2}{3}e^{-2t/3} \\ e^{-7t/3} & e^{-2t/3} \end{bmatrix}$$

and its inverse

$$\Phi(t)^{-1} = \begin{bmatrix} -\frac{6}{5}e^{7t/3} & \frac{4}{5}e^{7t/3} \\ \frac{6}{5}e^{2t/3} & \frac{1}{5}e^{2t/3} \end{bmatrix}.$$

By the variation of parameters formula

$$\begin{aligned} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} &= \Phi(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \Phi(t) \int \Phi(t)^{-1} \begin{bmatrix} \frac{1}{100} \\ 0 \end{bmatrix} dt \\ &= \begin{bmatrix} -\frac{1}{6}e^{-7t/3} & \frac{2}{3}e^{-2t/3} \\ e^{-7t/3} & e^{-2t/3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} -\frac{1}{6}e^{-7t/3} & \frac{2}{3}e^{-2t/3} \\ e^{-7t/3} & e^{-2t/3} \end{bmatrix} \int \begin{bmatrix} -\frac{6}{5}e^{7t/3} & \frac{4}{5}e^{7t/3} \\ \frac{6}{5}e^{2t/3} & \frac{1}{5}e^{2t/3} \end{bmatrix} \begin{bmatrix} \frac{1}{100} \\ 0 \end{bmatrix} dt. \end{aligned}$$

After a computation, we obtain

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{6}e^{-7t/3} & \frac{2}{3}e^{-2t/3} \\ e^{-7t/3} & e^{-2t/3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{9}{700} \\ \frac{9}{700} \end{bmatrix}.$$

We can now impose the initial conditions at $t = 0$. This results in the linear system of algebraic equations

$$\begin{cases} -\dfrac{1}{6}c_1 + \dfrac{2}{3}c_2 + \dfrac{9}{700} = 1 \\ \qquad c_1 + c_2 + \dfrac{9}{700} = 0, \end{cases}$$

which has the solution $c_1 = -2091/1750$ and $c_2 = 591/500$. The solution of the initial value problem is thus

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{6}e^{-7t/3} & \frac{2}{3}e^{-2t/3} \\ e^{-7t/3} & e^{-2t/3} \end{bmatrix} \begin{bmatrix} \frac{-2091}{1750} \\ \frac{591}{500} \end{bmatrix} + \begin{bmatrix} \frac{9}{700} \\ \frac{9}{700} \end{bmatrix}.$$

Note that in the long run (for $t \to \infty$), the system goes to a steady state $(u, v) = (9/700, 9/700)$. Can you show this result without solving the system? Can you find the solution of system (7.19) without using variation of parameters? Hint: The general solution of a linear system is always given by the general homogeneous solution plus a particular solution. □

Homework Assignments

Find the solutions to the initial value problems:

7.1.

$$\begin{cases} x' = 4x - 6y + 10 \\ y' = x - y, \end{cases}$$

with

$$x(0) = 0, \qquad y(0) = 0.$$

7.2.

$$\begin{cases} x' = 2y + e^t \\ y' = -x - 3y + 3e^t, \end{cases}$$

with

$$x(0) = 0, \qquad y(0) = 1.$$

Find the general solution of the following systems:

7.3.

$$\begin{cases} x' = x - y \\ y' = 3x + 5y + t. \end{cases}$$

7.4.

$$\begin{cases} x' = -y \\ y' = x + \cos(t). \end{cases}$$

7.5.

$$\begin{cases} x' = 4x - 2y + 8 \\ y' = 6x - 4y + 2e^t. \end{cases}$$

7.6.
Find the general solution of the following system for $t > 0$.

$$\begin{cases} x' = y \\ y' = -4x - 4y + t^{-2}e^{-2t}. \end{cases}$$

8 Qualitative Methods

Suppose a point is moving on the plane and its position coordinates x and y, which are functions of time t, satisfy the equations in the system

$$\begin{aligned} x' &= ax + by, \\ y' &= cx + dy, \end{aligned} \tag{8.1}$$

or equivalently,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{8.2}$$

In this section we are interested in the geometric behavior of all such parametric curves, which are called trajectories of the system (8.1). For simplicity we will assume that the coefficient matrix of our homogeneous linear system has only non-zero eigenvalues.

The essential idea that connects the system (8.1) to the geometry of its solutions is very simple. A solution of our system of differential equations can be viewed as a parametric curve in the plane whose position vector at time t is

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

The left-hand side of (8.2), namely $\mathbf{x}'(t)$, is the velocity of the moving point at time t. The differential equation states that this velocity is given as a function of the position; that is, the velocity at time t is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{8.3}$$

Thus, if we were to plot the vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{8.4}$$

(translated so that its tail is at the point (x, y)) for each point (x, y) of the plane, then the solutions of the differential equation are exactly those parametric curves that pass through each point (x, y) with velocity vector (8.4).

Using a computer with software such as Mathematica, Maple, or Matlab (or with pencil and paper), we can plot the vector field described in the last paragraph at a finite number of points. We can also plot a few of the solutions of the corresponding system of differential equations to illustrate their geometry. Such a picture of the trajectories of a system of differential equations is called a *phase portrait*; it should have enough trajectories plotted so that we can tell at a glance the geometry of all trajectories.

Example. We will illustrate the connection between the vector field defined by the right-hand side and the solutions of the system

$$\begin{cases} x' = y, \\ y' = 2x. \end{cases} \tag{8.5}$$

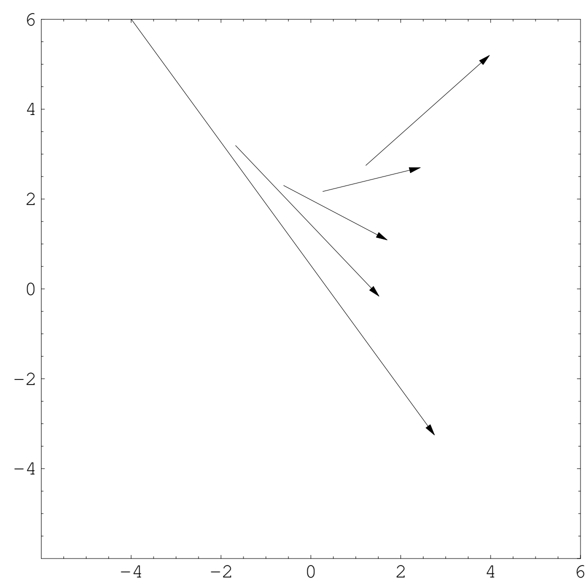

Figure 8.1: Five vectors generated by the right-hand side of the system (8.5)

Figure 8.1 is the plot of exactly five velocity vectors in the plane generated from the right-hand side of the system (8.5). Note that if the tail is at (x, y), then the head is at the point $(x, y) + (y, 2x)$. For example, one of the vectors has tail at (approximately) $(1.22, 2.75)$ and head at the point $(3.97, 5.19)$.

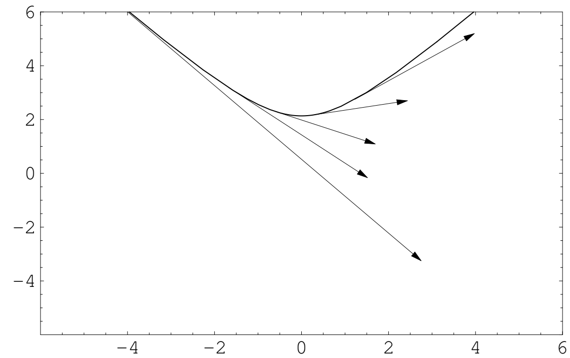

Figure 8.2: The five vectors generated by the right-hand side of the system (8.5) together with the trajectory starting at the point $(-6, 8.75)$ at time $t = 0$.

