Planar Systems of Differential Equations

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Contents

1 Introduction

A system of differential equations is a set of equations involving the derivatives of several functions of the same independent variable. Systems of differential equations are used to model many physical situations. For example, the following system of differential equations arises in the study of predator-prey interactions in ecology:

$$\begin{aligned}\frac{dx}{dt} &= x(\alpha - \beta y),\\ \frac{dy}{dt} &= y(-\gamma + \delta x),\end{aligned} \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta$ are positive constants and x and y are functions of the same independent variable t. In system (1.1), $x(t)$ and $y(t)$ represent the populations of the prey and predator species, respectively, at time t. Systems of differential equations are also used to model electrical circuits. For example, let L, C and R be the inductance, capacitance and resistance, respectively, in the parallel LCR circuit shown in Figure 1.1. Assume that these quantities L, C and R are held constant. Let V be the voltage drop across the capacitor and I be the current through the inductor. Then V and I satisfy the system of differential equations

$$\begin{aligned}\frac{dI}{dt} &= \frac{V}{L},\\ \frac{dV}{dt} &= -\frac{I}{C} - \frac{V}{RC}.\end{aligned} \tag{1.2}$$

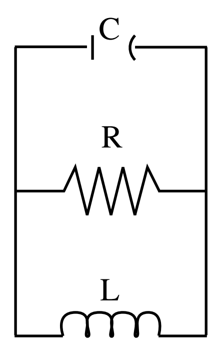


Figure 1.1: A parallel LCR circuit

In these notes we will focus on a special type of system of differential equations, namely the linear 2×2 systems. These are systems of differential equations that can be written in the form

$$\begin{aligned}\frac{dx}{dt} &= ax + by + f(t),\\ \frac{dy}{dt} &= cx + dy + g(t),\end{aligned} \tag{1.3}$$

where $x(t)$ and $y(t)$ are functions of the independent variable t, and a, b, c and d are constant real numbers and f and g are continuous functions on some open interval I of the real numbers. For example,

$$\begin{aligned} \frac{dx}{dt} &= 2x + 7y + t^2, \\ \frac{dy}{dt} &= 3x + 12y + e^t, \end{aligned} \tag{1.4}$$

is a 2×2 linear system of differential equations. We choose to focus on this type of system because (1) the theory is accessible to students who have taken only Math 1500, (2) this subject provides a good introduction to the theory of higher-dimensional systems, and (3) the linearizations of many nonlinear systems that involve only two functions of one independent variable are linear 2×2 systems, which provide good first-order approximations to the local behavior of the nonlinear system. Linear 2×2 systems also arise directly in applications as the following example shows.

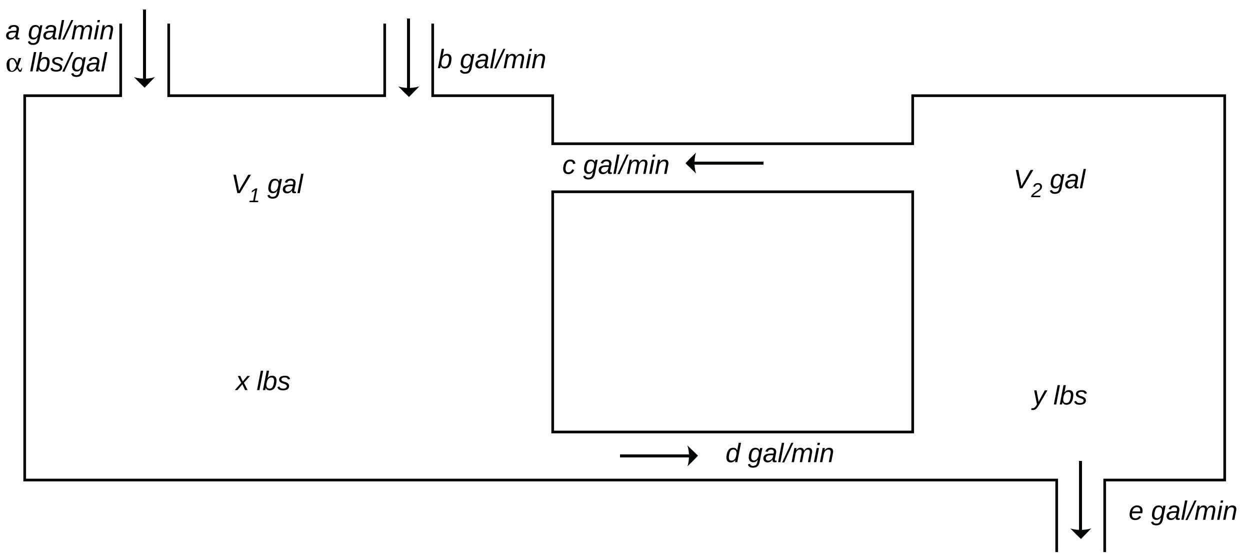


Figure 1.2: Tanks with pipes

Example. *Many industrial processes involve "mixing." Consider, for instance, the arrangement of tanks and pipes depicted in Figure 1.2. In this case, liquid is pumped between the tanks at the rates shown. Also, liquid enters the first tank, which has liquid volume V_1, from two sources. The first source pumps in pure water at the rate of b gal/min; the second source enters at the rate of a gal/min and contains a certain chemical (solute) at the concentration of α lbs/gal. The solution enters and leaves the first tank at the rates c and d, and it leaves the second tank, which has liquid volume V_2, at the rate e. The liquid volume of each tank is assumed to remain constant. We are also given the initial amounts of the chemical in each tank. The problem is to determine the amount of the chemical in each of the tanks as a function of time.*

The mathematical model is based on conservation of mass. Note first that the rates at which the solution enters and leaves the tanks cannot be arbitrary; the amount of solution coming in must equal the amount going out. Equivalently, the sum of the rates in must equal the sum of the rates out. For the first tank we must have

$$a + b + c = d$$

and for the second

$$d = c + e.$$

These relations are important in the analysis of the system. In particular, we must have $d > c$ to be in a physically realistic situation.

We will use conservation of mass again to set up our differential equations. Let x and y denote *the amount measured in lbs* of the chemical dissolved in the first and second tanks, respectively. The differential equations are simply a statement of conservation of mass:

$$\text{rate of change of amount with respect to time} = \text{rate in} - \text{rate out}. \tag{1.5}$$

Consider the first tank. The rate of change of the amount with respect to time is dx/dt. The "rate in" means the rate at which the chemical enters the tank. This must be expressed in units of lbs/min. The rate in from the outside source is simply

$$a \text{ gal/min times } \alpha \text{ lbs/gal} = a\alpha \text{ lbs/min}.$$

The rate of increase due to the pipe entering from the second tank must be

$$c \text{ gal/min times a factor measured in lbs/gal}.$$

We now come to an essential point: In the second tank y denotes the number of lbs of the chemical. By our assumptions, the volume in the second tank remains constant. Also, we will assume that all mixing in the tanks is instantaneous, so that the concentration is the same everywhere in the tank. Under these assumptions, the appropriate factor measured in lbs/gal is y/V_2. The "rate in" via the pipe from the second tank is yc/V_2 lbs/min. Likewise, the rate out is xd/V_1 lbs/min. These facts and the conservation law (1.5) lead to the differential equation

$$\dot{x} = (a\alpha + \frac{c}{V_2}y) - \frac{d}{V_1}x$$

By a similar procedure (taking into account the relation $e = d - c$), we find that

$$\dot{y} = \frac{d}{V_1}x - \frac{d}{V_2}y.$$

This system together with the initial conditions takes the form

$$\frac{dx}{dt} = -\frac{d}{V_1}x + \frac{c}{V_2}y + a\alpha,$$

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d}{V_1}x - \frac{d}{V_2}y, \\
x(0) &= x_0, \\
y(0) &= y_0.
\end{aligned} \tag{1.6}$$

You will soon learn how to solve such systems. A special case of our mixing problem will be solved in the example in Section 7.

In Sections 4–7 we will divide 2×2 systems into 4 classes, and for each of these 4 classes, we shall give one method of finding the general solution of systems of differential equations in the class. To accomplish this task, we shall first need some concepts from matrix theory and linear algebra, which we will describe in Section 2.

We have already encountered in disguise a special type of linear 2×2 systems in Chapter 3 of Boyce and DiPrima: the second order linear differential equations with constant coefficients.

Example. *Consider the second order differential equation*

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 4x = 3e^{2t}. \tag{1.7}$$

We can rewrite this equation as a linear 2×2 system by introducing the function $y = \frac{dx}{dt}$ so that (1.7) becomes

$$\begin{aligned}
\frac{dx}{dt} &= 0x + y + 0, \\
\frac{dy}{dt} &= 4x + 3y + 3e^{2t},
\end{aligned} \tag{1.8}$$

which is in the form (1.3). By convention, we drop the terms with zero coefficients and write this system in the form

$$\begin{aligned}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= 4x + 3y + 3e^{2t},
\end{aligned}$$

Note that the second equation in system (1.8) follows from equation (1.7) if we take into account that $y = \frac{dx}{dt}$.

In general, given a linear second order differential equation

$$\alpha\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \gamma x = f(t) \tag{1.9}$$

where α, β and γ are constant real numbers with $\alpha \neq 0$, we can, by putting $y = \frac{dx}{dt}$, rewrite it as a linear 2×2 system in the form of (1.3):

$$\begin{aligned}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= -\frac{\gamma}{\alpha}x - \frac{\beta}{\alpha}y + \frac{1}{\alpha}f(t).
\end{aligned} \tag{1.10}$$

Again the second equation in system (1.10) follows from (1.9) by taking into account that $y = \frac{dx}{dt}$. So the theory of linear 2×2 systems gives us another way of looking at linear second order differential equations with constant coefficients.

Homework Assignments

1.1. Rewrite the second-order differential equation

$$2\frac{d^2x}{dx^2} + 4\frac{dx}{dt} - 5x = te^{-3t}$$

as a 2×2 system of differential equations.

1.2. Rewrite the linear 2×2 system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= 3x - y + 4e^t \end{aligned}$$

as a linear second-order differential equation.

1.3. Using the change of variables

$$x = u + 2v, \qquad y = 3u + 4v$$

Show that the linear 2×2 system of differential equations

$$\begin{aligned} \frac{du}{dt} &= 5u + 8v \\ \frac{dv}{dt} &= -u - 2v \end{aligned}$$

can be rewritten as a linear second-order differential equation.

2 Some Concepts from Matrix Theory and Linear Algebra

2.1 Notations and Basic Definitions

(1) We shall write a vector on the plane as a "column vector" with square brackets. For example, the vector that starts at the origin $(0, 0)$ and terminates at the point $(1, 3)$ will be denoted by $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (see Figure 2.1).

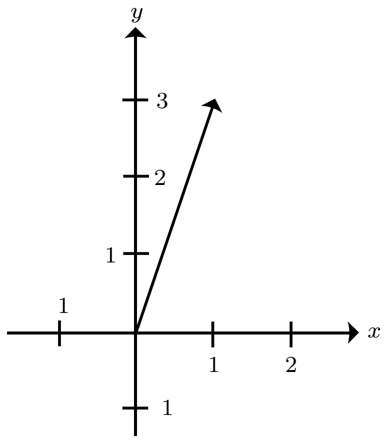


Figure 2.1: A vector in the plane

(2) A 2×2 matrix, with real entries, is a rectangular array of real numbers arranged in two columns and two rows. So all 2×2 matrices can be written in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.1}$$

where a, b, c and d are real numbers. For example

$$\begin{bmatrix} 2 & 1.7 \\ 6.2 & -4.3 \end{bmatrix} \tag{2.2}$$

is a 2×2 matrix.

Two 2×2 matrices are considered equal if all four of their corresponding entries are equal. We can add or subtract two 2×2 matrices in an obvious way: If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$B = \begin{bmatrix} \sigma & \omega \\ \tau & \rho \end{bmatrix},$$

then $A \pm B$ is the matrix

$$A \pm B = \begin{bmatrix} a \pm \sigma & b \pm \omega \\ c \pm \tau & d \pm \rho \end{bmatrix}.$$

For example, we have that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}.$$

If λ is a real number and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then we can multiply A, on the left, by λ. The result is the 2×2 matrix

$$\lambda A = \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix};$$

i.e., to multiply a matrix A by a real number λ, we just multiply every entry of A by λ. For example,

$$7 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 21 & 28 \end{bmatrix}.$$

(3) The result of multiplying a 2×2 matrix (on the right side) by a plane vector is a plane vector defined by:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \sigma \\ \tau \end{bmatrix} = \begin{bmatrix} a\sigma + b\tau \\ c\sigma + d\tau \end{bmatrix}. \tag{2.3}$$

For example,

$$\begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \times 6 + 4 \times 7 \\ 1 \times 6 + 5 \times 7 \end{bmatrix} = \begin{bmatrix} 46 \\ 41 \end{bmatrix}. \tag{2.4}$$

Remark. *By (2.3) we see that a linear 2×2 system of differential equations*

$$\begin{aligned} \frac{dx}{dt} &= ax + by, \\ \frac{dy}{dt} &= cx + dy, \end{aligned} \tag{2.5}$$

can be written in the form

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{2.6}$$

The matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.7}$$

is called the coefficient matrix of the linear system of differential equations.

(4) The result of multiplying two 2×2 matrices is a 2×2 matrix defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \sigma & \omega \\ \tau & \rho \end{bmatrix} = \begin{bmatrix} a\sigma + b\tau & a\omega + b\rho \\ c\sigma + d\tau & c\omega + d\rho \end{bmatrix}. \tag{2.8}$$

For example,

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} &= \begin{bmatrix} 1 \times 5 + 2 \times 7 & 1 \times 6 + 2 \times 8 \\ 3 \times 5 + 4 \times 7 & 3 \times 6 + 4 \times 8 \end{bmatrix} \\ &= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}. \end{aligned} \tag{2.9}$$

Remark. *Note that matrix multiplication is not commutative; i.e., in general*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \sigma & \omega \\ \tau & \rho \end{bmatrix} \neq \begin{bmatrix} \sigma & \omega \\ \tau & \rho \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{2.10}$$

For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \tag{2.11}$$

but

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}. \tag{2.12}$$

Hence, in this case

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \neq \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}. \tag{2.13}$$

On the other hand, matrix multiplication is associative; i.e., if $A, B,$ and C are 2×2 matrices, then

$$(AB)C = A(BC). \tag{2.14}$$

(5) In general we shall denote matrices by capital letters like A, B, C etc. and their entries by the corresponding small letters with subscripts. For example, if A is a 2×2 matrix, then its entries will be a_{11}, a_{12}, a_{21} and a_{22} arranged as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{2.15}$$

The matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.16}$$

will be denoted by "I" and is called the identity matrix. This matrix has the property that for any 2×2 matrix A, we have

$$AI = IA = A. \tag{2.17}$$

2.2 Determinants, Inverses, Linear Dependence, Eigenvalues and Eigenvectors

(1) Consider the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{2.18}$$

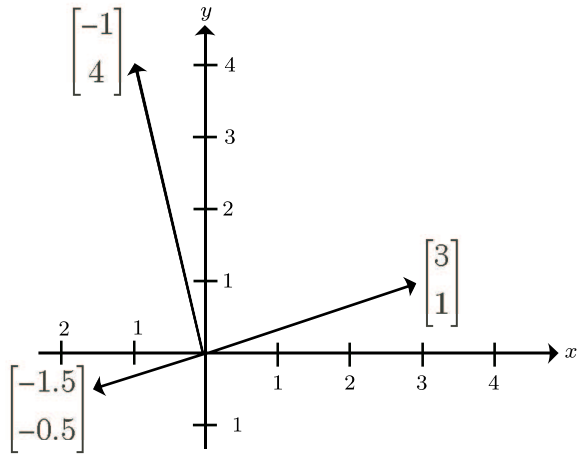


Figure 2.2: Three vectors in the plane

The determinant of A, denoted by $|A|$, is the real number defined by

$$|A| = a_{11}a_{22} - a_{12}a_{21}. \tag{2.19}$$

For example,

$$\left|\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right| = 2 \times 5 - 3 \times 4 = -2. \tag{2.20}$$

(2) Two plane vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent if there exist real numbers α and β, NOT BOTH ZERO, such that

$$\alpha\mathbf{a} + \beta\mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.21}$$

The plane vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent if and only if they lie on the same line when the initial points of these vectors are placed at the origin $(0, 0)$. For example, referring to Figure 2.2, the vectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix}$ are linearly dependent because when their starting points are placed at the origin $(0, 0)$, these two vectors lie on the same line. On the other hand, the vectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ are linearly independent, because when we place their starting points at $(0, 0)$, these two vectors lie on different lines.

