CONSECUTIVE PRIMES AND BEATTY SEQUENCES

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Abstract. Fix irrational numbers \( \alpha, \hat{\alpha} > 1 \) of finite type and real numbers \( \beta, \hat{\beta} \geq 0 \), and let \( B \) and \( \hat{B} \) be the Beatty sequences
\[
B := ([\alpha m + \beta])_{m \in \mathbb{N}} \quad \text{and} \quad \hat{B} := ([\hat{\alpha} m + \hat{\beta}])_{m \in \mathbb{N}}.
\]
In this note, we study the distribution of pairs \((p, p^\sharp)\) of consecutive primes for which \( p \in B \) and \( p^\sharp \in \hat{B} \). Under a strong (but widely accepted) form of the Hardy-Littlewood conjectures, we show that
\[
\left| \{ p \leq x : p \in B \text{ and } p^\sharp \in \hat{B} \} \right| = (\alpha \hat{\alpha}^{-1}) \pi(x) + O(x(\log x)^{-3/2+\epsilon}).
\]

1. Introduction

For any given real numbers \( \alpha > 0 \) and \( \beta \geq 0 \), the associated (generalized) Beatty sequence is defined by
\[
B_{\alpha,\beta} := \left( \lfloor \alpha m + \beta \rfloor \right)_{m \in \mathbb{N}},
\]
where \( \lfloor t \rfloor \) is the largest integer not exceeding \( t \). If \( \alpha \) is irrational, it follows from a classical exponential sum estimate of Vinogradov [7] that \( B_{\alpha,\beta} \) contains infinitely many prime numbers; in fact, one has
\[
\# \{ \text{prime } p \leq x : p \in B_{\alpha,\beta} \} \sim \alpha^{-1} \pi(x) \quad (x \to \infty),
\]
where \( \pi(x) \) is the prime counting function.

Throughout this paper, we fix two (not necessarily distinct) irrational numbers \( \alpha, \hat{\alpha} > 1 \) and two (not necessarily distinct) real numbers \( \beta, \hat{\beta} \geq 0 \), and denote
\[
\mathcal{B} := B_{\alpha,\beta} \quad \text{and} \quad \hat{\mathcal{B}} := B_{\hat{\alpha},\hat{\beta}}. \quad (1.1)
\]
Our aim is to study the set of primes \( p \in \mathcal{B} \) for which the next larger prime \( p^\sharp \) lies in \( \hat{\mathcal{B}} \). The results we obtain are conditional, relying only on the Hardy-Littlewood conjectures in the following strong form. Let \( \mathcal{H} \) be a finite subset of \( \mathbb{Z} \), and let \( 1_\mathcal{P} \) denote the indicator function of the primes. The Hardy-Littlewood conjecture for \( \mathcal{H} \) asserts that the estimate
\[
\sum_{n \leq x} \prod_{h \in \mathcal{H}} 1_\mathcal{P}(n + h) = \mathcal{S}(\mathcal{H}) \int_2^x \frac{du}{(\log u)|\mathcal{H}|} + O(x^{1/2+\epsilon}) \quad (1.2)
\]
holds for any fixed \( \epsilon > 0 \), where \( \mathcal{S}(\mathcal{H}) \) is the singular series given by
\[
\mathcal{S}(\mathcal{H}) := \prod_p \left( 1 - \frac{|(\mathcal{H} \mod p)|}{p} \right) \left( 1 - \frac{1}{p} \right)^{-|\mathcal{H}|}.
\]
Our main result is the following.

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Theorem 1.1. Fix irrational numbers $\alpha, \hat{\alpha} > 1$ of finite type and real numbers $\beta, \hat{\beta} \geq 0$, and let $B$ and $\hat{B}$ be the Beatty sequences given by (1.1). For every prime $p$, let $p^\sharp$ denote the next larger prime. Suppose that the Hardy-Littlewood conjecture (1.2) holds for every finite subset $H$ of $\mathbb{Z}$. Then, for any fixed $\varepsilon > 0$, the counting function

$$\pi(x; B, \hat{B}) := \left| \{ p \leq x : p \in B \text{ and } p^\sharp \in \hat{B} \} \right|$$

satisfies the estimate

$$\pi(x; B, \hat{B}) = (\alpha \hat{\alpha})^{-1} \pi(x) + O(x^3/2 + \varepsilon),$$

where the implied constant depends only on $\alpha, \hat{\alpha}$ and $\varepsilon$.