Figure 8.2 depicts the vectors in Figure 8.1 together with the trajectory of the system (8.5) with the initial conditions $x(0) = -6$ and $y(0) = 8.75$. Note that the velocity vectors along the trajectory depicted in Figure 8.2 have different lengths, which correspond to the speed of the particle at different points. The speed of the particle might be important for some purposes, but it is not relevant to the phase portrait, which is meant to show the qualitative behavior of the solutions of the system of differential equations. Thus, when we draw the vector field, it is preferable to draw only the direction field; that is, all the vectors are taken to have the same length.

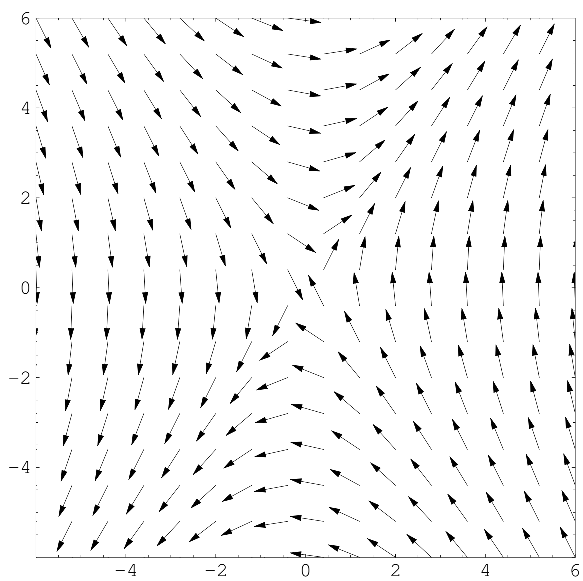

Figure 8.3: The direction field on a 16×16 grid for the system (8.5).

A direction field plot that shows a grid of normalized velocity vectors (i.e. vectors of unit length in the directions of the corresponding velocity vectors) for the system (8.5) is depicted in Figure 8.3. This plot suggests the general qualitative behavior of the system: Most trajectories starting near the upper left move toward the origin for a while and eventually leave the depicted region in the direction of the upper right or lower left. Other trajectories enter from the lower right and leave the depicted region in the direction of the upper right or lower left. The different behaviors must be separated by some special trajectories. In this case the separating trajectories lie on the invariant lines through the origin determined by the eigenvectors of the coefficient matrix. A phase portrait of system (8.5) is depicted in Figure 8.4.

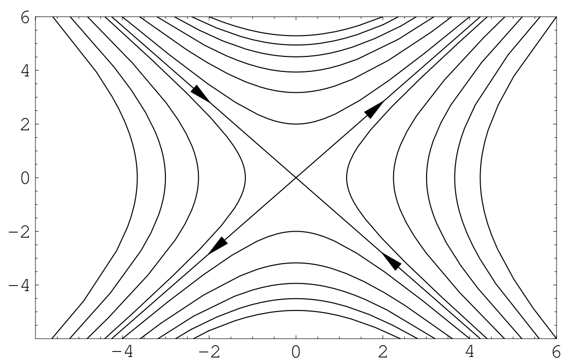

Figure 8.4: The phase portrait for the system (8.5).

Using the methods discussed in previous sections, we can draw the phase portrait by hand. To do so, let us first determine the eigenvalues and the associated eigenvectors for the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}. \tag{8.6}$$

The eigenvalues of A are $\pm\sqrt{2}$. Two associated eigenvectors are

$$\mathbf{u} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix},$$

respectively. The general solution is therefore

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-\sqrt{2}\,t} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} + c_2 e^{\sqrt{2}\,t} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}. \tag{8.7}$$

To obtain the phase portrait, we first sketch the trajectory for the solution with $c_1 = 1$ and $c_2 = 0$; that is, the solution

$$x(t) = e^{-\sqrt{2}\,t}, \qquad y(t) = -\sqrt{2}e^{-\sqrt{2}\,t}.$$

Notice that this solution lies on the straight line through the origin given by $y = -\sqrt{2}\,x$ and that as $t \to \infty$ we have $x(t) \to 0$ and $y(t) \to 0$. Moreover, for this

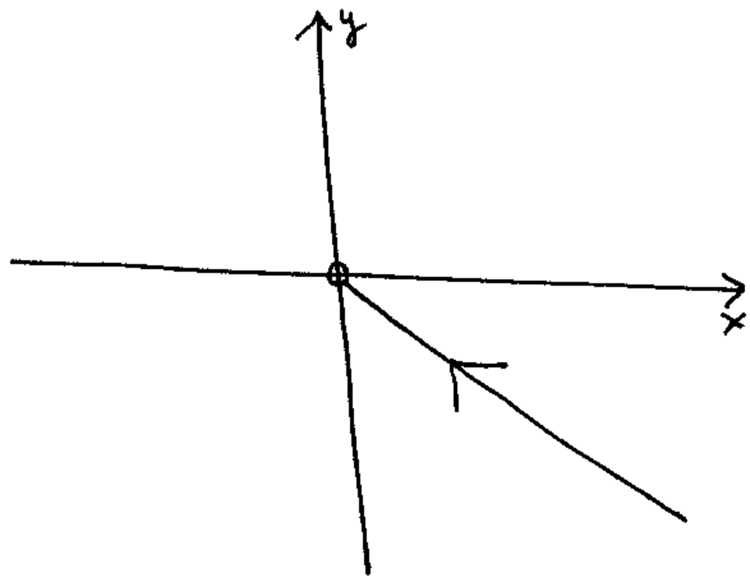

Figure 8.5: The trajectory for $x(t) = e^{-\sqrt{2}\,t}$, $y(t) = -\sqrt{2}e^{-\sqrt{2}\,t}$

solution $x(t)$ is always positive and $y(t)$ is always negative. The trajectory lies in the fourth quadrant and looks like the trajectory in Figure 8.5.

A similar method shows us that if exactly one of the parameters c_1 and c_2 are zero, then the trajectory will be a half-line. By adding the trajectories for the three cases (c_1, c_2) equal to $(-1, 0)$, $(0, 1)$ and $(0, -1)$ and the origin, which is itself a trajectory (Why?), we end up with Figure 8.6.

The trajectories in Figure 8.6 form the "skeleton" of our phase portrait. Note that straight line trajectories are easy to find. They are the lines passing through the origin in the directions of real eigenvectors corresponding to real eigenvalues. You might miss these special trajectories by using a direction field.

To fill in the rest of the phase portrait, you can draw a few more trajectories that indicate the behavior of solutions with initial conditions not on the skeleton. Notice, in this case, that as $t \to \infty$ the function $e^{-\sqrt{2}\,t} \to 0$. Thus the first term in the solution (8.7) becomes very small (and less important in the graph) as $t \to \infty$; that is,

$$c_1 e^{-\sqrt{2}\,t} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} + c_2 e^{\sqrt{2}\,t} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \sim c_2 e^{\sqrt{2}\,t} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}.$$

Similarly, for $t \to -\infty$, we have

$$c_1 e^{-\sqrt{2}\,t} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} + c_2 e^{\sqrt{2}\,t} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \sim c_1 e^{-\sqrt{2}\,t} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}.$$

The phase portrait should reflect this (see Figure 8.7).