Example. *The vectors* $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ *are linearly dependent because*

$$1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \left(-\frac{1}{3}\right)\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.22}$$

Clearly both vectors lie on the same line if their initial points are both placed at the origin $(0, 0)$.

Remark. *Why did we formulate the definition of linear dependence in the form of* (2.21)*, if being linearly dependent just means that the two vectors lie on the same line. This definition is given because it can be easily extended to higher dimensional vectors and to sets of vectors with more than two elements. It is only for vectors in two and three dimensions that linear dependence has a simple geometric interpretation.*

The following fact gives us a relationship between the concepts of linear dependence and determinants.

Fact. Let $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be plane vectors. These vectors are linearly independent if and only if

$$\left| \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \right| \neq 0.$$

(3) Let A be a 2×2 matrix given by (2.15). Let λ be a variable, which may be real or complex. The determinant of the matrix $A - \lambda I$ is given by

$$\begin{aligned} |A - \lambda I| &= \left| \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \right| \\ &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}. \end{aligned} \tag{2.23}$$

Hence the determinant $|A - \lambda I|$ is a quadratic polynomial in λ. The quadratic equation

$$0 = |A - \lambda I| = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} \tag{2.24}$$

is called the *characteristic equation* and its roots are called *eigenvalues* of the matrix A. An easy way to remember the characteristic equation is to notice that it has the form

$$\lambda^2 - \text{trace}(A)\lambda + \text{determinant}(A) = 0, \tag{2.25}$$

where the trace of a matrix is defined to be the sum of its diagonal elements.

Since the eigenvalues of a matrix A are roots of the quadratic equation (2.24) with real coefficients, there are three possibilities:

(i) A has two distinct real eigenvalues;

(ii) A has only one real eigenvalue, which is a double root of the characteristic equation (i.e., (2.24) can be written in the form $(\lambda - E)^2 = 0$ where E is a real number);

(iii) A has two complex eigenvalues that are complex conjugates of each other.

Example. *Find the eigenvalues of the matrix*

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}. \tag{2.26}$$

The eigenvalues of A are the roots of the equation

$$\begin{aligned} 0 = |A - \lambda I| &= \left| \begin{bmatrix} 4-\lambda & -2 \\ 3 & -3-\lambda \end{bmatrix} \right| \\ &= (4-\lambda)(-3-\lambda) + 6 \\ &= \lambda^2 - \lambda - 6. \end{aligned} \tag{2.27}$$

By solving the characteristic equation

$$0 = \lambda^2 - \lambda - 6, \tag{2.28}$$

we see that the eigenvalues of A are 3 *and* -2*.*

Example. *Find the eigenvalues of the matrix*

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}. \tag{2.29}$$

The eigenvalues of A are the roots of the equation

$$\begin{aligned} 0 = |A - \lambda I| &= \left| \begin{bmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{bmatrix} \right| \\ &= (\lambda+1)^2 + 4 \\ &= \lambda^2 + 2\lambda + 5. \end{aligned} \tag{2.30}$$

Using the quadratic formula to solve $\lambda^2 + 2\lambda + 5 = 0$, we see that the eigenvalues of A are the complex conjugates $-1 + 2i$ and $-1 - 2i$.

(4) Let A be a 2×2 matrix, and let E be an eigenvalue of A. An eigenvector of A associated to the eigenvalue E is a NON-ZERO plane vector $\mathbf{v}$ satisfying the equation $A\mathbf{v} = E\mathbf{v}$, or equivalently,

$$(A - EI)\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.31}$$

Example. *Find all the eigenvectors associated to the eigenvalues of the matrix*

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}.$$

In a previous example we found that the eigenvalues of A are 3 *and* -2*. By* (2.31) *the eigenvectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ associated to the eigenvalue* 3 *satisfy the equation*

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (A - 3I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} v_1 - 2v_2 \\ 3v_1 - 6v_2 \end{bmatrix}. \end{aligned} \tag{2.32}$$

Hence we must have

$$\begin{aligned} 0 &= v_1 - 2v_2, \\ 0 &= 3v_1 - 6v_2. \end{aligned} \tag{2.33}$$

Solving the simultaneous equations in (2.33)*, we see that* $v_1 = 2\alpha$ *and* $v_2 = \alpha$*, where* α *is an arbitrary number. Thus, every eigenvector associated to the eigenvalue* 3 *must be of the form* $\begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix}$ *for some non-zero number* α*.*

Similarly the eigenvectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ *associated to the eigenvalue* -2 *satisfy the equation*

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (A + 2I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} 6v_1 - 2v_2 \\ 3v_1 - v_2 \end{bmatrix}. \end{aligned} \tag{2.34}$$

Hence we must have

$$\begin{aligned} 0 &= 6v_1 - 2v_2, \\ 0 &= 3v_1 - v_2. \end{aligned} \tag{2.35}$$

Thus every eigenvector associated with the eigenvalue -2 *must be of the form* $\begin{bmatrix} \alpha \\ 3\alpha \end{bmatrix}$ *for some non-zero number* α*.*

Example. *Find all the eigenvectors associated with the eigenvalues of the matrix*

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}. \tag{2.36}$$

In a previous example we found that the eigenvalues of A are $-1 + 2i$ and $-1 - 2i$. By (2.31) the eigenvectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ associated to $-1 + 2i$ satisfy the equation

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (A - (-1 + 2i)I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} -2iv_1 + 2v_2 \\ -2v_1 - 2iv_2 \end{bmatrix}. \end{aligned} \tag{2.37}$$

Hence we have

$$\begin{aligned} 0 &= -2iv_1 + 2v_2, \\ 0 &= -2v_1 - 2iv_2 \end{aligned} \tag{2.38}$$

Solving (2.38) we see that all eigenvectors associated to the eigenvalue $-1+2i$ must be of the form $\begin{bmatrix} \alpha \\ i\alpha \end{bmatrix}$ for some non-zero number α.

Similarly the eigenvectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ associated to $-1-2i$ satisfy the equation

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (A - (-1-2i)I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} 2iv_1 + 2v_2 \\ -2v_1 + 2iv_2 \end{bmatrix}. \end{aligned} \tag{2.39}$$

Equivalently, we have the system of equations

$$\begin{aligned} 0 &= 2iv_1 + 2v_2, \\ 0 &= -2v_1 + 2iv_2. \end{aligned} \tag{2.40}$$

Thus all the eigenvectors associated to the eigenvalue $-1-2i$ must be of the form $\begin{bmatrix} \alpha \\ -i\alpha \end{bmatrix}$ for some non-zero number α.

Remark. *The last two examples show that every eigenvalue of the matrix A is associated with infinitely many eigenvectors. This is in fact always true. If $\mathbf{v}$ is an eigenvector associated to the eigenvalue E, then for every non-zero number α, the vector $\alpha\mathbf{v}$ is also an eigenvector of E (you should verify this for yourself). Also if $\mathbf{u}$ and $\mathbf{v}$ are both eigenvectors associated to the eigenvalue E, then $\mathbf{u}+\mathbf{v}$ is an eigenvector associated to E as long as $\mathbf{u}+\mathbf{v} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (verify this for yourself).*

(5) Fact. Let A be a 2×2 matrix with two distinct eigenvalues (real or complex) E_1 and E_2 and let $\mathbf{u}$ and $\mathbf{v}$ be eigenvectors associated to E_1 and E_2, respectively. Then the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent; i.e., if α and β are real numbers and if

$$\alpha\mathbf{u} + \beta\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.41}$$

then BOTH α and β must be zero.

(6) Consider the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{2.42}$$

The following fact gives us a condition under which there exists another matrix B such that

$$AB = BA = I. \tag{2.43}$$

Fact. There exists a 2×2 matrix B such that (2.43) holds if and only if

$$|A| \neq 0; \tag{2.44}$$

i.e., if and only if

$$ad - bc \neq 0. \tag{2.45}$$

Remark. *Given a matrix A, there can be at most one matrix B that satisfies* (2.43). *The matrix A is called invertible if such a matrix B exists and B, denoted by A^{-1}, is called the inverse of A. The entries of A^{-1} are*

$$\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}. \tag{2.46}$$

If A^{-1} exists, then, by the fact above, we have $ad - bc \neq 0$. Hence the entries of A^{-1} in (2.46) *are well defined. You should check that*

$$\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = I. \tag{2.47}$$

Homework Assignments

2.1. Draw the vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -6 \\ -5 \end{bmatrix}$ on the same diagram.

2.2. Evaluate the following

1. $\begin{bmatrix} 1 & 3 \\ -4 & 6 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$
2. $\begin{bmatrix} 2.5 & -6 \\ 5 & -3 \end{bmatrix} - \begin{bmatrix} -6 & 0 \\ 2 & -7 \end{bmatrix}$
3. $(-4) \begin{bmatrix} -6 & 20 \\ 6 & -3 \end{bmatrix}$
4. $e^{2t} \begin{bmatrix} t^2 e^{3t} & 6 \\ -7 & -6t \end{bmatrix}$, where t is a real number
5. $\begin{bmatrix} 6 & -2 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \end{bmatrix}$
6. $e^{5t} \begin{bmatrix} -t^2 & e^{6t} \\ 2 & 6t \end{bmatrix} \begin{bmatrix} t^4 \\ -t \end{bmatrix}$, where t is a real number
7. $\begin{bmatrix} 6 & -5 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -5 & 1 \\ 3 & -4 \end{bmatrix}$

8. $\begin{bmatrix} -4 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & -7 \end{bmatrix}$

9. $\begin{bmatrix} e^{2t}t^2 & 6t \\ 5e^{4t} & 3t \end{bmatrix} \begin{bmatrix} e^{-4t} & e^{4t} \\ te^{3t} & t^{-1} \end{bmatrix}$, where t is a real number, $t \neq 0$

2.3. Calculate the determinants of the following matrices.

1. $\begin{bmatrix} 6 & 7 \\ -4 & 3 \end{bmatrix}$

2. $\begin{bmatrix} -5 & 4 \\ -2 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -3 \\ -4 & 0 \end{bmatrix}$

2.4. Determine whether the following pairs of vectors are linearly dependent. If they are linearly dependent, find real numbers α and β, not both zero, such that

$$\alpha \mathbf{u} + \beta \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

1. $\mathbf{u} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} 2c^2 \\ c \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2c^3 \\ c \end{bmatrix}$, where c is a non-zero real number

3. $\mathbf{u} = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} \sqrt{8} \\ 2 \end{bmatrix}$

4. $\mathbf{u} = \begin{bmatrix} e^{5c} \\ e^{2c} \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} e^{3c} \\ 1 \end{bmatrix}$, where c is a real number

2.5. For each of the following matrices, find all eigenvalues and associated eigenvectors.

1. $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -3 \\ \frac{2}{3} & 3 \end{bmatrix}$

3. $\begin{bmatrix} 8 & 9 \\ -4 & -4 \end{bmatrix}$

2.6. For each of the following matrices, determine whether it is invertible. If it is invertible, find its inverse.

1. $\begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}$

2. $\begin{bmatrix} -7 & \sqrt{2} \\ 6 & -4 \end{bmatrix}$

3. $\begin{bmatrix} e^{2t} & t \\ t^2 e^t & \sqrt{t} \end{bmatrix}$, where t is a non-zero real number.

3 General Theory of Linear 2×2 Systems

In this section we shall give some general results concerning systems of differential equations that can be written in the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by + f(t), \\ \frac{dy}{dt} &= cx + dy + g(t), \end{aligned} \tag{3.1}$$

where we assume that a, b, c and d are real numbers and the functions f and g are continuous on an open interval I. If both f and g are identically zero, then the system (3.1) is said to be homogeneous, otherwise it is said to be nonhomogeneous.

(1) Fact. Let t_0 be a real number in the interval I and let x_0 and y_0 be two real numbers. There exists one and only one solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ of the system (3.1) on I that satisfies the initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0. \tag{3.2}$$

(2) Fact (Principle of Superposition). Let $\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ and $\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$ be solutions of the homogeneous linear 2×2 system

$$\begin{aligned} \frac{dx}{dt} &= ax + by, \\ \frac{dy}{dt} &= cx + dy, \end{aligned}$$

and let α and β be real numbers. Then,

$$\alpha\mathbf{u}(t) + \beta\mathbf{v}(t)$$

is a solution of the homogeneous system.

(3) Definition. Let I be an open interval on the real number line and let $\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ and $\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$ be two vector-valued functions defined for all t in I. We say that these two vector-valued functions are linearly dependent on I if there exist two real numbers α and β, NOT BOTH ZERO, such that

$$\alpha \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \beta \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.3}$$

for all t in I.

Remark. *In Section 2, (2), we defined the concept of linear dependence for vectors. Here we have defined linear dependence for vector-valued functions.*

(4) Fact. For every homogeneous linear 2×2 system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= ax + by, \\ \frac{dy}{dt} &= cx + dy, \end{aligned} \tag{3.4}$$

on an open interval I, there exist two solutions $\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ and $\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$ such that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

(5) Fact. If $\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ and $\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$ are two linearly independent solutions of the system (3.4) on an open interval I, then every solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ of (3.4) on I can be written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + c_2 \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \tag{3.5}$$

where c_1 and c_2 are real numbers.

Remark. *An expression that contains arbitrary constants in such a way that every solution of the system is obtained for some choice of these constants is called a general solution of the system.*

From the Facts **(4)** and **(5)**, we see that system (3.4) is solved once we find two linearly independent solutions. This brings up an interesting question:

How can we determine that two solutions of (3.4) are linearly independent?

The following fact provides an answer.

(6) Fact. Let $\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ and $\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$ be two solutions of system (3.4), both defined for all t in I. Their Wronskian function $W(t)$ is defined by

$$W(t) = \left| \begin{bmatrix} u_1(t) & v_1(t) \\ u_2(t) & v_2(t) \end{bmatrix} \right| = u_1(t)v_2(t) - u_2(t)v_1(t). \tag{3.6}$$

The functions $\mathbf{u}$ and $\mathbf{v}$ are linearly independent if and only if, for some number t_0 in I, $W(t_0) \neq 0$. Moreover, if $W(t_0) \neq 0$, then $W(t) \neq 0$ for all t in I.