Our results are largely inspired by the recent breakthrough paper of Lemke Oliver and Soundararajan [3], which studies the surprisingly erratic distribution of pairs of consecutive primes amongst the $\phi(q)^2$ permissible reduced residue classes modulo $q$. In [3] a conjectural explanation for this phenomenon is given which is based on the strong form of the Hardy-Littlewood conjectures considered in this note, that is, under the hypothesis that the estimate (1.2) holds for every finite subset $H$ of $\mathbb{Z}$.

2. Preliminaries

2.1. Notation. The notation $[t]$ is used to denote the distance from the real number $t$ to the nearest integer; that is,

$$[t] := \min_{n \in \mathbb{Z}} |t - n| \quad (t \in \mathbb{R}).$$

We denote by $\lfloor t \rfloor$ and $\{ t \}$ the greatest integer $\leq t$ and the fractional part of $t$, respectively. We also write $e(t) := e^{2\pi it}$ for all $t \in \mathbb{R}$, as usual.

Let $\mathbb{P}$ denote the set of primes in $\mathbb{N}$. In what follows, the letter $p$ always denotes a prime number, and $p^\sharp$ is used to denote the smallest prime greater than $p$. In other words, $p$ and $p^\sharp$ are consecutive primes with $p^\sharp > p$. We also put

$$\delta_p := p^\sharp - p \quad (p \in \mathbb{P}).$$

For an arbitrary set $S$, we use $1_S$ to denote its indicator function:

$$1_S(n) := \begin{cases} 1 & \text{if } n \in S, \\ 0 & \text{if } n \notin S. \end{cases}$$

Throughout the paper, implied constants in symbols $O$, $\ll$ and $\gg$ may depend (where obvious) on the parameters $\alpha, \hat{\alpha}, \varepsilon$ but are absolute otherwise. For given functions $F$ and $G$, the notations $F \ll G$, $G \gg F$ and $F = O(G)$ are all equivalent to the statement that the inequality $|F| \leq c|G|$ holds with some constant $c > 0$.

2.2. Discrepancy. We recall that the discrepancy $D(M)$ of a sequence of (not necessarily distinct) real numbers $x_1, x_2, \ldots, x_M \in [0, 1)$ is defined by

$$D(M) := \sup_{\mathcal{I} \subseteq [0,1)} \left| \frac{V(\mathcal{I}, M)}{M} - |\mathcal{I}| \right|,$$  \hspace{1cm} (2.1)
where the supremum is taken over all intervals \( I = (b,c) \) contained in \([0,1)\), the quantity \( V(I,M) \) is the number of positive integers \( m \leq M \) such that \( x_m \in I \), and \( |I| = c - b \) is the length of \( I \).

For any irrational number \( a \) we define its type \( \tau = \tau(a) \) by the relation

\[
\tau := \sup \{ t \in \mathbb{R} : \lim \inf \frac{n^t \lfloor an \rfloor}{n} = 0 \}.
\]

Using Dirichlet’s approximation theorem, one sees that \( \tau \geq 1 \) for every irrational number \( a \). Thanks to the work of Khinchin [1] and Roth [5,6] it is known that \( \tau = 1 \) for almost all real numbers (in the sense of the Lebesgue measure) and for all irrational algebraic numbers, respectively.

For a given irrational number \( a \), it is well known that the sequence of fractional parts \( \{a\}, \{2a\}, \{3a\}, \ldots \), is uniformly distributed modulo one (see, for example, [2, Example 2.1, Chapter 1]). When \( a \) is of finite type, this statement can be made more precise. By [2, Theorem 3.2, Chapter 2] we have the following result.

**Lemma 2.1.** Let \( a \) be a fixed irrational number of finite type \( \tau \). For every \( b \in \mathbb{R} \) the discrepancy \( D(a,b)(M) \) of the sequence of fractional parts \( \{am+b\}_{m=1}^M \) satisfies the bound

\[
D(a,b)(M) \leq M^{-1/\tau+o(1)} \quad (M \to \infty),
\]

where the function implied by \( o(\cdot) \) depends only on \( a \).