The phase portrait (obtained with computer graphics) of this system depicted in Figure 8.4 is typical for all linear 2×2 systems where the eigenvalues are real, distinct, and of opposite sign. The origin $(0, 0)$ is called a *saddle point* in this case. Note that

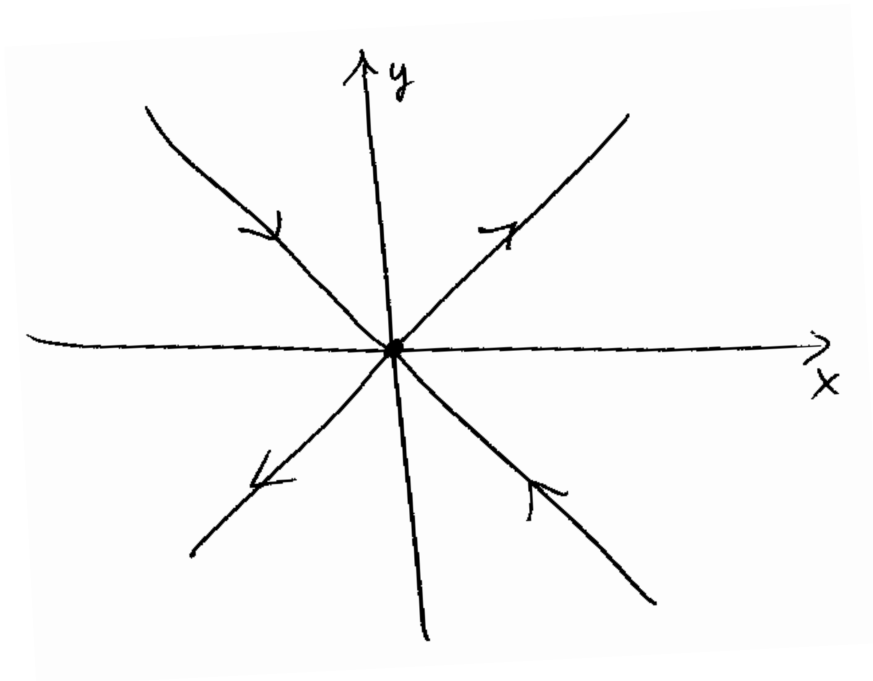

Figure 8.6: Hand drawn straight line trajectories for system (8.5)

in Figure 8.4 the straight line trajectories that tend towards and tend away from the origin $(0, 0)$ are in the directions of the eigenvectors. These lines are called the *stable* and *unstable manifolds*, respectively, of the saddle point. Solutions starting on the stable manifold—there are infinitely many such solutions—approach the origin as time increases to ∞; the solutions starting on the unstable manifold approach the origin as time decreases to $-\infty$. This is easy to see from the general solutions of the differential equation. The solutions on the stable manifold are given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-\sqrt{2}\,t} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix},$$

where c_1 is a real number and $c_2 = 0$; the solutions on the unstable manifold are given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_2 e^{\sqrt{2}\,t} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix},$$

where c_2 is a real number and $c_1 = 0$. □

Example. Consider the system

$$\begin{cases} x' = -x - y, \\ y' = x - y. \end{cases} \tag{8.8}$$

Let's determine the phase portrait near the origin. This time we will draw the phase portrait by hand! First, we find the eigenvalues of the coefficient matrix

$$\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}.$$

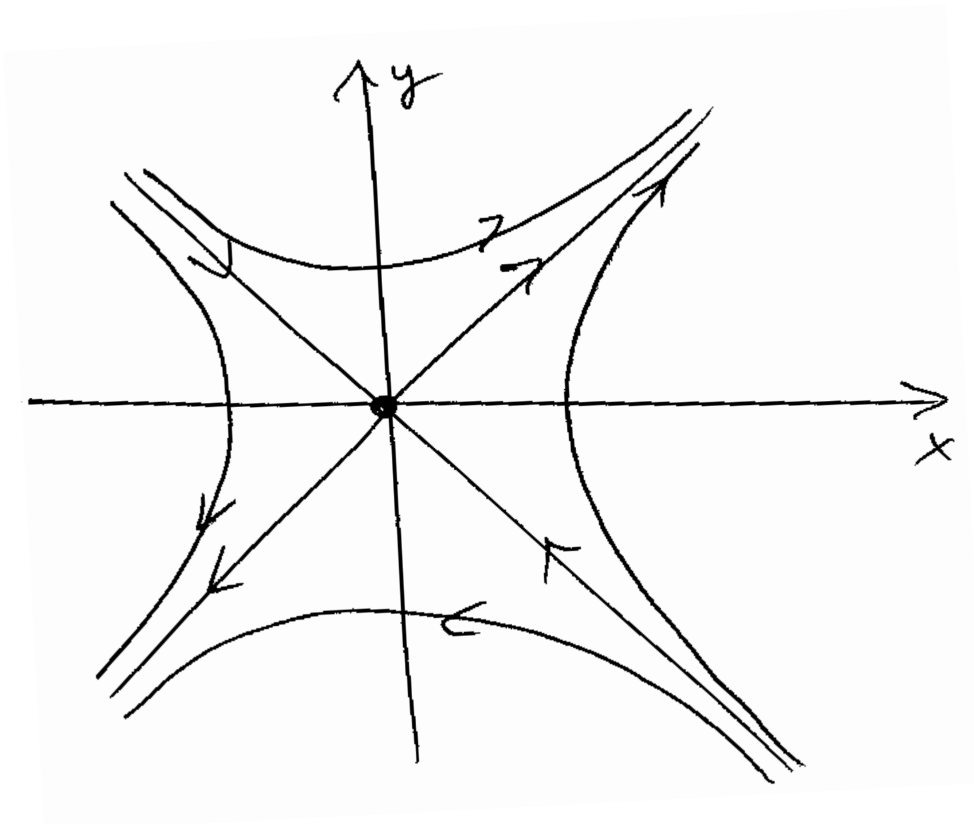

Figure 8.7: Hand drawn phase portrait for system (8.5)

The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$ and the eigenvalues are the complex numbers $-1 \pm i$. Using the methods we have learned, we find an eigenvector corresponding to the eigenvalue $-1+i$, which we choose to be $\begin{bmatrix} i \\ 1 \end{bmatrix}$. A complex solution is given by

$$e^{(-1+i)t} \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

By extracting its real and imaginary parts, the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{-t} \Big(c_1 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \Big),$$

which can also be written in the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{-t} \begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Because the functions sin and cos are 2π-periodic, every nonzero solution will spiral around the origin. Also, due to the presence of the exponential factor e^{-t} (corresponding to the *negative real part* of the eigenvalue) the solutions are asymptotic to the origin as $t \to \infty$. The key observation is that the eigenvalues are complex with *negative real parts*. The only remaining question is which way do the orbits spiral, clockwise or counterclockwise? To decide, consider the vector field (corresponding to the right-hand side of the system) along the coordinate axes. For example, we have $x = 0$ along the vertical axis; therefore, the vector field restricted to this set is given by $\begin{bmatrix} -y \\ -y \end{bmatrix}$. We are interested only in the first component of the vector field; it determines the direction that trajectories cross the y-axis. Here, the first component is negative for $y > 0$ and positive for $y < 0$. Hence, the trajectories spiral counterclockwise toward the origin. This type of rest point is called a (spiral) sink. A hand-drawn phase portrait is shown in Figure 8.8. Note that the axes are specified, the position of the trajectory that stays at the origin is indicated, the spiral trajectory has the correct qualitative behavior, and its direction is indicated by an arrow head along the trajectory. You can tell at a glance how all the trajectories behave—this is the purpose of drawing the phase portrait. □

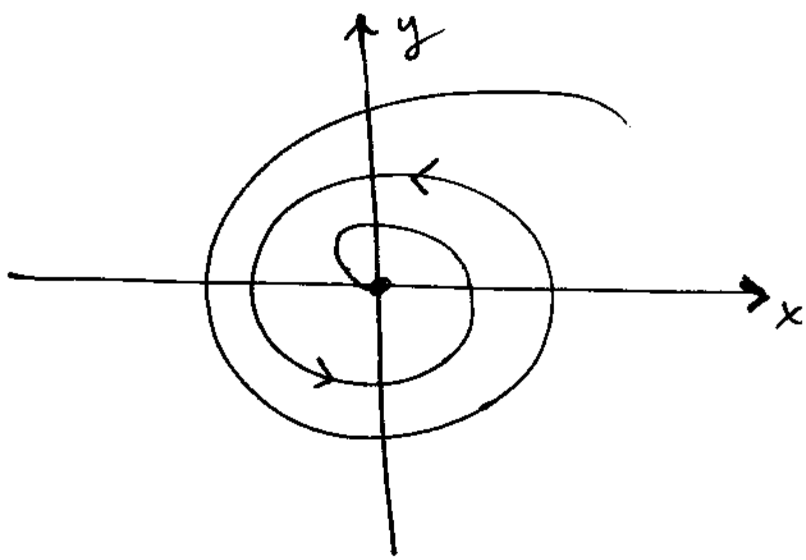

Figure 8.8: Phase portrait of system (8.8)

Example. Draw the phase portrait of the second-order differential equation

$$\ddot{x} - 5\dot{x} + 4x = 0;$$

that is, draw the phase portrait of an equivalent first-order system.