Example. *Consider the* 2×2 *system*

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{2}x + y, \\ \frac{dy}{dt} &= -x - \frac{1}{2}y. \end{aligned} \tag{3.7}$$

One can check that $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{bmatrix}$ and $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{bmatrix}$ are both solutions of the system (3.7). To find out if they are linearly independent we form their Wronskian function

$$\begin{aligned} W(t) &= \left| \begin{bmatrix} e^{-t/2}\cos t & e^{-t/2}\sin t \\ -e^{-t/2}\sin t & e^{-t/2}\cos t \end{bmatrix} \right| \\ &= e^{-t}(\cos^2 t + \sin^2 t) \\ &= e^{-t}. \end{aligned} \tag{3.8}$$

Since $W(t) \neq 0$, by the fact above, these two solutions are linearly independent. Thus, every solution of (3.7) can be written in the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{bmatrix} \tag{3.9}$$

where c_1 and c_2 are real numbers.

(7) Let us now consider a strategy for finding the general solution of the linear 2×2 system (3.4). Using matrix notation, system (3.4) can be written in the form

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}; \tag{3.10}$$

or, more conveniently, it can be written in the form

$$\frac{d\mathbf{p}}{dt} = A\mathbf{p}, \tag{3.11}$$

where $\mathbf{p} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{3.12}$$

is the coefficient matrix of the system (3.4).
The equation (3.11) should remind us of the first type of differential equation we learned to solve in this course: the first order linear differential equation of the form

$$\frac{dx}{dt} = ax \tag{3.13}$$

where a is a constant real number and x is a function of t. The general solution of (3.13) is $x = ce^{at}$, where c is an arbitrary real number. This suggests that we look for solutions of (3.4) of the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{3.14}$$

where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a constant plane vector and λ is a real number. Substituting (3.14) into (3.10), we obtain

$$\lambda e^{\lambda t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = e^{\lambda t} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \tag{3.15}$$

equivalently, we have

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$(A - \lambda I) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.16}$$

From (3.16) we see that if we have a solution of the form (3.14) of the system (3.4), then λ must be an eigenvalue of the coefficient matrix A and $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ must be an eigenvector associated with λ. Conversely, if λ is an eigenvalue of A and $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is an eigenvector associated to λ, then

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

is a solution of (3.4).

Now suppose that the coefficient matrix A has two distinct real eigenvalues E_1 and E_2 and let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be eigenvectors associated to E_1 and E_2 respectively. By the argument given above we see that both

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{E_1 t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{3.17}$$

and

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{E_2 t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{3.18}$$

are solutions of the system (3.4). Next, form the Wronskian function $W(t)$ of these two vector-valued functions to obtain

$$\begin{aligned} W(t) &= \left| \begin{bmatrix} e^{E_1 t} u_1 & e^{E_2 t} v_1 \\ e^{E_1 t} u_2 & e^{E_2 t} v_2 \end{bmatrix} \right| \\ &= e^{(E_1+E_2)t}(u_1 v_2 - u_2 v_1) \\ &\neq 0. \end{aligned} \tag{3.19}$$

(We have used Fact **(5)** on page 13 and the Fact on page 10, in the last inequality of (3.19).) Using Fact **(6)**, the solutions (3.17) and (3.18) are linearly independent. Thus by Fact **(5)**, the general solution of (3.4) is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{E_1 t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + c_2 e^{E_2 t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{3.20}$$

where c_1 and c_2 are arbitrary real numbers.

The above strategy for finding the solution of a linear 2×2 system always works if the coefficient matrix A has two distinct real eigenvalues. In this case, each pair of eigenvectors (consisting of one eigenvector corresponding to each eigenvalue) is linearly independent; hence, every vector in the plane can be expressed as a linear combination of eigenvectors. This strategy does not always work if A has only one eigenvalue because, in this case, there may not be two linearly independent eigenvectors. A new method is required to solve the corresponding systems of differential equations; it will be explained starting on page 26.

Homework Assignments

3.1. Verify Fact **(2)**.

3.2. Use the definition of linear dependence to determine whether the vector-valued functions $\mathbf{u}(t) = \begin{bmatrix} t^4 \\ t^2 \end{bmatrix}$ and $\mathbf{v}(t) = \begin{bmatrix} t^2 \\ 1 \end{bmatrix}$ are linearly dependent or linearly independent on the interval $(-10, 10)$. Then compute the determinant of

$$\begin{bmatrix} t^4 & t^2 \\ t^2 & 1 \end{bmatrix}.$$

The determinant looks a bit like a Wronskian. Have you found a contradiction? Why or why not?

3.3. Given that

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{4t} \begin{bmatrix} -3 \\ 3 \end{bmatrix} \tag{3.21}$$

and

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{4t} \begin{bmatrix} -3t+1 \\ 3t \end{bmatrix} \tag{3.22}$$

are solutions of the system

$$\begin{aligned} \frac{dx}{dt} &= x - 3y, \\ \frac{dy}{dt} &= 3x + 7y, \end{aligned}$$

determine whether these solutions are linearly independent.

3.4. Check that the functions given in (3.21) and (3.22) really are solutions of the system in exercise 3.3.

4 Case 1

In this section we consider linear 2×2 systems of differential equations of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by, \\ \frac{dy}{dt} &= cx + dy, \end{aligned} \tag{4.1}$$

where the coefficient matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{4.2}$$

has two distinct real eigenvalues. Let E_1 and E_2 be the two distinct real eigenvalues of A and let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be eigenvectors associated with E_1 and E_2, respectively. Then, the general solution of the system of differential equations (4.1) is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{E_1 t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + c_2 e^{E_2 t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{4.3}$$

where c_1 and c_2 are arbitrary constants.

Example. *Find the general solution of the system of differential equations*

$$\begin{aligned} \frac{dx}{dt} &= 3x - y, \\ \frac{dy}{dt} &= 4x - 2y. \end{aligned} \tag{4.4}$$

Also, find the solution that satisfies the initial condition

$$x(0) = 4, \quad y(0) = 3. \tag{4.5}$$

The coefficient matrix

$$A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \tag{4.6}$$

has eigenvalues 2 and -1. The real eigenvectors associated to the eigenvalue 2 all have the form $\begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$ and the eigenvectors associated to the eigenvalue -1 all have the form $\begin{bmatrix} \alpha \\ 4\alpha \end{bmatrix}$, where α is a non-zero real number. Taking $\alpha = 1$, we see that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ are eigenvectors associated to the eigenvalues 2 and -1, respectively. Thus, by (4.3), the general solution of (4.4) is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \tag{4.7}$$

where c_1 and c_2 are arbitrary constants. To find the solution of (4.4) that satisfies condition (4.5), we must choose the constants c_1 and c_2 in (4.7) so that (4.5) is satisfied. At $t = 0$, equation (4.7) becomes

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}. \tag{4.8}$$

This vector equation is equivalent to the system of equations

$$\begin{aligned} 4 &= c_1 + c_2, \\ 3 &= c_1 + 4c_2. \end{aligned} \tag{4.9}$$

By solving (4.9), we obtain $c_1 = \frac{13}{3}$ and $c_2 = \frac{-1}{3}$. Therefore, the particular solution satisfying (4.5) is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{13}{3} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{3} e^{-t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \tag{4.10}$$

or equivalently,

$$\begin{aligned} x(t) &= \frac{13}{3} e^{2t} - \frac{1}{3} e^{-t}, \\ y(t) &= \frac{13}{3} e^{2t} - \frac{4}{3} e^{-t}. \end{aligned} \tag{4.11}$$

Homework Assignments

4.1. Find the general solution of the system

$$\begin{aligned} \frac{dx}{dt} &= 4x + y \\ \frac{dy}{dt} &= 3x + 2y. \end{aligned}$$

4.2. Find the solution of the system

$$\frac{dx}{dt} = 3x + 2y$$
$$\frac{dy}{dt} = x + 2y$$

with $x(2) = 3$ and $y(2) = 1$.

4.3. Find the general solution of the system

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = x + y.$$

What are the possible behaviors of a solution of the system as $t \to \infty$?

4.4. (i) Rewrite the linear second order equation

$$2\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 0 \tag{4.12}$$

as a linear 2×2 system.

(ii) Show that the roots of the characteristic equation of (4.12) are the same as the eigenvalues of the corresponding linear 2×2 system.

(iii) Find the general solution of (4.12) by the method of Chapter 3 in Boyce and DiPrima.

(iv) Find the general solution of the linear 2×2 system corresponding to (4.12) and reconcile it to your answer to (iii).

4.5. For the system

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

describe the behavior of a solution as $t \to \infty$ without finding the general solution.

5 Case 2

In this section we consider linear 2×2 systems of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by, \\ \frac{dy}{dt} &= cx + dy, \end{aligned} \tag{5.1}$$

where the coefficient matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{5.2}$$

has two distinct complex eigenvalues E_1 and E_2. Since E_1 and E_2 must be complex conjugates (see Section 2.2, **(3)**), we have $E_1 = \sigma + i\tau$ and $E_2 = \sigma - i\tau$ for some pair of real numbers σ and τ with $\tau \neq 0$. Let $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, where a_1, a_2, b_1, b_2 are real numbers, be an eigenvector associated to E_1. The general solution of the system (5.1) is

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= c_1 e^{\sigma t} \left(\cos(\tau t) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \sin(\tau t) \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \\ &\quad + c_2 e^{\sigma t} \left(\sin(\tau t) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \cos(\tau t) \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \end{aligned} \tag{5.3}$$

where c_1 and c_2 are arbitrary real numbers.

Remark. *By the method in Section 3,* **(7)**

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{E_1 t} \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \tag{5.4}$$

is a complex solution of the system (5.1)*. Separating the right side of* (5.4) *into real and imaginary parts, we have*

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= e^{(\sigma + i\tau)t} \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \\ &= e^{\sigma t} (\cos(\tau t) + i \sin(\tau t)) \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \\ &= e^{\sigma t} \left(\cos(\tau t) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \sin(\tau t) \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \\ &\quad + i e^{\sigma t} \left(\sin(\tau t) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \cos(\tau t) \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right). \end{aligned} \tag{5.5}$$

(Notice that the real and imaginary parts of (5.5) are exactly the two summands in (5.3).) Since all the entries in the coefficient matrix are real numbers, we see (by writing (5.1) in the form (3.10)) that the real and imaginary parts of the solution (5.5) must each satisfy system (5.1). (Exercise: Convince yourself of this fact.) Hence, both are real solutions of (5.1). It can then be shown that these two real solutions are linearly independent. So, by Fact **(5)** stated in Section 3, the general solution of (5.1) is given by (5.3).

Example. *Find the general solution of the system*

$$\begin{aligned} \frac{dx}{dt} &= -x + 2y, \\ \frac{dy}{dt} &= -2x - y. \end{aligned} \tag{5.6}$$

The coefficient matrix is

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \tag{5.7}$$

and

$$|A - \lambda I| = \lambda^2 + 2\lambda + 5 = 0 \tag{5.8}$$

is its characteristic equation. The corresponding eigenvalues are $-1 \pm 2i$. To find an eigenvector associated to the eigenvalue $E = -1 + 2i$, we must solve one of the equivalent equations

$$(A - EI)u = 0, \quad (EI - A)u = 0, \text{ or } Au = Eu.$$

Employing the third choice for this example, we solve the equation

$$\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = (-1 + 2i) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{5.9}$$

or equivalently, the system

$$\begin{aligned} -u_1 + 2u_2 &= (-1 + 2i)u_1, \\ -2u_1 - u_2 &= (-1 + 2i)u_2. \end{aligned} \tag{5.10}$$

Both equations are equivalent to the single equation $u_2 = iu_1$, which has infinitely many solutions. A convenient choice for a solution is $u_1 = 1$ and $u_2 = i$; it corresponds to the complex solution

$$e^{(-1+2i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

of the homogeneous system. A fundamental set of real solutions is given by the real and imaginary parts of this complex solution. In fact, these solutions are

$$e^{-t} \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix}, \qquad e^{-t} \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}.$$

Thus, the general solution of (5.6) is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}. \tag{5.11}$$

Homework Assignments

5.1. Find the general solution of the system

$$\begin{aligned} \frac{dx}{dt} &= -x + y \\ \frac{dy}{dt} &= -4x - y. \end{aligned}$$

5.2. Solve the initial value problem

$$\frac{dx}{dt} = x + 4y$$
$$\frac{dy}{dt} = -25x + y$$
$$x(0) = 2000, \quad y(0) = 5.$$

5.3. Find the general solution of the system

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -4 & 10 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Describe all possible behaviors of solutions as $t \to \infty$.

5.4. The motion of a spring-mass system is described by the initial value problem

$$\begin{aligned} &mu'' + \gamma u' + ku = 0 \\ &u(t_0) = u_0, \quad u'(t_0) = v_0 \end{aligned} \tag{5.12}$$

where m, γ, k are the mass, damping constant and spring constant, respectively, of the system (see Boyce and DiPrima, Section 3.8). We assume that these quantities m, γ and k are all positive and that

$$\gamma < 2\sqrt{mk}.$$

(i) Rewrite (5.12) as an initial value problem of an equivalent linear 2×2 system of differential equations.

(ii) Find the general solution of the 2×2 system in (i) (i.e. ignore the initial conditions).

(iii) Describe the behavior of the solution of the initial value problem of (i) as $t \to \infty$.

5.5. Find the general solution of the system

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and describe the behaviors of its solutions as $t \to \infty$.

6 Case 3

In this section we consider linear 2×2 systems of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by, \\ \frac{dy}{dt} &= cx + dy, \end{aligned} \tag{6.1}$$

where the coefficient matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{6.2}$$

has only one eigenvalue E. There are two possibilities: we can choose two linearly independent eigenvectors corresponding to the eigenvalue E; or, no such pair of linearly independent eigenvectors exists.

In case a pair of linearly independent eigenvectors exists, we can proceed as if there were two distinct real eigenvalues. In case no such pair of eigenvalues exists, we must use a different method (which is based on the following fact).

Fact. Suppose that A is a 2×2 matrix with exactly one eigenvalue E. If the set of corresponding eigenvectors does not contain two linearly independent vectors, then for each eigenvector u, there is a vector v such that $(A - EI)v = u$.

For the new method, let $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be an eigenvector associated with the eigenvalue E. By the **Fact**, there is a vector $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that

$$(A - EI) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{6.3}$$

The general solution of (6.1) is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{Et} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + c_2 e^{Et} \left(t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right). \tag{6.4}$$

Explanation. Using the results in Section 3 (7), a solution of system (6.1) is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{Et} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{6.5}$$

Let us look for a second solution of the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{Et} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + te^{Et} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{6.6}$$

where $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is a vector to be determined. By substituting (6.6) in (6.1), note that $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ must satisfy the equation

$$(A - EI) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{6.7}$$

By the **Fact**, we can solve for $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. It is possible to prove (by computing their Wronskian) that the two solutions in (6.5) and (6.6), with $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ satisfying (6.7), are linearly independent. Thus, the general solution of (6.1) is indeed given by equation (6.4).