2.3. **Indicator function of a Beatty sequence.** As in §1 we fix (possibly equal) irrational numbers \( \alpha, \hat{\alpha} > 1 \) and (possibly equal) real numbers \( \beta, \hat{\beta} \geq 0 \), and we set

\[
B := B_{\alpha,\beta} \quad \text{and} \quad \hat{B} := B_{\hat{\alpha},\hat{\beta}}.
\]

In what follows we denote

\[
a := \alpha^{-1}, \quad \hat{a} := \hat{\alpha}^{-1}, \quad b := \alpha^{-1}(1 - \beta) \quad \text{and} \quad \hat{b} := \hat{\alpha}^{-1}(1 - \hat{\beta}).
\]

It is straightforward to show that

\[
1_B(m) = \psi_a(am + b) \quad \text{and} \quad 1_{\hat{B}}(m) = \psi_{\hat{a}}(\hat{a}m + \hat{b}) \quad (m \in \mathbb{N}), \quad (2.2)
\]

where for any \( t \in (0,1) \) we use \( \psi_t \) to denote the periodic function of period one defined by

\[
\psi_t(x) := \begin{cases} 
1 & \text{if } 0 < \{x\} \leq t, \\
0 & \text{if } t < \{x\} < 1 \text{ or } \{x\} = 0.
\end{cases}
\]

2.4. **Modified Hardy-Littlewood conjecture.** For their work on primes in short intervals, Montgomery and Soundararajan [4] have introduced the modified singular series

\[
\mathcal{G}_0(\mathcal{H}) := \sum_{\mathcal{T} \subseteq \mathcal{H}} (-1)^{|\mathcal{H}\setminus\mathcal{T}|} \mathcal{G}(\mathcal{T}),
\]

for which one has the relation

\[
\mathcal{G}(\mathcal{H}) = \sum_{\mathcal{T} \subseteq \mathcal{H}} \mathcal{G}_0(\mathcal{T}).
\]
Note that $\mathcal{S}(\emptyset) = \mathcal{S}_0(\emptyset) = 1$. The Hardy-Littlewood conjecture (1.2) can be reformulated in terms of the modified singular series as follows:

$$
\sum_{n \leq x} \prod_{h \in \mathcal{H}} \left( 1_{\mathcal{P}}(n + h) - \frac{1}{\log n} \right) = \mathcal{S}_0(\mathcal{H}) \int_2^x \frac{du}{(\log u)^{\mathcal{H}^c}} + O(x^{1/2+\varepsilon}). \quad (2.3)
$$

**Lemma 2.2.** We have

$$
\sum_{1 \leq t \leq h} \mathcal{S}_0(\{0, t\}) \ll h^{1/2+\varepsilon},
\sum_{1 \leq t \leq h} \mathcal{S}_0(\{t, h\}) \ll h^{1/2+\varepsilon},
\sum_{1 \leq t_1 < t_2 \leq h} \mathcal{S}_0(\{t_1, t_2\}) = -\frac{1}{2} h \log h + \frac{1}{2} Ah + O(h^{1/2+\varepsilon}),
$$

where $A := 2 - C_0 - \log 2\pi$ and $C_0$ denotes the Euler-Mascheroni constant.

**Proof.** Let us denote

$$
B := \sum_{1 \leq t \leq h-1} \mathcal{S}_0(\{0, t\}), \quad C := \sum_{1 \leq t \leq h-1} \mathcal{S}_0(\{t, h\}),
$$

and

$$
D_\pm := \sum_{1 \leq t_1 < t_2 \leq h+1} \mathcal{S}_0(\{t_1, t_2\})
$$

for either choice of the sign $\pm$. Clearly,

$$
\mathcal{S}_0(\{0, h\}) + B + C + D_- = D_+ \quad \text{and} \quad B = \sum_{1 \leq t \leq h-1} \mathcal{S}_0(\{0, h - t\}) = C.
$$

From [4, Equation (16)] we derive the estimates

$$
D_\pm = -\frac{1}{2} h \log h + \frac{1}{2} Ah + O(h^{1/2+\varepsilon}).
$$

Using the trivial bound $\mathcal{S}_0(\{0, h\}) \ll \log \log h$ and putting everything together, we finish the proof. \hfill \Box

2.5. **Technical lemmas.** Let $\nu(u) := 1 - 1/\log u$. Note that $\nu(u) \approx 1$ for $u \geq 3$.