The usual way to obtain an equivalent first-order system is to set $\dot{x} = y$ and then write the formula for

$$\dot{y} = \ddot{x} = -4x + 5\dot{x} = -4x + 5y.$$

In other words, the equivalent system is

$$\begin{cases} x' = y, \\ y' = -4x + 5y. \end{cases} \tag{8.9}$$

We will draw the phase portrait of this system.

The process is always the same: we consider the eigenvalues and eigenvectors of the system. In this case, the coefficient matrix is

$$\begin{bmatrix} 0 & 1 \\ -4 & 5 \end{bmatrix},$$

which has the characteristic equation $\lambda^2 - 5\lambda + 4 = 0$ and the eigenvalues 1 and 4. Corresponding eigenvectors are given by

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

respectively. Hence the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

In this case there are two straight line solutions corresponding to the two eigenvectors. Also, all trajectories (except the one that stays at the origin) move away from the origin due to the exponential factors, which both grow without bound as $t \to \infty$. The key point is that both eigenvalues have positive real parts. In this example, the eigenvalues are positive real numbers. This type of rest point is called a (nodal) source (see Figure 8.9). □

Some authors make finer distinctions (for example, they define proper and improper nodes); but, for most applications, the most important features of rest points are captured by the basic classification: source, sink, or saddle, which is determined by the signs of the real parts of the corresponding eigenvalues. If both eigenvalues have positive real parts the rest point is a source. If both eigenvalues have negative real parts the rest point is a sink. And, if there is one positive and one negative eigenvalue, the rest point is a saddle.

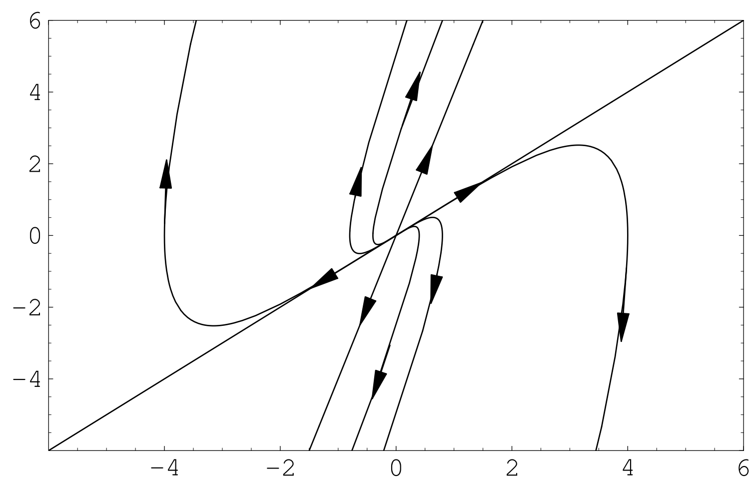

Figure 8.9: Phase portrait of system (8.9)

Homework Assignments

For each of the following systems, determine the nature of the origin $(0,0)$ i.e., whether it is a saddle point, a node, etc., and whether the trajectories tend to move towards or away from the origin as $t \to \infty$. For each system sketch the trajectories near $(0,0)$.

8.1.

$$\begin{cases} x' = 2x - y \\ y' = x + 2y. \end{cases}$$

8.2.

$$\begin{cases} x' = -6x + y \\ y' = x - 6y. \end{cases}$$

8.3.

$$\begin{cases} x' = -7x + 10y \\ y' = -5x + 8y. \end{cases}$$

8.4.

$$\begin{cases} x' = y \\ y' = x - y. \end{cases}$$

8.5.

$$\begin{cases} x' = -10x - y \\ y' = x - 10y. \end{cases}$$

9 Linearization of Nonlinear Systems at Isolated Rest Points

In this section we study how linear 2×2 systems can be used to approximate the local behavior of the trajectories of solutions to nonlinear systems.
We will only consider nonlinear systems of the form

$$\begin{cases} x' = f(x,y), \\ y' = g(x,y), \end{cases} \tag{9.1}$$

where f and g are twice continuously differentiable functions of two variables x and y. In this case, if initial conditions are given at some time t_0, say

$$x(t_0) = x_0, \qquad y(t_0) = y_0,$$

then *the system has a unique solution with the given initial conditions.*

Definition of Rest Point. A point (x_0, y_0) on the plane is said to be a rest point of the system (9.1) if

$$f(x_0, y_0) = g(x_0, y_0) = 0. \tag{9.2}$$

Definition of Isolated Rest Point. A rest point (x_0, y_0) of (9.1) is called isolated if there is a disk D centered at (x_0, y_0) such that the only rest point in D is (x_0, y_0).

Example. Consider Duffing's equation with damping

$$x'' + \epsilon x' - x + x^3 = 0, \tag{9.3}$$

where ϵ is a nonnegative real number. This second-order equation is equivalent to the first-order system

$$\begin{cases} x' = y, \\ y' = -\epsilon y + x - x^3. \end{cases} \tag{9.4}$$

Let's find the rest points of system (9.4) They will be solutions of the simultaneous system of algebraic equations

$$y = 0, \qquad -\epsilon y + x - x^3 = 0.$$

Thus the rest points are $(0, 0)$, $(1, 0)$ and $(-1, 0)$. □

To determine the local phase portrait of a system, such as (9.1), near one of its isolated rest points, the basic tool is a corresponding linear system that closely approximates the nonlinear system near this rest point. It turns out that the local phase portrait of the nonlinear system near the rest point is qualitatively the same as the phase portrait of this special linear system. This is one reason why we have studied the phase portraits of linear systems.

To be more precise, we first define the appropriate linear system to study at a rest point of a nonlinear system.

(2) Linearization. The linearization of the system (9.1) at the rest point (x_0, y_0) is defined to be the homogeneous linear system

$$\begin{cases} \dfrac{du}{dt} = \left[\dfrac{\partial f}{\partial x}(x_0, y_0)\right] u + \left[\dfrac{\partial f}{\partial y}(x_0, y_0)\right] v, \\ \dfrac{dv}{dt} = \left[\dfrac{\partial g}{\partial x}(x_0, y_0)\right] u + \left[\dfrac{\partial g}{\partial y}(x_0, y_0)\right] v. \end{cases} \tag{9.5}$$

Example. Continuing with the last example (Duffing's equation (9.4)), let's determine the linearizations at its rest points.

We simply compute the (Jacobian) matrix

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix} \tag{9.6}$$

at each rest point and use the definition of the linearization. For Duffing's equation, the Jacobian matrix at (x_0, y_0) is

$$\begin{bmatrix} 0 & 1 \\ 1 - 3x_0^2 & -\epsilon \end{bmatrix} \tag{9.7}$$

The linearized system at the origin is

$$\begin{cases} u' = v, \\ v' = u - \epsilon v. \end{cases} \tag{9.8}$$

The linearized system at $(\pm 1, 0)$ is

$$\begin{cases} u' = v, \\ v' = -2u - \epsilon v. \end{cases} \tag{9.9}$$

□

Example. Let's linearize the system

$$\begin{cases} x' = x - y + x(1 - x^2 - y^2), \\ y' = x + y + y(1 - x^2 - y^2), \end{cases} \tag{9.10}$$

at $(0, 0)$. The origin is an isolated rest point of the system. With

$$f(x, y) = x - y + x(1 - x^2 - y^2), \qquad g(x, y) = x + y + y(1 - x^2 - y^2),$$

we have

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2 - 3x^2 - y^2, & \frac{\partial f}{\partial y} &= -1 - 2xy, \\ \frac{\partial g}{\partial x} &= 1 - 2xy, & \frac{\partial g}{\partial y} &= 2 - x^2 - 3y^2. \end{aligned}$$

Thus, the linearization of (9.10) at the origin is

$$\begin{cases} u' = 2u - v, \\ v' = u + 2v. \end{cases} \tag{9.11}$$

□

The next result tells us that in certain cases the linearized system is a good approximation to the nonlinear system at a rest point.