Example. *Find the general solution of the system*

$$\begin{aligned}\frac{dx}{dt} &= -4x - y,\\ \frac{dy}{dt} &= 4x - 8y.\end{aligned} \tag{6.8}$$

First find the eigenvalues of the coefficient matrix

$$A = \begin{bmatrix} -4 & -1 \\ 4 & -8 \end{bmatrix} \tag{6.9}$$

by solving the quadratic equation

$$\begin{aligned}|A - \lambda I| &= \left| \begin{bmatrix} -4-\lambda & -1 \\ 4 & -8-\lambda \end{bmatrix} \right| \\ &= \lambda^2 + 12\lambda + 36 \\ &= (\lambda+6)^2 \\ &= 0.\end{aligned} \tag{6.10}$$

In this case $\lambda = -6$ is the only eigenvalue of A. To find the associated eigenvectors, solve the equation

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (A - (-6)I) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{6.11}$$

or equivalently, the single equation

$$2u_1 - u_2 = 0. \tag{6.12}$$

All the real eigenvectors associated to -6 are of the form $\begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix}$, where α can be any non-zero real number. In particular, with $\alpha = 1$, the vector

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is an eigenvector associated to -6. By (6.3), to find the vector $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, we solve

$$\begin{aligned}\begin{bmatrix} 1 \\ 2 \end{bmatrix} &= (A + 6I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},\end{aligned} \tag{6.13}$$

or equivalently,

$$2v_1 - v_2 = 1. \tag{6.14}$$

The solution $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ can be any vector of the form $\begin{bmatrix} \beta \\ 2\beta - 1 \end{bmatrix}$, where β is a real number. With $\beta = 1$, we get $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, by (6.4), the general solution of (6.8) is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t} \left(t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right). \tag{6.15}$$

Homework Assignments

Find the general solution of the following systems:

6.1.
$x' = 4x - y$
$y' = x + 2y$

6.2.
$x' = -3x - 8y$
$y' = 2x + 5y$

6.3.
$x' = -\frac{1}{2}x + \frac{1}{2}y$
$y' = -\frac{9}{2}x - \frac{7}{2}y$

6.4.
$x' = 3x$
$y' = 3y$

Find the solution to the initial value problems:

6.5.
$x' = \frac{1}{2}x + \frac{1}{2}y$
$y' = -2x - \frac{3}{2}y$
$x(0) = -5,\ y(0) = 6$

6.6.
$x' = 5x + y$
$y' = -4x + y$
$x(0) = 4,\ y(0) = 2$

6.7. Find all the (real) values of s so that every solution of the system
$x' = sx - y$
$y' = x + (2 + s)y$
has $\lim_{t\to\infty} x(t) = 0$ *and* $\lim_{t\to\infty} y(t) = 0$.

7 Solutions of Nonhomogeneous Systems

In this section we consider linear 2×2 systems of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by + f(t), \\ \frac{dy}{dt} &= cx + dy + g(t), \end{aligned} \tag{7.1}$$

where the functions f and g are continuous on an open interval I.

Fact. Suppose that

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

are two linearly independent solutions of the homogeneous system

$$\begin{aligned} \frac{dx}{dt} &= ax + by, \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

corresponding to (7.1) and

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \tag{7.2}$$

is a solution of (7.1). Then every solution of (7.1) on the interval I can be written in the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + c_2 \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \tag{7.3}$$

where c_1 and c_2 are some constant real numbers.

Notice that the first two terms of (7.3) form the general solution of the corresponding homogeneous 2×2 system of (7.1). Hence (7.3) tells us that the general solution of the nonhomogeneous system (7.1) is the sum of a particular solution and the general solution of the corresponding homogeneous system. Since we already know how to find the general solution of the corresponding homogeneous system of (7.1), if we can find a solution of (7.1), then we can construct the general solution of (7.1). In this section we will learn a method, called variation of parameters, for finding a solution of (7.1).

Description of the Method. Using the methods in Sections 4, 5 and 6, find two linearly independent solutions

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

of the homogeneous system corresponding to (7.1) and form the matrix

$$\Phi(t) = \begin{bmatrix} u_1(t) & v_1(t) \\ u_2(t) & v_2(t) \end{bmatrix}. \tag{7.4}$$

In general, a matrix of functions is called a fundamental matrix for the homogeneous system corresponding to (7.1) if its columns are linearly independent solutions of the homogeneous system. (There are infinitely many fundamental matrices, depending on which two linearly independent solutions we use.) Because the solutions

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

are linearly independent, $\Phi(t)$ is an invertible matrix. Particular solutions of (7.1) are given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \Phi(t) \int \Phi(t)^{-1} \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} dt. \tag{7.5}$$

(The notation on the right-hand side of equation (7.5) means we must first find an antiderivative of the vector function

$$t \mapsto \Phi(t)^{-1} \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

and then multiply our choice of antiderivative by the fundamental matrix $\Phi(t)$). The general solution of (7.1) is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + c_2 \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} + \Phi(t) \int \Phi(t)^{-1} \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} dt, \tag{7.6}$$

or equivalently,

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \Phi(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \Phi(t) \int \Phi(t)^{-1} \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} dt.$$

This important result is called the variation of parameters formula.

Explanation. Since

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

are solutions of the homogeneous system, we have

$$\begin{bmatrix} \frac{du_1}{dt} & \frac{dv_1}{dt} \\ \frac{du_2}{dt} & \frac{dv_2}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u_1(t) & v_1(t) \\ u_2(t) & v_2(t) \end{bmatrix}, \tag{7.7}$$

or in a more compact form

$$\frac{d}{dt}\Phi(t) = A\Phi(t), \tag{7.8}$$

where A is the coefficient matrix.

Let us look for a particular solution of (7.1) in the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \Phi(t) \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) & v_1(t) \\ u_2(t) & v_2(t) \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \tag{7.9}$$

where the functions $w_1(t)$ and $w_2(t)$ will be determined. Choose t_0 in the interval I. Let us determine the solution of (7.1) that vanishes at t_0. By (7.9) and using the product rule for differentiation, we have

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \frac{d}{dt}(\Phi(t)) \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + \Phi(t) \begin{bmatrix} \frac{dw_1}{dt} \\ \frac{dw_2}{dt} \end{bmatrix}. \tag{7.10}$$

Because $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a solution of (7.1) and by (7.8), substituting in (7.10) yields

$$A\Phi(t) \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = A\Phi(t) \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + \Phi(t) \begin{bmatrix} \frac{dw_1}{dt} \\ \frac{dw_2}{dt} \end{bmatrix}. \tag{7.11}$$

Hence, we have

$$\Phi(t) \frac{d}{dt} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}. \tag{7.12}$$

Multiplying both sides of (7.12) by $\Phi^{-1}(t)$ and integrating, we obtain

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \int \Phi(t)^{-1} \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} dt. \tag{7.13}$$

Substituting (7.13) into (7.9), we get the particular solution of (7.1) that is given by (7.5).

Example. *Solve the initial value problem*

$$\begin{aligned} \frac{dx}{dt} &= 4x - 2y + 4te^{6t}, \\ \frac{dy}{dt} &= 3x - 3y - 5t, \\ x(0) &= 4, \quad y(0) = 3. \end{aligned} \tag{7.14}$$

In the examples in Section 2 we showed that the coefficient matrix

$$\begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix} \tag{7.15}$$

of the system (7.14) has eigenvalues 3 and -2 and that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are eigenvectors associated to 3 and -2 respectively. Hence

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{7.16}$$

and

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{-2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{7.17}$$

are linearly independent solutions of the corresponding homogenous system of (7.14). Thus, by (7.4), a fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 2e^{3t} & e^{-2t} \\ e^{3t} & 3e^{-2t} \end{bmatrix} \tag{7.18}$$

with inverse (recall (2.46))

$$\begin{aligned}\Phi(t)^{-1} &= \begin{bmatrix} \frac{3e^{-2t}}{5e^t} & \frac{-e^{-2t}}{5e^t} \\ \frac{-e^{3t}}{5e^t} & \frac{2e^{3t}}{5e^t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{5}e^{-3t} & -\frac{1}{5}e^{-3t} \\ -\frac{1}{5}e^{2t} & \frac{2}{5}e^{2t} \end{bmatrix}.\end{aligned} \tag{7.19}$$

Thus,

$$\begin{aligned}\int \Phi(t)^{-1} \begin{bmatrix} 4te^{6t} \\ -5t \end{bmatrix} dt &= \int \begin{bmatrix} \frac{3}{5}e^{-3t} & -\frac{1}{5}e^{-3t} \\ -\frac{1}{5}e^{2t} & \frac{2}{5}e^{2t} \end{bmatrix} \begin{bmatrix} 4te^{6t} \\ -5t \end{bmatrix} dt \\ &= \int \begin{bmatrix} \frac{12}{5}te^{3t} + te^{-3t} \\ -\frac{4}{5}te^{8t} - 2te^{2t} \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{12}{5}\left(\frac{1}{3}te^{3t} - \frac{1}{9}e^{3t}\right) + \left(-\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t}\right) \\ -\frac{4}{5}\left(\frac{1}{8}te^{8t} - \frac{1}{64}e^{8t}\right) - 2\left(\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t}\right) \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{3}\left(t + \frac{1}{3}\right)e^{-3t} + \frac{4}{5}\left(t - \frac{1}{3}\right)e^{3t} \\ -\left(t - \frac{1}{2}\right)e^{2t} - \frac{1}{10}\left(t - \frac{1}{8}\right)e^{8t} \end{bmatrix}\end{aligned} \tag{7.20}$$

and a solution of (7.14) is, by (7.5),

$$\begin{aligned}&\begin{bmatrix} 2e^{3t} & e^{-2t} \\ e^{3t} & 3e^{-2t} \end{bmatrix} \begin{bmatrix} -\frac{1}{3}\left(t + \frac{1}{3}\right)e^{-3t} + \frac{4}{5}\left(t - \frac{1}{3}\right)e^{3t} \\ -\left(t - \frac{1}{2}\right)e^{2t} - \frac{1}{10}\left(t - \frac{1}{8}\right)e^{8t} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{2}{3}\left(t + \frac{1}{3}\right) + \frac{8}{5}\left(t - \frac{1}{3}\right)e^{6t} - \left(t - \frac{1}{2}\right) - \frac{1}{10}\left(t - \frac{1}{8}\right)e^{6t} \\ -\frac{1}{3}\left(t + \frac{1}{3}\right) + \frac{4}{5}\left(t - \frac{1}{3}\right)e^{6t} - 3\left(t - \frac{1}{2}\right) - \frac{3}{10}\left(t - \frac{1}{8}\right)e^{6t} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{5}{3}t + \frac{5}{18} + \left(\frac{3}{2}t - \frac{25}{48}\right)e^{6t} \\ -\frac{10}{3}t + \frac{25}{18} + \left(\frac{1}{2}t - \frac{11}{48}\right)e^{6t} \end{bmatrix}.\end{aligned} \tag{7.21}$$

By (7.6), the general solution of (7.14), is

$$\begin{aligned}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= c_1e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2e^{-2t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &\quad + \begin{bmatrix} -\frac{5}{3}t + \frac{5}{18} + \left(\frac{3}{2}t - \frac{25}{48}\right)e^{6t} \\ -\frac{10}{3}t + \frac{25}{18} + \left(\frac{1}{2}t - \frac{11}{48}\right)e^{6t} \end{bmatrix}.\end{aligned} \tag{7.22}$$

When $t = 0$, we have $x(0) = 4$ and $y(0) = 3$. Hence

$$\begin{aligned}4 &= 2c_1 + c_2 + \frac{5}{18} - \frac{25}{48} \\ 3 &= c_1 + 3c_2 + \frac{25}{18} - \frac{11}{48}.\end{aligned} \tag{7.23}$$

It follows that $c_1 = \frac{98}{45}$ and $c_2 = -\frac{9}{80}$. Substituting these into (7.22) we obtain the solution of the initial value problem

$$\begin{aligned}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \frac{98}{45}e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{9}{80}e^{-2t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &\quad + \begin{bmatrix} -\frac{5}{3}t + \frac{5}{18} + \left(\frac{3}{2}t - \frac{25}{48}\right)e^{6t} \\ -\frac{10}{3}t + \frac{25}{18} + \left(\frac{1}{2}t - \frac{11}{48}\right)e^{6t} \end{bmatrix}.\end{aligned} \tag{7.24}$$

Example. *Let us solve the special case of the tank mixing problem* (1.6) *given by*

$$\begin{aligned} \dot{u} &= -u + \frac{2}{9}v + \frac{1}{100}, \\ \dot{v} &= 2u - 2v, \\ u(0) &= 1, \\ v(0) &= 0. \end{aligned} \tag{7.25}$$

The original tank mixing problem (1.6) has several parameters. You might ask if the special form of (7.25) represents a reasonable example. You can skip this paragraph if you are interested only in the solution method. We will introduce a new (but easy) technique that might convince you that our special case is reasonable. We will also see that this technique tames the zoo of parameters in the general case. The idea here is to make the system *dimensionless* by a change of variables. This should always be done when solving real physical problems. The change of variables is of the form

$$u = \frac{x}{A}, \qquad v = \frac{y}{B}, \qquad s = \frac{t}{\lambda}$$

where A and B are measured in pounds and λ is measured in minutes. We start (using (1.6)) as follows:

$$\frac{du}{ds} = \frac{du}{dt}\frac{dt}{ds} = \frac{\lambda}{A}\frac{dx}{dt} = \frac{\lambda}{A}\left(-\frac{Ad}{V_1}u + \frac{cB}{V_2}v + a\alpha\right).$$

After cleaning up the algebra and doing the same procedure for the variable v, we arrive at the system

$$\begin{aligned} \frac{du}{ds} &= -\frac{\lambda d}{V_1}u + \frac{cB\lambda}{AV_2}v + \frac{a\alpha\lambda}{A}, \\ \frac{dv}{ds} &= \frac{A\lambda d}{BV_1}u - \frac{\lambda d}{V_2}v. \end{aligned}$$

We now make some choices for the variables A, B and λ. To make the coefficient of u in the first equation -1, we choose $\lambda = V_1/d$. It is also convenient to choose $A = \alpha V_1$ and $B = \alpha V_2$. Note that these quantities have the correct dimensions. After substitution into the system, we obtain the dimensionless system

$$\begin{aligned} \frac{du}{ds} &= -u + \beta v + \epsilon, \\ \frac{dv}{ds} &= \gamma u - \gamma v, \end{aligned} \tag{7.26}$$

where

$$\beta = \frac{c}{d}, \qquad \epsilon = \frac{a}{d}, \qquad \gamma = \frac{V_1}{V_2}.$$

It is now clear that the important parameters are the dimensionless ratios given in the last display, which makes perfect sense! In a real physical application, we must remember also to change the initial conditions to dimensionless form. Note that we can recover the solution in the original variables from a solution of the dimensionless system by a change of variables. After this explanation, it should be obvious that system (7.25) is a reasonable special case. Of course, system (7.26) is also in the best form for a more complete analysis of the original system. You are invited to find the general solution of the dimensionless system.