**Lemma 2.3.** Let $c > 0$ be a constant, and suppose that $f$ is a function such that $|f(h)| \leq h^c$ for all $h \geq 1$. Then, uniformly for $3 \leq u \leq x$ and $\lambda \in \mathbb{R}$ we have

$$
\sum_{h \leq (\log x)^3} f(h)\nu(u)^h e(\lambda h) = \sum_{h \geq 1} f(h)\nu(u)^h e(\lambda h) + O(e(x^{-1})).
$$

**Proof.** Write $\nu(u)^h = e^{-h/H}$ with $H := -(\log \nu(u))^{-1}$. Since $H \leq \log u$ for $u \geq 3$, for any $h > (\log x)^3$ we have $h/H \geq h^{2/3}$ as $u \leq x$; therefore,

$$
\left| \sum_{h > (\log x)^3} f(h)\nu(u)^h e(\lambda h) \right| \leq \sum_{h > (\log x)^3} h^c e^{-h^{2/3}} \leq x^{-1} \sum_{h > (\log x)^3} h^c e^{h^{1/3} - h^{2/3}} \ll e x^{-1},
$$

and the result follows. \hfill \Box
The next statement is an analogue of [3, Proposition 2.1] and is proved using similar methods.

**Lemma 2.4.** Fix $\theta \in [0, 1]$ and $\vartheta = 0$ or 1. For all $\lambda \in \mathbb{R}$ and $u \geq 3$, let

$$R_{\theta, \vartheta; \lambda}(u) := \sum_{h \geq 1 \atop 2 \mid h} h^{\theta} (\log h)^{\vartheta} \nu(u)^h e(\lambda h),$$

$$S_{\lambda}(u) := \sum_{h \geq 1 \atop 2 \mid h} \mathcal{S}_0(\{0, h\}) \nu(u)^h e(\lambda h).$$

When $\lambda = 0$ we have the estimates

$$R_{\theta, 0; 0}(u) = \frac{1}{2} \Gamma(1 + \theta)(\log u)^{1+\theta} + O(1),$$

$$R_{\theta, 1; 0}(u) = \frac{1}{2} (\log 2) \Gamma(1 + \theta)(\log u)^{1+\theta} + O(1),$$

$$S_0(u) = \frac{1}{2} \log u - \frac{1}{2} \log \log u + O(1).$$

On the other hand, if $\lambda$ is such that $|\lambda| \geq (\log u)^{-1}$, then

$$\max \{ |R_{\theta, \vartheta; \lambda}(u)|, |S_{\lambda}(u)| \} \ll \lambda^{-4}.$$

**Proof.** We adapt the proof of [3, Proposition 2.1]. As in Lemma 2.3 we write $\nu(u)^h = e^{-h/H}$ with $H := -(\log \nu(u))^{-1}$. We simplify the expressions $R_{\theta, \vartheta; \lambda}(u)$, $S_{\lambda}(u)$ and $T_{\lambda}(u)$ by writing

$$\nu(u)^h e(\lambda h) = e^{-h/H_{\lambda}} \quad \text{with} \quad H_{\lambda} := \frac{H}{1 - 2\pi i \lambda H}.$$

Since $\Re(h/H_{\lambda}) = h/H > 0$ for any positive integer $h$, using the Cahen-Mellin integral we have

$$R_{\theta, \vartheta; \lambda}(u) = \sum_{h \geq 1 \atop 2 \mid h} h^{\theta} (\log h)^{\vartheta} e^{-h/H_{\lambda}} = \frac{1}{2\pi i} \int_{4-i\infty}^{4+i\infty} \left( \sum_{h \geq 1 \atop 2 \mid h} h^{\theta} (\log h)^{\vartheta} \right) \Gamma(s) H_{\lambda}^s ds.$$

In particular,

$$R_{\theta, 0; \lambda}(u) = \frac{2^\theta}{2\pi i} \int_{4-i\infty}^{4+i\infty} 2^{-s} \zeta(s - \theta) \Gamma(s) H_{\lambda}^s ds \quad (2.4)$$

and

$$R_{\theta, 1; \lambda}(u) = R_{\theta, 0; \lambda}(u) \log 2 - \frac{2^\theta}{2\pi i} \int_{4-i\infty}^{4+i\infty} 2^{-s} \zeta'(s - \theta) \Gamma(s) H_{\lambda}^s ds. \quad (2.5)$$