Fact (Hartman–Grobman Theorem). Suppose that a nonlinear system of differential equations

$$\begin{cases} x' = f(x, y), \\ y' = g(x, y), \end{cases} \tag{9.12}$$

has an isolated rest point at (x_0, y_0). Let

$$\begin{cases} u' = au + bv, \\ v' = cu + dv, \end{cases} \tag{9.13}$$

be its linearization at (x_0, y_0). If the eigenvalues of the coefficient matrix of (9.13) have non-zero real part then the nonlinear system (9.12) near (x_0, y_0) and the linear system (9.13) near $(0, 0)$ have the "same" phase portrait.

Remark. *If there is an eigenvalue of the coefficient matrix of* (9.13) *with zero real part, then no conclusion can be drawn about the the behavior of the trajectories of the solutions of the nonlinear system* (9.12) *through its linearization* (9.13).

Remark. *In other words, if one of the two situations (i) and (ii) of (a) arises, then the trajectories of the nonlinear system in some small neighborhood of* (x_0, y_0) *behave "similarly" to those of the linearization near* $(0, 0)$. *"Similarly" means, for example, that if* $(0, 0)$ *is a saddle point of the linearization, then* (x_0, y_0) *is also a saddle point of the nonlinear system; i.e., near* (x_0, y_0) *the trajectories of the nonlinear system will look like those near a linear saddle point.*

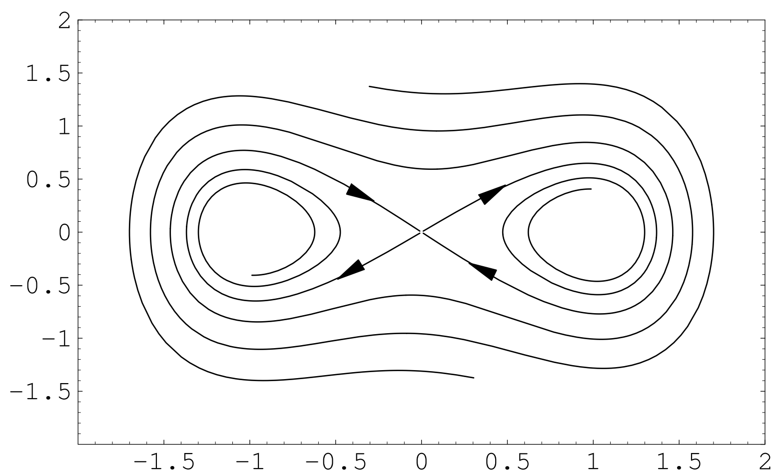

Figure 9.1: The phase portrait of Duffing's equation (9.4) with $\epsilon = 0.1$.

Example. Continuing with Duffing's equation (9.4), determine the local phase portrait at each of its rest points.

At the rest point $(0, 0)$, the coefficient matrix of the linearization (9.8) is

$$\begin{bmatrix} 0 & 1 \\ 1 & -\epsilon \end{bmatrix}$$

The eigenvalues of this matrix are

$$\lambda_\pm = \frac{-\epsilon \pm \sqrt{\epsilon^2 + 4}}{2}.$$

Recall that $\epsilon \geq 0$. It is easy to see that the eigenvalues are real. There is one positive eigenvalue λ_+ and one negative eigenvalue λ_-. Thus, the linearization has a saddle point at the origin. According to the Hartman–Grobman theorem, the nonlinear system also has a saddle point at the origin.

At the rest points $(\pm 1, 0)$, the coefficient matrix of the linearization (9.9) is

$$\begin{bmatrix} 0 & 1 \\ -2 & -\epsilon \end{bmatrix}$$

The eigenvalues of this matrix are

$$\frac{-\epsilon \pm \sqrt{\epsilon^2 - 8}}{2}.$$

For $\epsilon = 0$, the eigenvalues are pure imaginary. In this case the Hartman–Grobman theorem does not apply. If $0 < \epsilon < \sqrt{8}$, the eigenvalues are complex both with negative real parts and if $\epsilon > \sqrt{8}$, then the eigenvalues are both real and negative. The Hartman–Grobman theorem applies in these cases; the rest point is a sink. The nature of the sink will change from a spiral sink to a nodal sink as ϵ increases through $\sqrt{8}$. □

Example. Consider the nonlinear system (9.10) again. Determine its local phase portrait at the origin.

The linearization at the isolated rest point $(0, 0)$ is given by (9.11). The eigenvalues of the coefficient matrix of (9.11) are $2 \pm i$. So the origin $(0, 0)$ is a spiral source for the linearization. Thus, according to the Hartman–Grobman theorem, the rest point $(0, 0)$ of the nonlinear system (9.10) is a spiral source. □

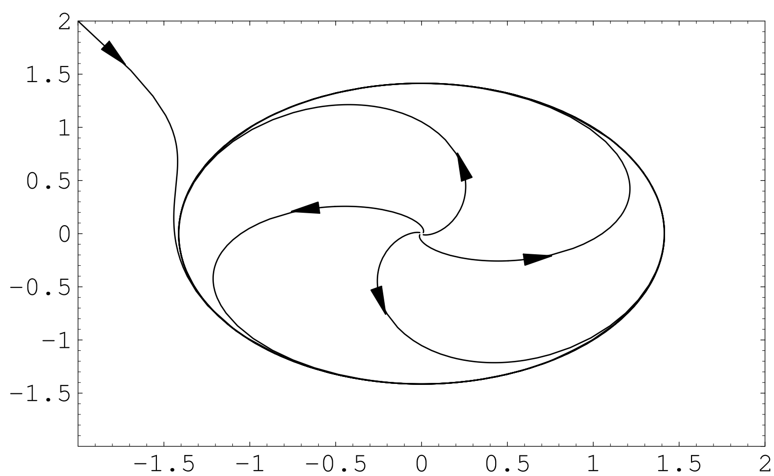

Figure 9.2: Trajectories of system (9.10). Note the source at the origin and the stable periodic trajectory, which is called a limit cycle. It turns out that all trajectories, except for the rest point, are attracted to the limit cycle.

Example. A typical problem in mathematical ecology is the formulation and analysis of population models. A famous example is the Volterra model for the populations of a predator and its prey. A similar model was proposed earlier by Lotka. Thus the differential equations model we will derive is often called the Lotka–Volterra model. Let us denote by x the size of the population of a certain predator and by y the size of the population of its prey. We assume that the predator and its prey are living together in some environment. If there were no predators, the population of the prey species would grow exponentially except that the population growth would be limited by the food supply in the environment. For this reason, we will model the growth of the prey species by the (logistic) differential equation

$$y' = ay - by^2,$$

where a and b are positive real parameters to be determined by studying the reproduction rate of the prey species and the carrying capacity of the environment. More precisely, the growth rate is a and the carrying capacity is a/b. Why? If there were no prey species the predator would die out at a certain rate. Thus, we model the decay of the predator population by

$$x' = -cx,$$

where $c > 0$ is the decay rate. The interaction (predator meets prey) must be a function of the two variables x and y that has at least one special property: it must be zero (no interaction) if one of the populations is zero. The simplest function with this property is xy; we take it to be the interaction term.

Interaction increases the population of predators—they eat the prey—and decreases the population of the prey. So, the Lotka–Volterra model with logistic prey growth is

$$\begin{cases} x' = -cx + dxy, \\ y' = ay - by^2 - fxy. \end{cases} \tag{9.14}$$

What is the fate of the predators and the prey if their populations evolve according to the model (9.14)?