We will solve system (7.25) using variation of parameters. The coefficient matrix for the homogeneous system is

$$\begin{bmatrix} -1 & \frac{2}{9} \\ 2 & -2 \end{bmatrix}.$$

Its eigenvalues are $-7/3$ and $-2/3$, and its corresponding eigenvectors are

$$\begin{bmatrix} -\frac{1}{6} \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}.$$

Using these results, we have—by choosing the columns to be independent solutions—the fundamental matrix given by

$$\Phi(t) = \begin{bmatrix} -\frac{1}{6}e^{-7t/3} & \frac{2}{3}e^{-2t/3} \\ e^{-7t/3} & e^{-2t/3} \end{bmatrix}$$

and its inverse

$$\Phi(t)^{-1} = \begin{bmatrix} -\frac{6}{5}e^{7t/3} & \frac{4}{5}e^{7t/3} \\ \frac{6}{5}e^{2t/3} & \frac{1}{5}e^{2t/3} \end{bmatrix}.$$

By the variation of parameters formula

$$\begin{aligned} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} &= \Phi(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \Phi(t) \int \Phi(t)^{-1} \begin{bmatrix} \frac{1}{100} \\ 0 \end{bmatrix} dt \\ &= \begin{bmatrix} -\frac{1}{6}e^{-7t/3} & \frac{2}{3}e^{-2t/3} \\ e^{-7t/3} & e^{-2t/3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{6}e^{-7t/3} & \frac{2}{3}e^{-2t/3} \\ e^{-7t/3} & e^{-2t/3} \end{bmatrix} \int \begin{bmatrix} -\frac{6}{5}e^{7t/3} & \frac{4}{5}e^{7t/3} \\ \frac{6}{5}e^{2t/3} & \frac{1}{5}e^{2t/3} \end{bmatrix} \begin{bmatrix} \frac{1}{100} \\ 0 \end{bmatrix} dt. \end{aligned}$$

After a computation, we obtain

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{6}e^{-7t/3} & \frac{2}{3}e^{-2t/3} \\ e^{-7t/3} & e^{-2t/3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{9}{700} \\ \frac{9}{700} \end{bmatrix}.$$

We can now impose the initial conditions at $t = 0$. This results in the linear system of algebraic equations

$$\begin{aligned} -\frac{1}{6}c_1 + \frac{2}{3}c_2 + \frac{9}{700} &= 1 \\ c_1 + c_2 + \frac{9}{700} &= 0, \end{aligned}$$

which has the solution $c_1 = -2091/1750$ and $c_2 = 591/500$. The solution of the initial value problem is

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{6}e^{-7t/3} & \frac{2}{3}e^{-2t/3} \\ e^{-7t/3} & e^{-2t/3} \end{bmatrix} \begin{bmatrix} \frac{-2091}{1750} \\ \frac{591}{500} \end{bmatrix} + \begin{bmatrix} \frac{9}{700} \\ \frac{9}{700} \end{bmatrix}.$$

Note that in the long run (for $t \to \infty$), the system goes to a steady state $(u, v) = (9/700, 9/700)$. Can you show this result without solving the system? Can you find the solution of system (7.25) without using variation of parameters? Hint: The general solution of a linear system is always given by the general homogeneous solution plus a particular solution.

Homework Assignments

Find the solutions to the initial value problems:

7.1.
$x' = 4x - 6y + 10$
$y' = x - y$
$x(0) = 0,\ y(0) = 0$

7.2.
$x' = 2y + e^t$
$y' = -x - 3y + 3e^t$
$x(0) = 0,\ y(0) = 1$

Find the general solution of the following systems:

7.3.
$x' = x - y$
$y' = 3x + 5y + t$

7.4.
$x' = -y$
$y' = x + \cos(t)$

7.5.
$x' = 4x - 2y + 8$
$y' = 6x - 4y + 2e^t$

7.6.
Find the general solution of the following system for $t > 0$.
$x' = y$
$y' = -4x - 4y + t^{-2}e^{-2t}$
This problem is related to problem 7, Section 3.7 of Boyce & Diprima (7th edition). How are the problems related, and how are the answers related?

8 Qualitative Methods

Suppose a point is moving on the plane and its position coordinates x and y, which are functions of time t, satisfy the equations in the system

$$\begin{aligned} \frac{dx}{dt} &= ax + by, \\ \frac{dy}{dt} &= cx + dy, \end{aligned} \tag{8.1}$$

or equivalently,

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{8.2}$$

In this section we are interested in the geometric behavior of all such parametric curves, which are called trajectories of the system (8.1). For simplicity we will assume that the coefficient matrix of our homogeneous linear system has only non-zero eigenvalues.

The essential idea that connects the system (8.1) to the geometry of its solutions is very simple. A solution of our system of differential equations can be viewed as a parametric curve in the plane whose position vector at time t is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. The left-hand side of (8.2), namely $\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}$, evaluated at time t, is the velocity of the moving point. The differential equation states that this velocity is given as a function of the position; that is, the velocity at time t is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{8.3}$$

Thus, if we were to plot the vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{8.4}$$

(translated so that its tail is at the point (x, y)) for each point (x, y) of the plane, then the solutions of the differential equation are exactly those parametric curves that pass through each point (x, y) with velocity vector (8.4).

Using a computer with software such as Mathematica, Maple, or Matlab (or with pencil and paper), we can plot the vector field described in the last paragraph at a finite number of points. We can also plot a few of the solutions of the corresponding system of differential equations to illustrate their geometry. Such a picture of the trajectories of a system of differential equations is called a *phase portrait*; it should have enough trajectories plotted so that we can tell at a glance the geometry of all trajectories.

Example. *We will illustrate the connection between the vector field defined by the right-hand side and the solutions of the system*

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= 2x. \end{aligned} \tag{8.5}$$

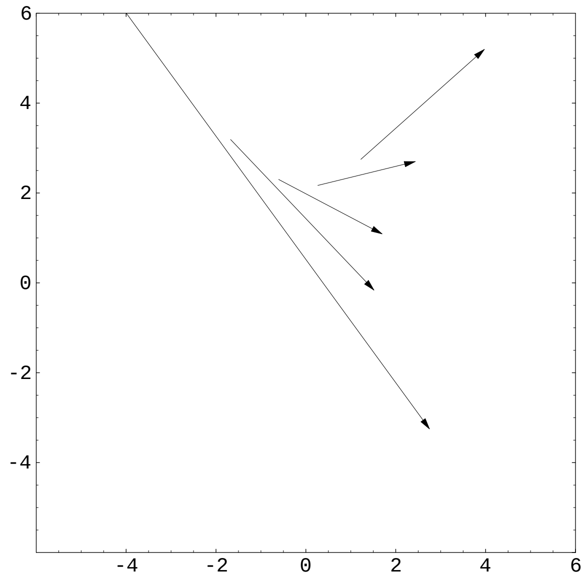


Figure 8.1: Five vectors generated by the right-hand side of the system (8.5)

Figure 8.1 is the plot of exactly five velocity vectors in the plane generated from the right-hand side of the system (8.5). Note that if the tail is at (x, y), then the head is at the point $(x, y) + (y, 2x)$. For example, one of the vectors has tail at (approximately) $(1.22, 2.75)$ and head at the point $(3.97, 5.19)$.

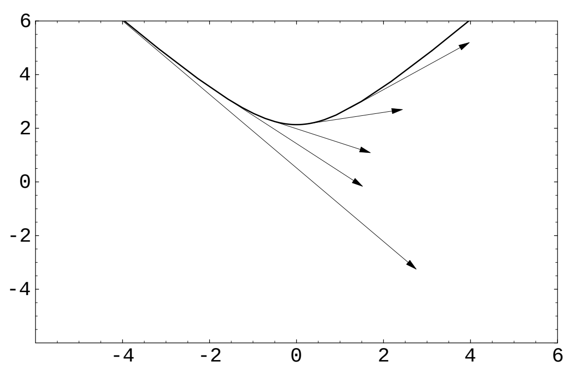


Figure 8.2: The five vectors generated by the right-hand side of the system (8.5) together with the trajectory starting at the point $(-6, 8.75)$ at time $t = 0$.

Figure 8.2 depicts the vectors in Figure 8.1 together with the trajectory of the system (8.5) with the initial conditions $x(0) = -6$ and $y(0) = 8.75$. Note that the velocity vectors along the trajectory depicted in Figure 8.2 have different lengths, which correspond to the speed of the particle at different points. The speed of the particle might be important for some purposes, but it is not relevant to the phase portrait, which is meant to show the qualitative behavior of the solutions of the system of differential equations. Thus, when we draw the vector field, it is preferable to draw only the direction field; that is, all the vectors are taken to have the same length.

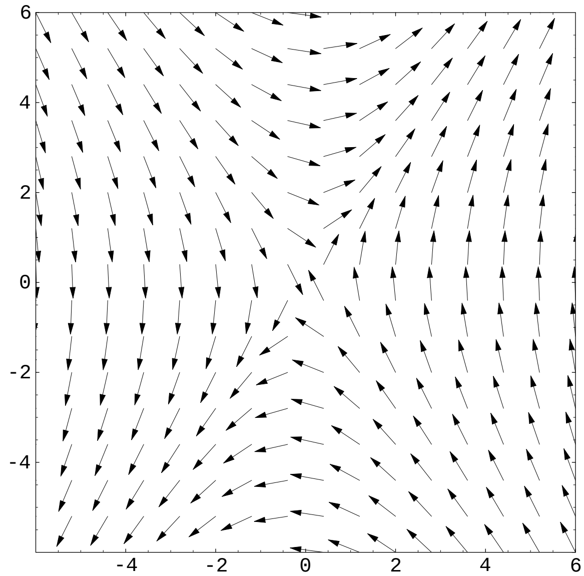


Figure 8.3: The direction field on a 16×16 grid for the system (8.5).

A direction field plot that shows a grid of normalized velocity vectors (i.e. vectors of unit length in the directions of the corresponding velocity vectors) for the system (8.5) is depicted in Figure 8.3. This plot suggests the general qualitative behavior of the system: Most trajectories starting near the upper left move toward the origin for a while and eventually leave the depicted region in the direction of the upper right or lower left. Other trajectories enter from the lower right and leave the depicted region in the direction of the upper right or lower left. The different behaviors must be separated by some special trajectories. In this case the separating trajectories lie on the invariant lines through the origin determined by the eigenvectors of the coefficient matrix. A phase portrait of system (8.5) is depicted in Figure 8.4.

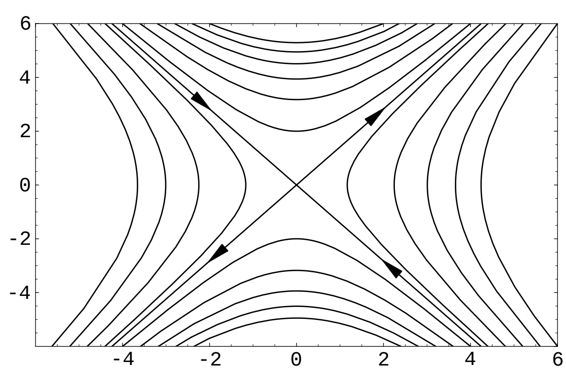


Figure 8.4: The phase portrait for the system (8.5).

Using the methods discussed in previous sections, we can draw the phase portrait by hand. To do so, let us first determine the eigenvalues and the associated eigenvectors for the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}. \tag{8.6}$$

The eigenvalues of A are $\pm\sqrt{2}$. The associated eigenvectors are $\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$, respectively. The general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-\sqrt{2}t} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} + c_2 e^{\sqrt{2}t} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}. \tag{8.7}$$

To obtain the phase portrait, we first sketch the trajectory for the solution with $c_1 = 1$ and $c_2 = 0$; that is, the solution

$$x(t) = e^{-\sqrt{2}t}, \qquad y(t) = -\sqrt{2}e^{-\sqrt{2}t}.$$

Notice that this solution lies on the straight line through the origin given by $y = -\sqrt{2}x$ and that as $t \to \infty$ we have $x(t) \to 0$ and $y(t) \to 0$. Moreover, for this solution $x(t)$ is always positive and $y(t)$ is always negative. The trajectory lies in the fourth quadrant and looks like the trajectory in Figure 8.5.

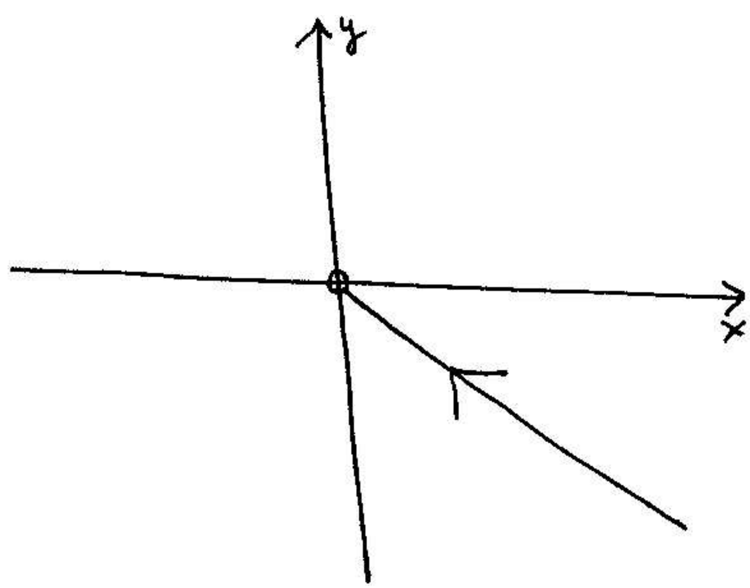


Figure 8.5: The trajectory for $x(t) = e^{-\sqrt{2}t}$, $y(t) = -\sqrt{2}e^{-\sqrt{2}t}$

A similar method shows us that if exactly one of the parameters c_1 and c_2 are zero, then the trajectory will be a half-line. By adding the trajectories for the three cases (c_1, c_2) equal to $(-1, 0)$, $(0, 1)$ and $(0, -1)$ and the origin, which is itself a trajectory (Why?), we end up with Figure 8.6.
The trajectories in Figure 8.6 form the "skeleton" of our phase portrait. Note that straight line trajectories are easy to find. They are the lines passing through the

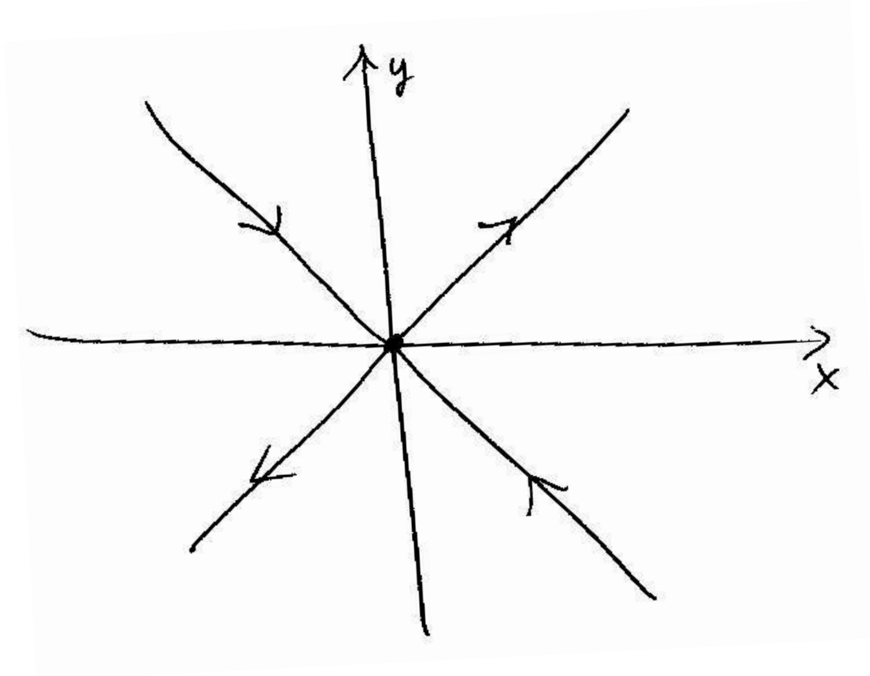


Figure 8.6: Hand drawn straight line trajectories for system (8.5)

origin in the directions of real eigenvectors corresponding to real eigenvalues. You might miss these special trajectories by using a direction field.

To fill in the rest of the phase portrait, you can fill a few more trajectories that indicate the behavior of trajectories not on the skeleton. Notice, in this case, that as $t \to \infty$ the function $e^{-\sqrt{2}t} \to 0$. Thus the first term in the solution (8.7) becomes very small (and less important in the graph) as $t \to \infty$; that is,

$$c_1 e^{-\sqrt{2}t} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} + c_2 e^{\sqrt{2}t} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \sim c_2 e^{\sqrt{2}t} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}.$$

Similarly, for $t \to -\infty$, we have

$$c_1 e^{-\sqrt{2}t} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} + c_2 e^{\sqrt{2}t} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \sim c_1 e^{-\sqrt{2}t} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}.$$

The phase portrait should reflect this (see Figure 8.7).