Of course, the answer to our question will depend on the choice of the parameters a, b, c, d and f and the initial populations. For simplicity, we will discuss the special case

$$\begin{cases} x' = \dfrac{1}{10}x(-20 + 5y), \\ y' = \dfrac{1}{10}y(4 - \dfrac{1}{4}y - 50x), \end{cases} \tag{9.15}$$

where x and y measure the populations in units of thousands of individuals (for example, $x = 3$ means a population of 3000 predators). The reader is invited to investigate other cases.

Since x and y represent populations, these quantities are non-negative. So, for this application, we are interested only in the trajectories in the closed first quadrant. Our model would not be satisfactory if there were a trajectory, starting in the first quadrant, that eventually crosses the x or y-axis. Indeed, if this happened, then at least one of the populations would become negative! As we will see, this is not the case.

The nonlinear system has three isolated rest points: $(0, 0)$, $(0, 16)$ and $(3/50, 4)$. The rest point at the origin represents the steady state where there are no predators and no prey. Of course, if there are zero populations initially, the populations remain zero for all time. The rest point at $(0, 16)$ represents the situation where there are no predators and the prey species reaches an equilibrium according to the carrying capacity of the environment. We will soon determine the meaning of the third rest point.

By physical reasoning, it is obvious that if there are no predators at the initial time, then there are no predators forever. This would mean that if a trajectory starts on the y-axis, then it remains on the y-axis for all time. But, this is not correct reasoning. Our model is a mathematical construct; maybe it does not reflect exactly the physical situation. Also, the purpose of the model is to derive physical conclusions. We must reason from the model, not from the physical situation that it is supposed to represent! If the initial population is $(0, y_0)$, we must show mathematically that $x(t) = 0$ for all t. This is easy. Simply note that the system

$$x' = 0, \qquad y' = \frac{1}{10}y\left(4 - \frac{1}{4}y\right)$$

has a unique solution of the form $(x(t), y(t))$ where $x(t)$ is zero and $y(0) = y_0$. This solution is a solution of system (9.15). A similar argument shows that a solution which starts on the x-axis stays on the x-axis for all time. These results show that a trajectory that starts in the first quadrant must stay in the first quadrant. If it did not, it would have to cross one of the axes. But, these are both invariant sets (a trajectory that starts on one of these sets stays on that set); trajectories cannot cross in the phase plane. If they did, the uniqueness of solutions of ODEs would be violated.

To determine the fate of the populations, we will determine the local phase portraits at the rest points.

The coefficient matrix of the linearized system corresponding to system (9.15) is

$$\begin{bmatrix} -2 + \frac{1}{2}y & \frac{1}{2}x \\ -5y & \frac{2}{5} - \frac{1}{20}y - 5x \end{bmatrix}.$$

At the rest point $(0, 0)$, this matrix has one positive and one negative eigenvalue. Hence, this rest point is a saddle. Moreover, the eigenvectors are in the directions of the coordinate axes, in concert with the invariance of these sets. The flow is approaching the rest point along the x-axis and moving away from the origin along the y-axis. In this special case, as we have already seen, the nonlinear system also has the coordinate axes as invariant sets. (Warning: In general, the straight-line solutions of a linearization may not be invariant sets for the corresponding non-linear system. The Lotka–Volterra system is special in this regard.)

The rest point at $(0, 16)$ is also a saddle with eigenvalues $-2/5$ and 6. The eigenvectors corresponding to $-2/5$ are parallel to the y-axis, which corresponds to the invariance of this axis. The eigenvectors corresponding to the positive eigenvalue 6 are parallel to the vector $\begin{bmatrix} \frac{2}{25} \\ -1 \end{bmatrix}$. This tells us something important: If both populations sizes are positive (that is, the evolving point representing the populations is in the open first quadrant), then it is not possible for either species to become extinct. Indeed, a trajectory in the first quadrant cannot approach a point on either axis. The only possibilities are that it approach one of the rest points (at $(0, 0)$ or $(0, 16)$).

But, they are both saddle points. Thus, (by the Hartman–Grobman theorem) the only solutions that approach them must lie on one of their stable manifolds. Since their stable manifolds lie on the coordinate axes, our solution in the open first quadrant does not lie on either of these invariant manifolds.

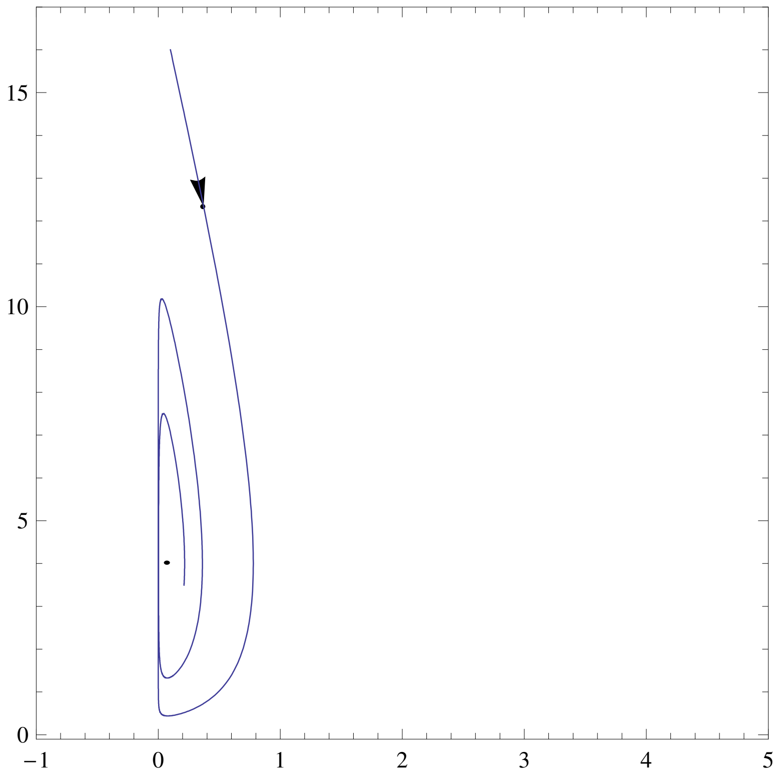

Figure 9.3: Phase portrait of the predator-prey model (9.15). The trajectory with initial conditions $x(0) = 0.1$ and $y(t) = 16.0$ and the rest point at $(3/50, 4)$ are shown. The trajectory spirals toward the rest point.

The rest point at $(3/50, 4)$ is a sink; in fact, the eigenvalues are complex and both have real part $-1/20$. Thus, it seems reasonable to conclude that for arbitrary nonzero initial populations of predators and prey, the population will evolve toward the steady state solution $(3/50, 4)$. For example, if our measurements for x and y are in units of thousands of individuals, then the populations—in our fictitious example—will evolve to the steady state of 60 predators and 4000 prey. Our analysis shows that if we start near this steady state, we will evolve toward this steady state. With more work it is possible to prove that all trajectories starting in the first quadrant do in fact evolve to this sink (see Figure 9.3). □

Homework Assignments

For each of the following nonlinear systems find all the rest points and determine which rest points are isolated.

9.1.

$$\begin{cases} x' = -x + xy, \\ y' = 2y - y^2 - xy. \end{cases}$$

9.2.

$$\begin{cases} x' = x^2 + xy, \\ y' = xy + y^2. \end{cases}$$

9.3.

$$\begin{cases} x' = y, \\ y' = -y - \sin x. \end{cases}$$

For each of the following nonlinear systems linearize the system around its isolated rest points in the phase plane. If possible determine the nature of the rest points from their linearizations (i.e. whether the trajectories near the rest point look like those near a saddle point, node, etc; and whether the trajectories near the rest point tend towards or tend away from it.)

9.4.

$$\begin{cases} x' = -x + xy, \\ y' = 2y - y^2 - xy. \end{cases}$$

9.5.

$$\begin{cases} x' = y, \\ y' = -y - \sin x. \end{cases}$$

9.6. The first-order system corresponding to

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x - x^3 = 0.$$

9.7. For the predator-prey model (9.14), assume that $f = d$. What relation among the parameters a, b, c, and d must be satisfied for there to be a rest point in the first quadrant?

9.8. (a) Determine the local phase portraits near the rest points of the system

$$x' = y, \quad y' = -y + x - x^2.$$

(b) Draw a plausible global phase portrait. (c) What is the fate of the trajectory with initial conditions $x(0) = 1/2$ and $y(0) = 0$ as $t \to \infty$?

9.9. Show that system (9.10) has a stable periodic orbit. Hint: Change to polar coordinates.

10 Answers to Selected Exercises

Page 5

1.1

$$x' = y, \qquad y' = \frac{5}{2}x - 2y + \frac{1}{2}te^{-3t}$$

1.2

$$\frac{d^2x}{dt^2} + x' - 3x = 4e^t$$

1.3

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 2x = 0$$

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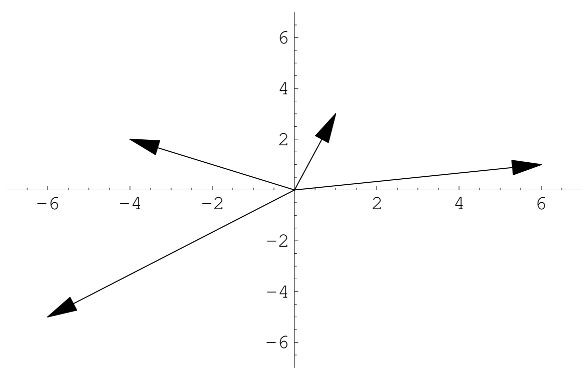

2.1

2.2 1. $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 8.5 & -6 \\ 3 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 24 & -80 \\ -24 & 12 \end{bmatrix}$

4. $\begin{bmatrix} t^2e^{5t} & 6e^{2t} \\ -7e^{2t} & -6te^{2t} \end{bmatrix}$

5. $\begin{bmatrix} 46 \\ -43 \end{bmatrix}$

6. $\begin{bmatrix} -t^6e^{5t} - te^{11t} \\ 2t^4e^{5t} - 6t^2e^{5t} \end{bmatrix}$

7. $\begin{bmatrix} -45 & 26 \\ -29 & 16 \end{bmatrix}$

8. $\begin{bmatrix} -11 & -37 \\ 4 & 18 \end{bmatrix}$

9. $\begin{bmatrix} t^2e^{-2t}+6t^2e^{3t} & t^2e^{6t}+6 \\ 5+3t^2e^{3t} & 5e^{8t}+3 \end{bmatrix}$

2.3
1. 46
2. 13
3. -12

2.4
1. linearly independent
2. linearly independent, except when $c=1$, then linearly dependent, $\alpha=1$, $\beta=-1$
3. linearly dependent, e.g. $\alpha=-2$, $\beta=1$
4. linearly dependent, e.g. $\alpha=1$, $\beta=-e^{2c}$

2.5
1. eigenvalues 3, 2, eigenvectors: $\begin{bmatrix} -2\alpha \\ \alpha \end{bmatrix}$, $\begin{bmatrix} -\alpha \\ \alpha \end{bmatrix}$ respectively
2. eigenvalues $2+i$, $2-i$; eigenvectors $\begin{bmatrix} (-1+i)3\alpha \\ 2\alpha \end{bmatrix}$, $\begin{bmatrix} (-1-i)3\alpha \\ 2\alpha \end{bmatrix}$ respectively
3. eigenvalues 2, 2; eigenvectors $\begin{bmatrix} -3\alpha \\ 2\alpha \end{bmatrix}$

2.6
1. not invertible
2. invertible, $(28-6\sqrt{2})^{-1}\begin{bmatrix} -4 & -\sqrt{2} \\ -6 & -7 \end{bmatrix}$
3. invertible, $(\sqrt{t}e^{2t}-t^3e^t)^{-1}\begin{bmatrix} \sqrt{t} & -t \\ -t^2e^t & e^{2t} \end{bmatrix}$

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3.2 linearly independent, determinant is 0

3.3 linearly independent

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4.1

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^t\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

4.2

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{4}{3}e^{-8}e^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}e^{-2}e^t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

4.3

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1e^{\left(\frac{3+\sqrt{5}}{2}\right)t}\begin{bmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix} + c_2e^{\left(\frac{3-\sqrt{5}}{2}\right)t}\begin{bmatrix} 1 \\ -\left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix}.$$

Except for the trivial (zero) solution, the length of the solution vector (that is, $\sqrt{x(t)^2+y(t)^2}$) grows without bound as $t\to\infty$.

4.4(i)

$$x' = 0 \cdot x + y$$
$$y' = 3x + \frac{1}{2}y.$$

4.4(ii)

$$r_1 = -\frac{3}{2} \text{ and } r_2 = 2.$$

4.4(iii)

$$x(t) = c_1 e^{-\frac{3}{2}t} + c_2 e^{2t}.$$

4.4(iv)

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-\frac{3}{2}t} \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

4.5 $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$.

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5.2

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-t} \left\{ \cos(2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(2t) \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\} + c_2 e^{-t} \left\{ \sin(2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(2t) \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$$

5.3

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 2000 e^{t} \left\{ \cos(10t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(10t) \begin{bmatrix} 0 \\ \frac{5}{2} \end{bmatrix} \right\} + 2 e^{t} \left\{ \sin(10t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(10t) \begin{bmatrix} 0 \\ \frac{5}{2} \end{bmatrix} \right\}$$

5.4

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{t} \left\{ \cos(5t) \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} - \sin(5t) \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \right\} + c_2 e^{t} \left\{ \sin(5t) \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + \cos(5t) \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \right\}$$

Except for the zero solution, $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ spirals away from the origin toward "infinity" as $t \to \infty$.

5.5(i)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$x(t_0) = u_0, \quad y(t_0) = v_0.$$

5.5(ii)

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-\frac{\gamma}{2m}t} \Bigg\{ & \cos\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t\right) \begin{bmatrix} 1 \\ -\frac{\gamma}{2m} \end{bmatrix} \\ & - \sin\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t\right) \begin{bmatrix} 0 \\ \frac{\sqrt{4mk-\gamma^2}}{2m} \end{bmatrix} \Bigg\} \\ & + c_2 e^{-\frac{\gamma}{2m}t} \Bigg\{ \sin\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t\right) \begin{bmatrix} 1 \\ -\frac{\gamma}{2m} \end{bmatrix} \\ & + \cos\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t\right) \begin{bmatrix} 0 \\ \frac{\sqrt{4mk-\gamma^2}}{2m} \end{bmatrix} \Bigg\} \end{aligned}$$

5.5(iii)

$$\begin{bmatrix} u \\ u' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ as } t \to \infty$$

5.6

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 & \left\{ \cos(4t) \begin{bmatrix} 1 \\ \frac{2}{5} \end{bmatrix} + \sin(4t) \begin{bmatrix} 0 \\ \frac{4}{5} \end{bmatrix} \right\} \\ & + c_2 \left\{ \sin(4t) \begin{bmatrix} 1 \\ \frac{2}{5} \end{bmatrix} - \cos(4t) \begin{bmatrix} 0 \\ \frac{4}{5} \end{bmatrix} \right\} \end{aligned}$$

The trajectories go around the origin periodically with period $\pi/2$.