The phase portrait (obtained with computer graphics) of this system depicted in Figure 8.4 is typical for all linear 2×2 systems where the eigenvalues are real, distinct, and of opposite sign. The origin $(0, 0)$ is called a *saddle point* in this case. Note that in Figure 8.4 the straight line trajectories that tend towards and tend away from the origin $(0, 0)$ are in the directions of the eigenvectors. These lines are called the stable and unstable manifolds (respectively) of the saddle point. Solutions starting on the stable manifold—there are infinitely many such solutions—approach the origin

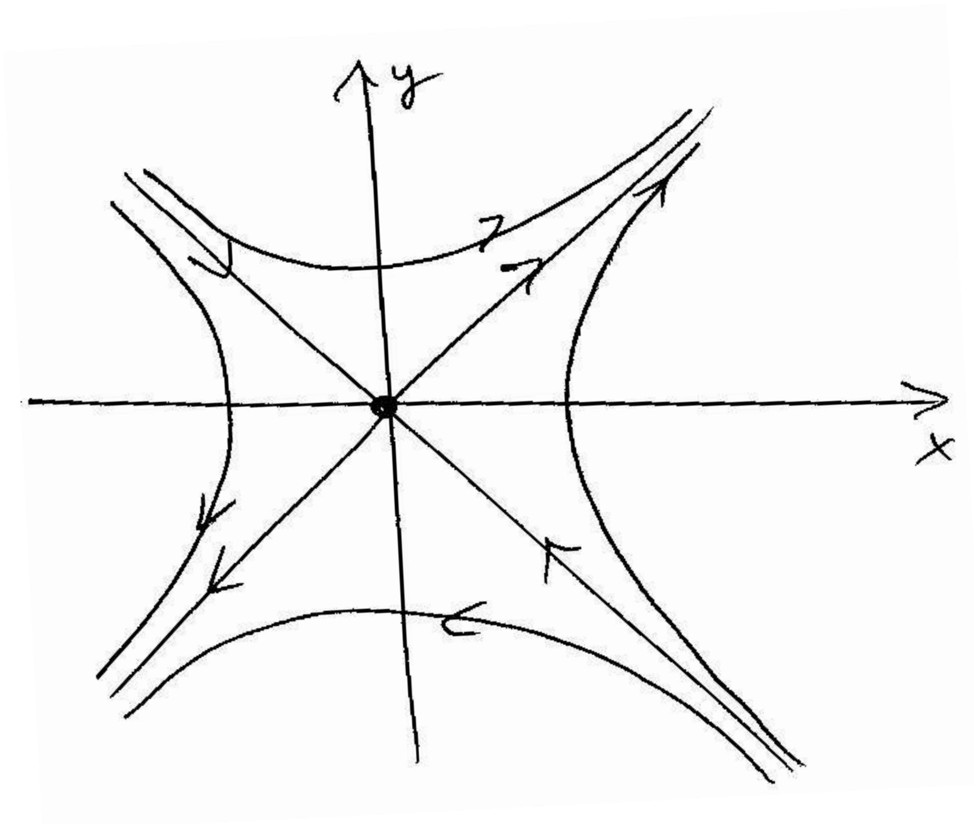


Figure 8.7: Hand drawn phase portrait for system (8.5)

as time increases to ∞; the solutions starting on the unstable manifold approach the origin as time decreases to $-\infty$. This is easy to see from the general solutions of the differential equation. The solutions on the stable manifold are given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-\sqrt{2}\,t} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix},$$

where c_1 is a real number and $c_2 = 0$; the solutions on the unstable manifold are given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_2 e^{\sqrt{2}\,t} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix},$$

where c_2 is a real number and $c_1 = 0$.

Example. *Consider the system*

$$\begin{aligned} \frac{dx}{dt} &= -x - y, \\ \frac{dy}{dt} &= x - y. \end{aligned} \tag{8.8}$$

Determine the phase portrait of the system near the origin.

This time we will draw the phase portrait by hand! First, we find the eigenvalues of the coefficient matrix

$$\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}. \tag{8.9}$$

The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$ and the eigenvalues are the complex numbers $-1 \pm i$. Using the methods we have learned, we find an eigenvector corresponding to the eigenvalue $-1 + i$, which we choose to be $\begin{bmatrix} i \\ 1 \end{bmatrix}$. A complex solution is given by

$$e^{(-1+i)t} \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

By extracting its real and imaginary parts, the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{-t} \Big(c_1 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \Big),$$

which can also be written in the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{-t} \begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Because the functions sin and cos are 2π-periodic, every nonzero solution will spiral around the origin. Also, due to the presence of the exponential factor e^{-t} (corresponding to the *negative real part* of the eigenvalue) the solutions are asymptotic to the origin as $t \to \infty$. The key observation is that the eigenvalues are complex with *negative real parts*. The only remaining question is which way do the orbits spiral, clockwise or counterclockwise? To decide, consider the vector field (corresponding to the right-hand side of the system) along the coordinate axes. For example, we have $x = 0$ along the vertical axis; therefore, the vector field restricted to this set is given by $\begin{bmatrix} -y \\ -y \end{bmatrix}$. We are interested only in the first component of the vector field; it determines the direction that trajectories cross the y-axis. Here, the first component is negative for $y > 0$ and positive for $y < 0$. Hence, the trajectories spiral counterclockwise toward the origin. This type of rest point is called a (spiral) sink. A hand-drawn phase portrait is shown in Figure 8.8. Note that the axes are specified, the position of the trajectory that stays at the origin is indicated, the spiral trajectory has the correct qualitative behavior, and its direction is indicated by an arrow head along the trajectory. You can tell at a glance how all the trajectories behave—this is the purpose of drawing the phase portrait.

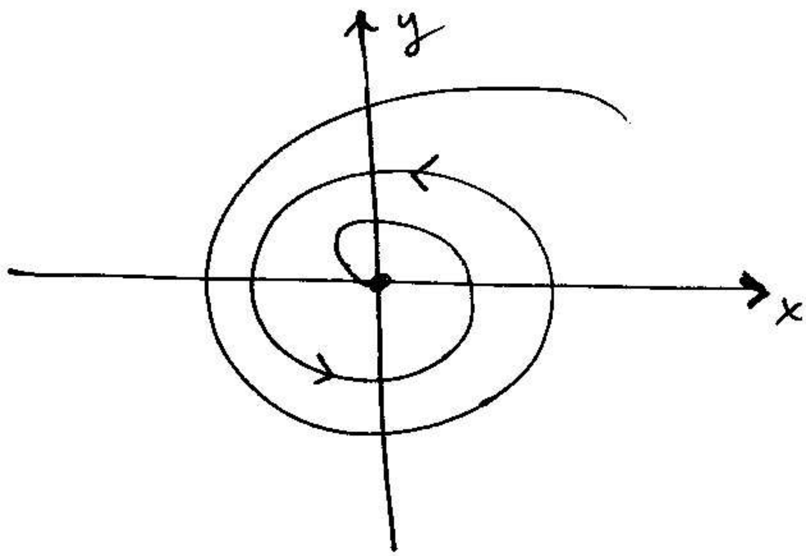


Figure 8.8: Phase portrait of system (8.8)

Example. *Draw the phase portrait of the second-order differential equation*

$$\ddot{x} - 5\dot{x} + 4x = 0;$$

that is, draw the phase portrait of an equivalent first-order system.

The usual way to obtain an equivalent first-order system is to set $\dot{x} = y$ and then write the formula for

$$\dot{y} = \ddot{x} = -4x + 5\dot{x} = -4x + 5y.$$

In other words, the equivalent system is

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -4x + 5y, \end{aligned} \tag{8.10}$$

We will draw the phase portrait of this system.

The process is always the same: we consider the eigenvalues and eigenvectors of the system. In this case, the coefficient matrix is

$$\begin{bmatrix} 0 & 1 \\ -4 & 5 \end{bmatrix},$$

which has the characteristic equation $\lambda^2 - 5\lambda + 4 = 0$ and the eigenvalues 1 and 4. Corresponding eigenvectors are given by

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

respectively. Hence the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

In this case there are two straight line solutions corresponding to the two eigenvectors. Also, all trajectories (except the one that stays at the origin) move away from the origin due to the exponential factors, which both grow without bound as $t \to \infty$. The key point is that both eigenvalues have positive real parts. In this example, the eigenvalues are positive real numbers. This type of rest point is called a (nodal) source (see Figure 8.9).

Note: Some authors make finer distinctions (for example, they define proper and improper nodes); but, for most applications, the most important features of rest points are captured by the basic classification: source, sink, or saddle, which is determined by the signs of the real parts of the corresponding eigenvalues. If both eigenvalues have positive real parts the rest point is a source. If both eigenvalues have negative real parts the rest point is a sink. And, if there is one positive and one negative eigenvalue, the rest point is a saddle.

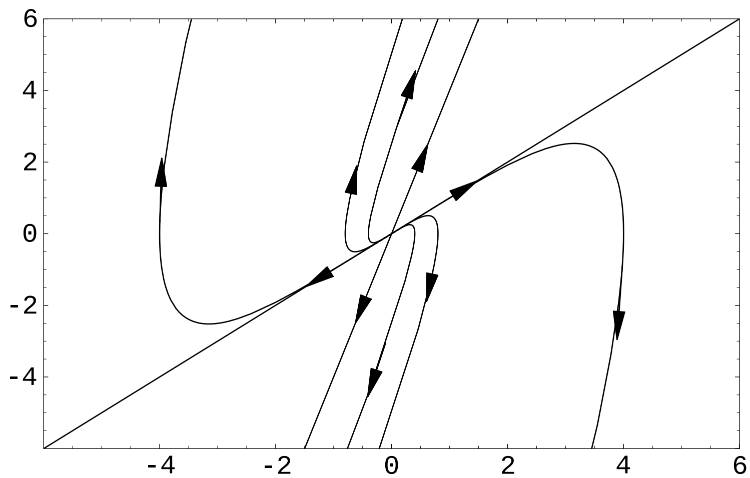


Figure 8.9: Phase portrait of system (8.10)

Homework Assignments

For each of the following systems, determine the nature of the origin $(0, 0)$ i.e., whether it is a saddle point, a node, etc., and whether the trajectories tend to move towards or away from the origin as $t \to \infty$. For each system sketch the trajectories near $(0, 0)$.

8.1.
$\frac{dx}{dt} = 2x - y$
$\frac{dy}{dt} = x + 2y$

8.2.
$\frac{dx}{dt} = -6x + y$
$\frac{dy}{dt} = x - 6y$

8.3.
$\frac{dx}{dt} = -7x + 10y$
$\frac{dy}{dt} = -5x + 8y$

8.4.
$\frac{dx}{dt} = y$
$\frac{dy}{dt} = x - y$

8.5.
$\frac{dx}{dt} = -10x - y$
$\frac{dy}{dt} = x - 10y$

9 Linearization of Nonlinear Systems at Isolated Rest Points

In this section we study how linear 2×2 systems can be used to approximate the local behavior of the trajectories of solutions to nonlinear systems.
We will only consider nonlinear systems of the form

$$\begin{aligned} \frac{dx}{dt} &= f(x, y), \\ \frac{dy}{dt} &= g(x, y), \end{aligned} \tag{9.1}$$

where f and g are twice continuously differentiable functions of two variables x and y. In this case, if initial conditions are given at some time t_0, say $x(t_0) = x_0$ and $y(t_0) = y_0$, then *the system has a unique solution with the given initial conditions.*

Definition of Rest Point. A point (x_0, y_0) on the plane is said to be a rest point of the system (9.1) if

$$f(x_0, y_0) = g(x_0, y_0) = 0. \tag{9.2}$$

Definition of Isolated Rest Point. A rest point (x_0, y_0) of (9.1) is called isolated if there is a disk D centered at (x_0, y_0) such that the only rest point in D is (x_0, y_0).

Example. *Consider Duffing's equation with damping*

$$\frac{d^2x}{dt^2} + \epsilon\frac{dx}{dt} - x + x^3 = 0, \tag{9.3}$$

where ϵ is a nonnegative real number. This second-order equation is equivalent to the first-order system

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= -\epsilon y + x - x^3.\end{aligned} \tag{9.4}$$

Find the rest points of system (9.4)

The rest points of system (9.4) are the solutions of the simultaneous system of algebraic equations

$$y = 0, \qquad -\epsilon y + x - x^3 = 0.$$

Thus the rest points are $(0, 0)$, $(1, 0)$ and $(-1, 0)$.

To determine the local phase portrait of a system (for instance, system (9.1)) near one of its isolated rest points, the basic tool is a corresponding linear system that closely approximates the nonlinear system near this rest point. It turns out that the local phase portrait of the nonlinear system near the rest point is qualitatively the same as the phase portrait of this special linear system. This is one reason why we have studied the phase portraits of linear systems.

To be more precise, we first define the appropriate linear system to study at a rest point of a nonlinear system.

(2) Linearization. The linearization of the system (9.1) at the rest point (x_0, y_0) is defined to be the homogeneous linear system

$$\begin{aligned}\frac{du}{dt} &= \left[\frac{\partial f}{\partial x}(x_0, y_0)\right] u + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right] v,\\ \frac{dv}{dt} &= \left[\frac{\partial g}{\partial x}(x_0, y_0)\right] u + \left[\frac{\partial g}{\partial y}(x_0, y_0)\right] v.\end{aligned} \tag{9.5}$$

Example. *Continuing with the last example (Duffing's equation* (9.4)*), determine the linearizations at its rest points.*

We simply compute the (Jacobian) matrix

$$\begin{bmatrix}\frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0)\\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)\end{bmatrix} \tag{9.6}$$

at each rest point and use the definition of the linearization. For Duffing's equation, the Jacobian matrix is

$$\begin{bmatrix}0 & 1\\ 1 - 3x_0^2 & -\epsilon\end{bmatrix} \tag{9.7}$$

The linearized system at the origin is

$$\begin{aligned}\frac{du}{dt} &= v,\\ \frac{dv}{dt} &= u - \epsilon v.\end{aligned} \tag{9.8}$$

The linearized system at $(\pm 1, 0)$ is

$$\begin{aligned}\frac{du}{dt} &= v,\\ \frac{dv}{dt} &= -2u - \epsilon v.\end{aligned} \tag{9.9}$$

Example. *Linearize the system*

$$\begin{aligned}\frac{dx}{dt} &= x - y + x(1 - x^2 - y^2),\\ \frac{dy}{dt} &= x + y + y(1 - x^2 - y^2),\end{aligned} \tag{9.10}$$

at $(0, 0)$.

The origin is an isolated rest point of the system. With $f(x, y) = x - y + x(1 - x^2 - y^2)$ and $g(x, y) = x + y + y(1 - x^2 - y^2)$, we have

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2 - 3x^2 - y^2, & \frac{\partial f}{\partial y} &= -1 - 2xy,\\ \frac{\partial g}{\partial x} &= 1 - 2xy, & \frac{\partial g}{\partial y} &= 2 - x^2 - 3y^2.\end{aligned} \tag{9.11}$$

Thus, the linearization of (9.10) at the origin is

$$\begin{aligned}\frac{du}{dt} &= 2u - v,\\ \frac{dv}{dt} &= u + 2v.\end{aligned} \tag{9.12}$$

The next result tells us that in certain cases the linearized system is a good approximation to the nonlinear system at a rest point.

Fact (Hartman-Grobman Theorem). Suppose that a nonlinear system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= f(x, y), \\ \frac{dy}{dt} &= g(x, y), \end{aligned} \tag{9.13}$$

has an isolated rest point at (x_0, y_0). Let

$$\begin{aligned} \frac{du}{dt} &= au + bv, \\ \frac{dv}{dt} &= cu + dv, \end{aligned} \tag{9.14}$$

be its linearization at (x_0, y_0).

(a) If

  (i) the eigenvalues of the coefficient matrix of (9.14) are real and not zero, or

  (ii) the eigenvalues of the coefficient matrix of (9.14) are complex with non-zero real parts,

  then the nonlinear system (9.13) near (x_0, y_0) and the linear system (9.14) near $(0, 0)$ have the "same" phase portrait.

(b) If zero is an eigenvalue of the coefficient matrix of (9.14) or if the eigenvalues are pure imaginary, then no conclusion can be drawn about the the behavior of the trajectories of the solutions of the nonlinear system (9.13) through its linearization (9.14).

Remark. *In other words, if one of the two situations (i) and (ii) of (a) arises, then the trajectories of the nonlinear system in some small neighborhood of* (x_0, y_0) *behave "similarly" to those of the linearization near* $(0, 0)$*. "Similarly" means, for example, that if* $(0, 0)$ *is a saddle point of the linearization, then* (x_0, y_0) *is also a saddle point of the nonlinear system; i.e., near* (x_0, y_0) *the trajectories of the nonlinear system will look like those near a linear saddle point.*

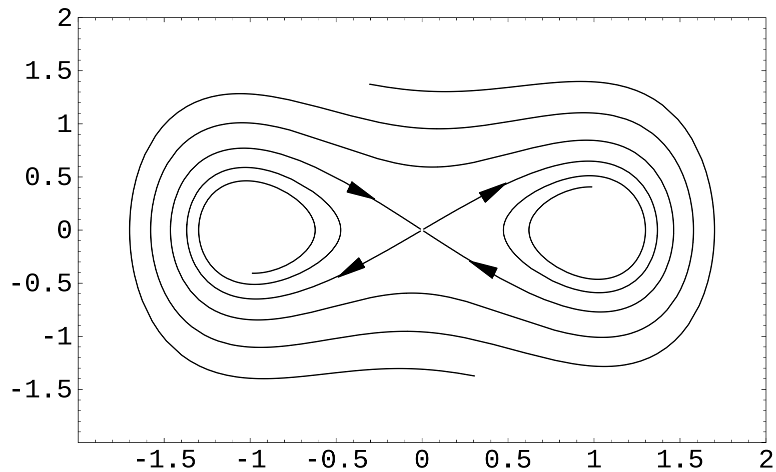


Figure 9.1: The phase portrait of Duffing's equation (9.4) with $\epsilon = 0.1$.

Example. *Continuing with Duffing's equation* (9.4), *determine the local phase portrait at each of its rest points.*

At the rest point $(0, 0)$, the coefficient matrix of the linearization (9.8) is

$$\begin{bmatrix} 0 & 1 \\ 1 & -\epsilon \end{bmatrix}$$

The eigenvalues of this matrix are

$$\frac{-\epsilon \pm \sqrt{\epsilon^2 + 4}}{2}.$$

Recall that $\epsilon \geq 0$. It is easy to see that the eigenvalues are real. There is one positive eigenvalue and one negative eigenvalue. Thus, the linearization has a saddle point at the origin. According to the Hartman-Grobman theorem, the nonlinear system also has a saddle point at the origin.

At the rest points $(\pm 1, 0)$, the coefficient matrix of the linearization (9.9) is

$$\begin{bmatrix} 0 & 1 \\ -2 & -\epsilon \end{bmatrix}$$

The eigenvalues of this matrix are

$$\frac{-\epsilon \pm \sqrt{\epsilon^2 - 8}}{2}.$$

For $\epsilon = 0$, the eigenvalues are pure imaginary. In this case the Hartman-Grobman theorem does not apply. If $0 < \epsilon < \sqrt{8}$, the eigenvalues are complex both with negative real parts and if $\epsilon > \sqrt{8}$, then the eigenvalues are both real and negative. The Hartman-Grobman theorem applies in these cases; the rest point is a sink. The nature of the sink will change from a spiral sink to a nodal sink as ϵ increases through $\sqrt{8}$.

Example. *Consider the nonlinear system* (9.10) *again. Determine its local phase portrait at the origin.*

The linearization at the isolated rest point $(0,0)$ is given by (9.12). The eigenvalues of the coefficient matrix of (9.12) are $2 \pm i$. So the origin $(0,0)$ is a spiral source for the linearization. Thus, according to the Hartman-Grobman theorem, the rest point $(0,0)$ of the nonlinear system (9.10) is a spiral source.

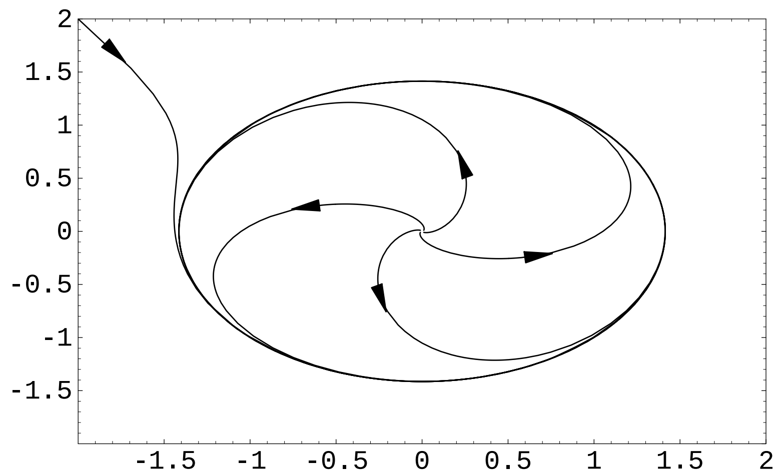


Figure 9.2: Trajectories of system (9.10). Note the source at the origin and the stable periodic trajectory, which is called a limit cycle. It turns out that all trajectories, except for the rest point, are attracted to the limit cycle.

Example. *A typical problem in mathematical ecology is the formulation and analysis of population models. A famous example is the Volterra model for the populations of a predator and its prey. A similar model was proposed earlier by Lotka. Thus the differential equations model we will derive is often called the Lotka-Volterra model. Let us denote by x the size of the population of a certain predator and by y the size of the population of its prey. We assume that the predator and its prey are living together in some environment. If there were no predators, the population of the prey species would grow exponentially except that the population growth would be limited by the food supply in the environment. For this reason, we will model the growth of the prey species by the (logistic) differential equation*

$$\frac{dy}{dt} = ay - by^2,$$

where a and b are positive real parameters to be determined by studying the reproduction rate of the prey species and the carrying capacity of the environment. More precisely, the growth rate is a and the carrying capacity is a/b. Why? If there were no prey species the predator would die out at a certain rate. Thus, we model the decay of the predator population by

$$\frac{dx}{dt} = -cx,$$

where $c > 0$ is the decay rate. The interaction (predator meets prey) must be a function of the two variables x and y that has at least one special property: it must be zero (no interaction) if one of the populations is zero. The simplest function with this property is xy; we take it to be the interaction term.

Interaction increases the population of predators—they eat the prey—and decreases the population of the prey. So, the Lotka-Volterra model with logistic prey growth is

$$\begin{aligned} \frac{dx}{dt} &= -cx + dxy, \\ \frac{dy}{dt} &= ay - by^2 - fxy. \end{aligned} \tag{9.15}$$

What is the fate of the predators and the prey if their populations evolve according to the model (9.15)*?*

Of course, the answer to our question will depend on the choice of the parameters a, b, c, d and f and the initial populations. For simplicity, we will discuss the special case

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{10}x(-20 + 5y), \\ \frac{dy}{dt} &= \frac{1}{10}y(4 - \frac{1}{4}y - 50x), \end{aligned} \tag{9.16}$$

where x and y measure the populations in units of thousands of individuals (for example, $x = 3$ means a population of 3000 predators). The reader is invited to investigate other cases.

Since x and y represent populations, these quantities are non-negative. So, for this application, we are interested only in the trajectories in the closed first quadrant. Our model would not be satisfactory if there were a trajectory, starting in the first quadrant, that eventually crosses the x or y-axis. Indeed, if this happened, then at least one of the populations would become negative! As we will see, this is not the case.

The nonlinear system has three isolated rest points: $(0, 0)$, $(0, 16)$ and $(3/50, 4)$. The rest point at the origin represents the steady state where there are no predators and no prey. Of course, if there are zero populations initially, the populations remain zero for all time. The rest point at $(0, 16)$ represents the situation where there are no predators and the prey species reaches an equilibrium according to the carrying capacity of the environment. We will soon determine the meaning of the third rest point.

By physical reasoning, it is obvious that if there are no predators at the initial time, then there are no predators forever. This would mean that if a trajectory starts on the

y-axis, then it remains on the y-axis for all time. But, this is not correct reasoning. Our model is a mathematical construct; maybe it does not reflect exactly the physical situation. Also, the purpose of the model is to derive physical conclusions. We must reason from the model, not from the physical situation that it is supposed to represent! If the initial population is $(0, y_0)$, we must show mathematically that $x(t) = 0$ for all t. This is easy. Simply note that the system

$$\frac{dx}{dt} = 0, \qquad \frac{dy}{dt} = \frac{1}{10}y(4 - \frac{1}{4}y)$$

has a unique solution of the form $(x(t), y(t))$ where $x(t)$ is zero and $y(0) = y_0$. This solution is a solution of system (9.16). A similar argument shows that a solution which starts on the x-axis stays on the x-axis for all time. These results show that a trajectory that starts in the first quadrant must stay in the first quadrant. If it did not, it would have to cross one of the axes. But, these are both invariant sets (a trajectory that starts on one of these sets stays on that set); trajectories cannot cross in the phase plane. If they did, the uniqueness of solutions of ODEs would be violated.

To determine the fate of the populations, we will determine the local phase portraits at the rest points.

The coefficient matrix of the linearized system corresponding to system (9.16) is

$$\begin{bmatrix} -2 + \frac{1}{2}y & \frac{1}{2}x \\ -5y & \frac{2}{5} - \frac{1}{20}y - 5x \end{bmatrix}.$$

At the rest point $(0, 0)$, this matrix has one positive and one negative eigenvalue. Hence, this rest point is a saddle. Moreover, the eigenvectors are in the directions of the coordinate axes, in concert with the invariance of these sets. The flow is approaching the rest point along the x-axis and moving away from the origin along the y-axis. In this special case, as we have already seen, the nonlinear system also has the coordinate axes as invariant sets. (Warning: In general, the straight-line solutions of a linearization may not be invariant sets for the corresponding non-linear system. The Lotka-Volterra system is special in this regard.)

The rest point at $(0, 16)$ is also a saddle with eigenvalues $-2/5$ and 6. The eigenvectors corresponding to $-2/5$ are parallel to the y-axis, which corresponds to the invariance of this axis. The eigenvectors corresponding to the positive eigenvalue 6 are parallel to the vector $\begin{bmatrix} \frac{2}{25} \\ -1 \end{bmatrix}$. This tells us something important: If both populations sizes are positive (that is, the evolving point representing the populations is in the open first quadrant), then it is not possible for either species to become extinct. Indeed, a trajectory in the first quadrant cannot approach a point on either axis. The only possibilities are that it approach one of the rest points (at $(0, 0)$ or $(0, 16)$). But, they are both saddle points. Thus, (by the Hartman-Grobman theorem) the only solutions that approach them must lie on one of their stable manifolds. Since their stable manifolds lie on the coordinate axes, our solution in the open first quadrant does not lie on either of these invariant manifolds.

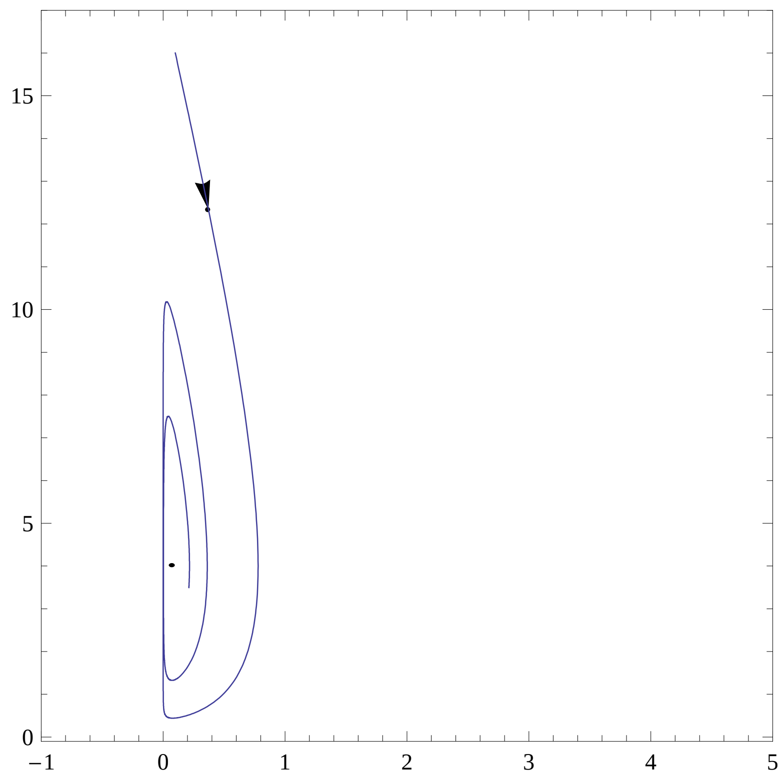


Figure 9.3: Phase portrait of the predator-prey model (9.16). The trajectory with initial conditions $x(0) = 0.1$ and $y(t) = 16.0$ and the rest point at $(3/50, 4)$ are shown. The trajectory spirals toward the rest point.

The rest point at $(3/50, 4)$ is a sink; in fact, the eigenvalues are complex and both have real part $-1/20$. Thus, it seems reasonable to conclude that for arbitrary nonzero initial populations of predators and prey, the population will evolve toward the steady state solution $(3/50, 4)$. For example, if our measurements for x and y are in units of thousands of individuals, then the populations—in our fictitious example—will evolve to the steady state of 60 predators and 4000 prey. Our analysis shows that if we start near this steady state, we will evolve toward this steady state. With more work it is possible to prove that all trajectories starting in the first quadrant do in fact evolve to this sink (see Figure 9.3).

Homework Assignments

For each of the following nonlinear systems find all the rest points and determine which rest points are isolated.

9.1.
$\frac{dx}{dt} = -x + xy,$
$\frac{dy}{dt} = 2y - y^2 - xy.$

9.2.
$\frac{dx}{dt} = x^2 + xy,$
$\frac{dy}{dt} = xy + y^2.$

9.3.
$\frac{dx}{dt} = y,$
$\frac{dy}{dt} = -y - \sin x.$

For each of the following nonlinear systems linearize the system around its isolated rest points in the phase plane. If possible determine the nature of the rest points from their linearizations (i.e. whether the trajectories near the rest point look like those near a saddle point, node, etc; and whether the trajectories near the rest point tend towards or tend away from it.)