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6.1 $x(t) = c_1 e^{3t} + c_2 e^{3t}(t+1)$, $y(t) = c_1 e^{3t} + c_2 e^{3t} t$

6.2 $x(t) = -2c_1 e^t + c_2 e^t(-2t + 1/2)$, $y(t) = c_1 e^t + c_2 e^t t$

6.3 $x(t) = c_1 e^{-2t} + c_2 e^{-2t}(t + 2/3)$, $y(t) = -3c_1 e^{-2t} + c_2 e^{-2t}(-3t)$

6.4 $x(t) = c_1 e^{3t},\ y(t) = c_2 e^{3t}$

6.5 $x(t) = e^{-t/2}\left(-5 - 2t\right),\ y(t) = e^{-t/2}\left(6 + 4t\right)$

6.6 $x(t) = e^{3t}(4 + 10t),\ y(t) = e^{3t}(2 - 20t)$

6.7 $s < -1$

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7.1 $x(t) = 5(1 - 4e^t + 3e^{2t}),\ y(t) = 5(1 - 2e^t + e^{2t})$

7.2 $x(t) = \frac{1}{3}(e^{-2t} - 6e^{-t} + 5e^{t}),\ y(t) = \frac{-e^{-2t}}{3} + e^{-t} + \frac{e^{t}}{3}$

7.3 $x(t) = c_1 e^{4t} + c_2 e^{2t} - \frac{3+4t}{32}$,
$y(t) = -c_1 3e^{4t} - c_2 e^{2t} + \frac{1-4t}{32}$

7.4 $x(t) = c_1 \cos t + c_2 \sin t - \frac{\cos^3 t}{2} - \sin t\left(\frac{t}{2} + \frac{\sin(2t)}{4}\right)$,
$y(t) = c_1 \sin t - c_2 \cos t - \frac{\cos^2 t \sin t}{2} + \cos t\left(\frac{t}{2} + \frac{\sin(2t)}{4}\right)$

7.5

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-2t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 + \frac{4}{3}e^{t} \\ -12 + 2e^{t} \end{bmatrix}$$

7.6

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-2t}\left(t\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + e^{-2t}\begin{bmatrix} -\log t - 1 \\ 2\log t + 2 - \frac{1}{t} \end{bmatrix}$$

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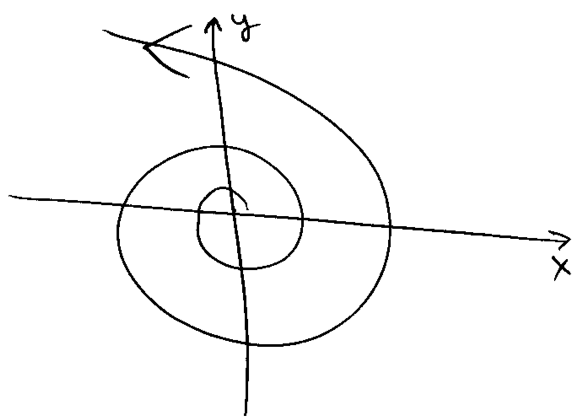
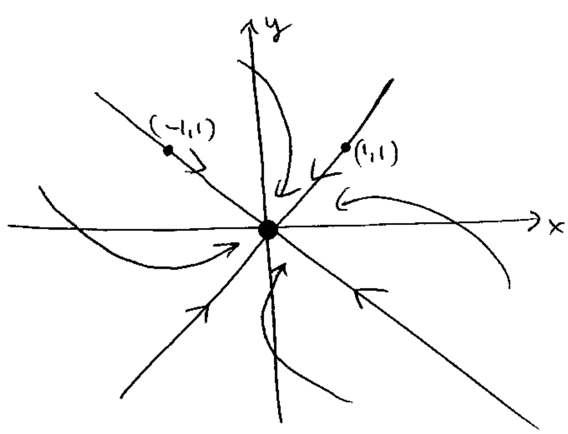
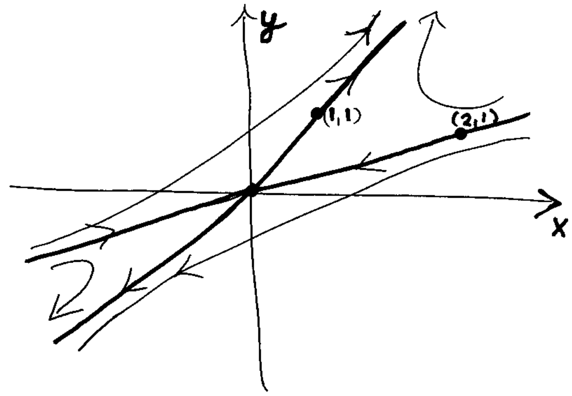
8.1 (Spiral) source; that is, except for the trajectory at the origin, all orbits spiral away from the origin.



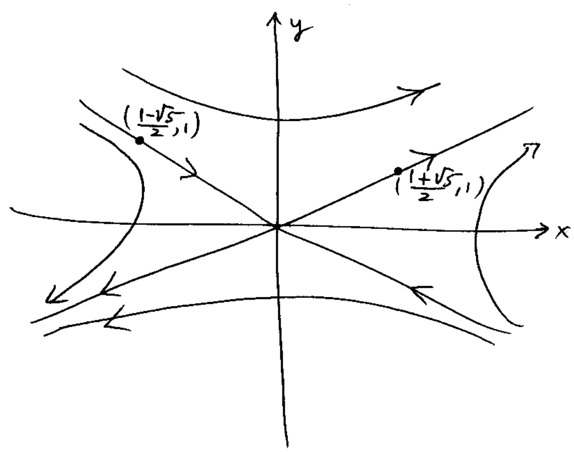
8.2 (Nodal) sink; that is, except for the trajectory at the origin, all orbits tend toward the origin and no orbit spirals around the origin.



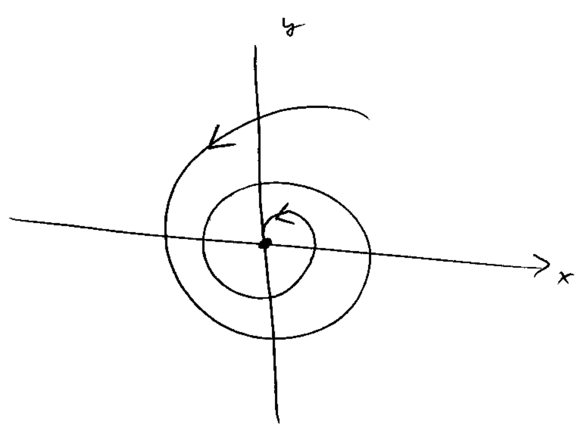
8.3 Saddle



8.4 Saddle



8.5 (Spiral) sink; that is, except for the trajectory at the origin, all orbits spiral toward the origin.



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9.1 The rest points are $(0,0)$, $(0,2)$ and $(1,1)$. All rest points are isolated.

9.2 All points on the line $y=-x$ are rest points. Hence, none of the rest points are isolated.

9.3 The rest points are $(n\pi,0)$, for all integers n. There are infinitely many rest points and they are all isolated.

9.4 There are three rest points: $(0,0)$ is a saddle, $(0,2)$ is a saddle, and $(1,1)$ is a spiral sink.

9.5 There are an infinite number of rest points $(n\pi,0)$, for all integers n. The rest points $(2k,0)$ are spiral sinks; the rest points $(2k+1,0)$ are saddles, for every integer k.

9.6 There are three rest points: $(0,0)$ is a spiral sink and $(0,\pm1)$ are saddles.

9.7 There is a rest point in the first quadrant whenever $ad-bc>0$.

9.8 There is a saddle point at the origin and a spiral sink at the point $(1,0)$. The trajectory starting at $(1/2,0)$ is asymptotic in positive time to the sink at $(1,0)$.

9.9 Note that $r^2=x^2+y^2$ and $\theta=\arctan(y/x)$. Differentiate these expressions and then substitute $x=r\cos\theta$ and $y=r\sin\theta$ to get

$$\dot{r}=r(2-r^2), \qquad \dot{\theta}=1.$$

Except for $r(0)=0$, every solution of the first equation is such that $\lim_{t\to\infty}=\sqrt{2}$. This follows from a qualitative analysis. It is also possible to solve this equation by separation of variables.