9.4.
$\frac{dx}{dt} = -x + xy,$
$\frac{dy}{dt} = 2y - y^2 - xy.$

9.5.
$\frac{dx}{dt} = y,$
$\frac{dy}{dt} = -y - \sin x.$

9.6. The first-order system corresponding to $\frac{d^2x}{dt^2} + \frac{dx}{dt} + x - x^3 = 0.$

9.7. For the predator-prey model (9.15), assume that $f = d$. What relation among the parameters a, b, c, and d must be satisfied for there to be a rest point in the first quadrant?

9.8. (a) Determine the local phase portraits near the rest points of the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -y + x - x^2.$$

(b) Draw a plausible global phase portrait. (c) What is the fate of the trajectory with initial conditions $x(0) = 1/2$ and $y(0) = 0$ as $t \to \infty$?

9.9. Show that system (9.10) has a stable periodic orbit. Hint: Change to polar coordinates.

10 Answers to Selected Exercises

Page 5

1.1

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = \frac{5}{2}x - 2y + \frac{1}{2}te^{-3t}$$

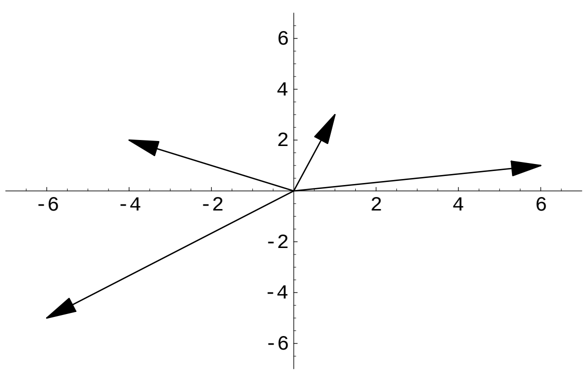
1.2

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 3x = 4e^t$$

1.3

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 2x = 0$$

Page 14



2.1

2.2 1. $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 8.5 & -6 \\ 3 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 24 & -80 \\ -24 & 12 \end{bmatrix}$

4. $\begin{bmatrix} t^2e^{5t} & 6e^{2t} \\ -7e^{2t} & -6te^{2t} \end{bmatrix}$

5. $\begin{bmatrix} 46 \\ -43 \end{bmatrix}$

6. $\begin{bmatrix} -t^6e^{5t} - te^{11t} \\ 2t^4e^{5t} - 6t^2e^{5t} \end{bmatrix}$

7. $\begin{bmatrix} -45 & 26 \\ -29 & 16 \end{bmatrix}$

8. $\begin{bmatrix} -11 & -37 \\ 4 & 18 \end{bmatrix}$

9. $\begin{bmatrix} t^2e^{-2t}+6t^2e^{3t} & t^2e^{6t}+6 \\ 5+3t^2e^{3t} & 5e^{8t}+3 \end{bmatrix}$

2.3 1. 46

2. 13

3. -12

2.4 1. linearly independent

2. linearly independent, except when $c = 1$, then linearly dependent, $\alpha = 1$, $\beta = -1$

3. linearly dependent, e.g. $\alpha = -2$, $\beta = 1$

4. linearly dependent, e.g. $\alpha = 1$, $\beta = -e^{2c}$

2.5 1. eigenvalues 3, 2, eigenvectors: $\begin{bmatrix} -2\alpha \\ \alpha \end{bmatrix}$, $\begin{bmatrix} -\alpha \\ \alpha \end{bmatrix}$ respectively

2. eigenvalues $2+i$, $2-i$; eigenvectors $\begin{bmatrix} (-1+i)3\alpha \\ 2\alpha \end{bmatrix}$, $\begin{bmatrix} (-1-i)3\alpha \\ 2\alpha \end{bmatrix}$ respectively

3. eigenvalues 2, 2; eigenvectors $\begin{bmatrix} -3\alpha \\ 2\alpha \end{bmatrix}$

2.6 1. not invertible

2. invertible, $(28-6\sqrt{2})^{-1}\begin{bmatrix} -4 & -\sqrt{2} \\ -6 & -7 \end{bmatrix}$

3. invertible, $(\sqrt{t}e^{2t}-t^3e^t)^{-1}\begin{bmatrix} \sqrt{t} & -t \\ -t^2e^t & e^{2t} \end{bmatrix}$

Page 20

3.2 linearly independent, determinant is 0

3.3 linearly independent

Page 22

4.1

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{t}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

4.2

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{4}{3}e^{-8}e^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}e^{-2}e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

4.3

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1e^{\left(\frac{3+\sqrt{5}}{2}\right)t}\begin{bmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix} + c_2e^{\left(\frac{3-\sqrt{5}}{2}\right)t}\begin{bmatrix} 1 \\ -\left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix}.$$

Except for the trivial (zero) solution, the length of the solution vector (that is, $\sqrt{x(t)^2+y(t)^2}$) grows without bound as $t \to \infty$.

4.4(i)

$$\frac{dx}{dt} = 0 \cdot x + y$$
$$\frac{dy}{dt} = 3x + \frac{1}{2}y.$$

4.4(ii)

$$r_1 = -\frac{3}{2} \text{ and } r_2 = 2.$$

4.4(iii)

$$x(t) = c_1 e^{-\frac{3}{2}t} + c_2 e^{2t}.$$

4.4(iv)

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-\frac{3}{2}t} \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

4.5 $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$.

Page 25

5.1

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-t} \left\{ \cos(2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(2t) \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\} + c_2 e^{-t} \left\{ \sin(2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(2t) \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$$

5.2

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 2000 e^{t} \left\{ \cos(10t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(10t) \begin{bmatrix} 0 \\ \frac{5}{2} \end{bmatrix} \right\} + 2e^{t} \left\{ \sin(10t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(10t) \begin{bmatrix} 0 \\ \frac{5}{2} \end{bmatrix} \right\}$$

5.3

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{t} \left\{ \cos(5t) \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} - \sin(5t) \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \right\} + c_2 e^{t} \left\{ \sin(5t) \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + \cos(5t) \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \right\}$$

Except for the zero solution, $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ spirals away from the origin toward "infinity" as $t \to \infty$.

5.4(i)

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$x(t_0) = u_0, \quad y(t_0) = v_0.$$

5.4(ii)

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= c_1 e^{-\frac{\gamma}{2m}t} \left\{ \cos\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t\right) \begin{bmatrix} 1 \\ -\frac{\gamma}{2m} \end{bmatrix} \right. \\ &\left. - \sin\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t\right) \begin{bmatrix} 0 \\ \frac{\sqrt{4mk-\gamma^2}}{2m} \end{bmatrix} \right\} \\ &+ c_2 e^{-\frac{\gamma}{2m}t} \left\{ \sin\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t\right) \begin{bmatrix} 1 \\ -\frac{\gamma}{2m} \end{bmatrix} \right. \\ &\left. + \cos\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t\right) \begin{bmatrix} 0 \\ \frac{\sqrt{4mk-\gamma^2}}{2m} \end{bmatrix} \right\} \end{aligned}$$

5.4(iii)

$$\begin{bmatrix} u \\ u' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ as } t \to \infty$$

5.5

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= c_1 \left\{ \cos(4t) \begin{bmatrix} 1 \\ \frac{2}{5} \end{bmatrix} + \sin(4t) \begin{bmatrix} 0 \\ \frac{4}{5} \end{bmatrix} \right\} \\ &+ c_2 \left\{ \sin(4t) \begin{bmatrix} 1 \\ \frac{2}{5} \end{bmatrix} - \cos(4t) \begin{bmatrix} 0 \\ \frac{4}{5} \end{bmatrix} \right\} \end{aligned}$$

The trajectories go around the origin periodically with period $\pi/2$.

Page 29

6.1 $x(t) = c_1 e^{3t} + c_2 e^{3t}(t+1)$, $y(t) = c_1 e^{3t} + c_2 e^{3t} t$

6.2 $x(t) = -2c_1 e^t + c_2 e^t(-2t + 1/2)$, $y(t) = c_1 e^t + c_2 e^t t$

6.3 $x(t) = c_1 e^{-2t} + c_2 e^{-2t}(t + 2/3)$, $y(t) = -3c_1 e^{-2t} + c_2 e^{-2t}(-3t)$

6.4 $x(t) = c_1 e^{3t},\ y(t) = c_2 e^{3t}$

6.5 $x(t) = e^{-t/2}\left(-5 - 2t\right),\ y(t) = e^{-t/2}\left(6 + 4t\right)$

6.6 $x(t) = e^{3t}(4 + 10t),\ y(t) = e^{3t}(2 - 20t)$

6.7 $s < -1$

Page 36

7.1 $x(t) = 5(1 - 4e^t + 3e^{2t}),\ y(t) = 5(1 - 2e^t + e^{2t})$

7.2 $x(t) = \frac{1}{3}(e^{-2t} - 6e^{-t} + 5e^{t}),\ y(t) = \frac{-e^{-2t}}{3} + e^{-t} + \frac{e^t}{3}$

7.3 $x(t) = c_1 e^{4t} + c_2 e^{2t} - \frac{3+4t}{32}$,
$y(t) = -c_1 3e^{4t} - c_2 e^{2t} + \frac{1-4t}{32}$

7.4 $x(t) = c_1 \cos t + c_2 \sin t - \frac{\cos^3 t}{2} - \sin t \left(\frac{t}{2} + \frac{\sin(2t)}{4}\right)$,
$y(t) = c_1 \sin t - c_2 \cos t - \frac{\cos^2 t \sin t}{2} + \cos t \left(\frac{t}{2} + \frac{\sin(2t)}{4}\right)$

7.5

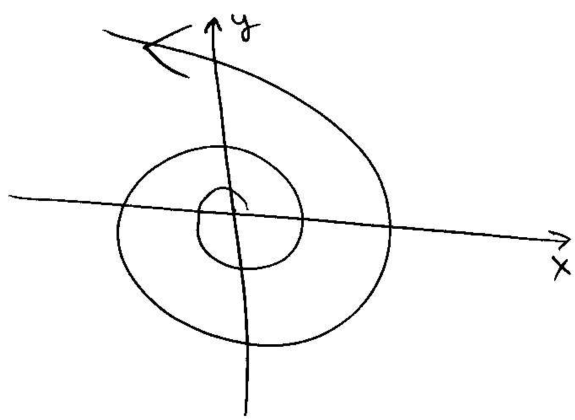
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 + \frac{4}{3}e^t \\ -12 + 2e^t \end{bmatrix}$$

7.6

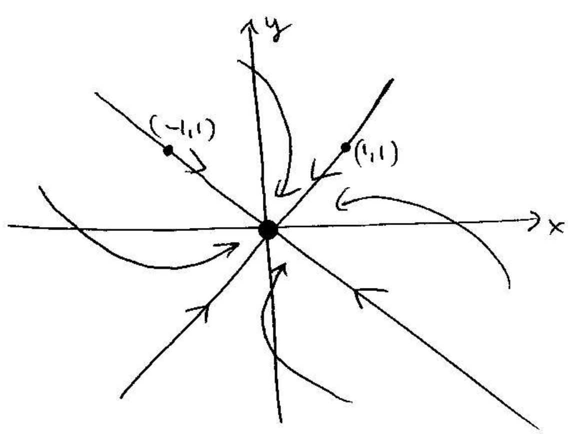
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-2t} \left(t \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + e^{-2t} \begin{bmatrix} -\log t - 1 \\ 2\log t + 2 - \frac{1}{t} \end{bmatrix}$$

Page 45

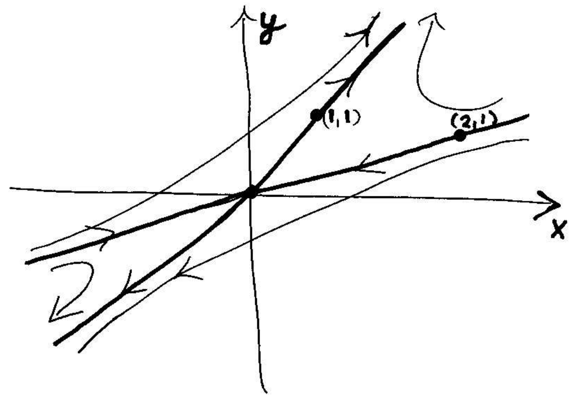
8.1 (Spiral) source; that is, except for the trajectory at the origin, all orbits spiral away from the origin.



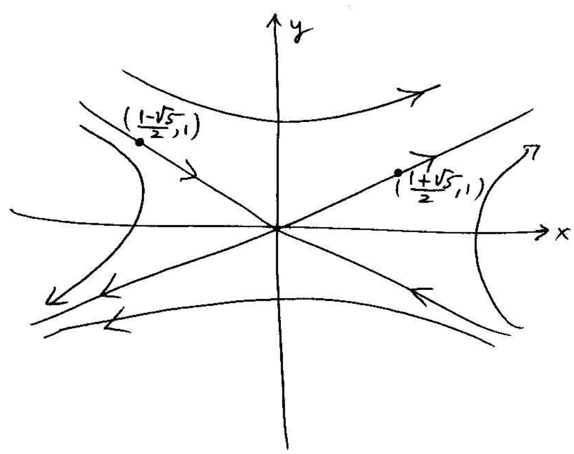
8.2 (Nodal) sink; that is, except for the trajectory at the origin, all orbits tend toward the origin and no orbit spirals around the origin.



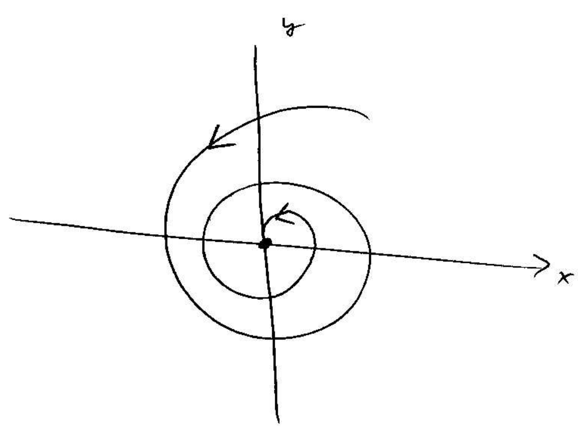
8.3 Saddle



8.4 Saddle



8.5 (Spiral) sink; that is, except for the trajectory at the origin, all orbits spiral toward the origin.



Page 55

9.1 The rest points are $(0,0)$, $(0,2)$ and $(1,1)$. All rest points are isolated.

9.2 All points on the line $y=-x$ are rest points. Hence, none of the rest points are isolated.

9.3 The rest points are $(n\pi,0)$, for all integers n. There are infinitely many rest points and they are all isolated.

9.4 There are three rest points: $(0,0)$ is a saddle, $(0,2)$ is a saddle, and $(1,1)$ is a spiral sink.

9.5 There are an infinite number of rest points $(n\pi,0)$, for all integers n. The rest points $(2k,0)$ are spiral sinks; the rest points $(2k+1,0)$ are saddles, for every integer k.

9.6 There are three rest points: $(0,0)$ is a spiral sink and $(0,\pm 1)$ are saddles.

9.7 There is a rest point in the first quadrant whenever $ad-bc>0$.

9.8 There is a saddle point at the origin and a spiral sink at the point $(1,0)$. The trajectory starting at $(1/2,0)$ is asymptotic in positive time to the sink at $(1,0)$.

9.9 Note that $r^2=x^2+y^2$ and $\theta=\arctan(y/x)$. Differentiate these expressions and then substitute $x=r\cos\theta$ and $y=r\sin\theta$ to get

$$\dot{r}=r(2-r^2), \qquad \dot{\theta}=1.$$

Except for $r(0)=0$, every solution of the first equation is such that $\lim_{t\to\infty}=\sqrt{2}$. This follows from a qualitative analysis. It is also possible to solve this equation by separation of variables.