When $\lambda \neq 0$ we have

$$|R_{\theta, 0; \lambda}(u)| \leq \frac{2^\theta |H_{\lambda}|^4}{2\pi} \int_{-\infty}^{\infty} |\zeta(4 - \theta + it) \Gamma(4 + it)| dt$$

$$\ll |H_{\lambda}|^4 \left( \frac{H^2}{1 + 4\pi^2 \lambda^2 H^2} \right)^2,$$

hence the bound $R_{\theta, 0; \lambda}(u) \ll \lambda^{-4}$ holds if $|\lambda| \geq (\log u)^{-1}$ since $H \approx \log u$ for $u \geq 3$. In the case that $\lambda = 0$, the stated estimate for $R_{\theta, 0; 0}(u)$ is obtained by shifting the line of integration in (2.4) to the line $\{ \Re(s) = -\frac{1}{3} \}$ (say), taking into account the residues of the poles of the integrand at $s = 1 + \theta$ and $s = 0$. 
Our estimates for $R_{\theta,1,\lambda}(u)$ are proved similarly, using (2.5) instead of (2.4) and taking into account that 
\[ \zeta'(s - \theta) = (s - 1 - \theta)^{-1} + O(1) \text{ for } s \text{ near } 1 + \theta. \]

Next, for all $\lambda \in \mathbb{R}$ and $u \geq 3$, let 
\[ T_\lambda(u) := \sum_{h \geq 1} \mathcal{G}(\{0, h\}) e^{-h/H_\lambda}. \]

Since $\mathcal{G}_0(\{0, h\}) = \mathcal{G}(\{0, h\}) - 1$ for all integers $h$, and $\mathcal{G}(\{0, h\}) = 0$ if $h$ is odd, it follows that 
\[ S_\lambda(u) = T_\lambda(u) - R_{0,0,\lambda}(u) = T_\lambda(u) - \frac{1}{2} \log u + O(1). \]

Hence, to complete the proof of the lemma, it suffices to show that 
\[ T_0(u) = \log u - \frac{1}{2} \log \log u + O(1) \quad \text{and} \quad T_\lambda(u) \ll \lambda^{-4} \text{ if } |\lambda| \geq (\log u)^{-1}. \]

As in the proof of [3, Proposition 2.1], we consider the Dirichlet series 
\[ F(s) := \sum_{h \geq 1} \frac{\mathcal{G}(\{0, h\})}{h^s}, \]

which can be expressed in the form 
\[ F(s) = \frac{\zeta(s)\zeta(s + 1)}{\zeta(2s + 2)} \prod_p \left( 1 - \frac{1}{(p - 1)^2} + \frac{2p}{(p - 1)^2(p^{s+1} + 1)} \right), \]

and the final product is analytic for $\Re(s) > -1$. Using the Cauchy-Mellin integral we have 
\[ T_\lambda(u) = \frac{1}{2\pi i} \int_{4-i\infty}^{4+i\infty} F(s)\Gamma(s)H_\lambda^s ds. \quad (2.6) \]

For $\lambda \neq 0$ we have 
\[ |T_\lambda(u)| \leq \frac{|H_\lambda|^4}{2\pi} \int_{-\infty}^{\infty} |F(4 + it)\Gamma(4 + it)| \, dt \ll |H_\lambda|^4 = \left( \frac{H^2}{1 + 4\pi^2\lambda^2H^2} \right)^2 \]

hence $T_\lambda(u) \ll \lambda^{-4}$ holds provided that $|\lambda| \geq (\log u)^{-1}$. For $\lambda = 0$, we shift the line of integration in (2.6) to the line $\{\Re(s) = -\frac{1}{3}\}$ (say), taking into account the double pole at $s = 0$ and the simple pole at $s = 1$. This leads to the stated estimate for $T_0(u)$. \qed

We also need the following integral estimate (proof omitted).

**Lemma 2.5.** For all $\lambda \in \mathbb{R}$ and $x \geq 3$, let 
\[ I_\lambda(x) := \int_3^x \frac{e(\lambda u)}{\nu(u)\log u} \, du. \]

When $\lambda = 0$ we have the estimate 
\[ I_0(x) = \frac{x}{\log x} + O\left( \frac{x}{(\log x)^2} \right), \]

whereas for any $\lambda \neq 0$ we have 
\[ I_\lambda(x) \ll |\lambda|^{-1}. \]
3. Proof of Theorem 1.1

For every even integer \( h \geq 2 \) we denote
\[
\pi_h(x; \mathcal{B}, \hat{\mathcal{B}}) := \left| \{ p \leq x : p \in \mathcal{B}, \ p^r \in \hat{\mathcal{B}} \text{ and } \delta_p = h \} \right| = \sum_{n \leq x} 1_{\mathcal{B}}(n)1_{\hat{\mathcal{B}}}(n + h)f_h(n),
\]
where
\[
f_h(n) := 1_{\mathcal{P}}(n)1_{\mathcal{P}}(n + h) \prod_{0 < t < h} \left( 1 - 1_{\mathcal{P}}(n + t) \right) = \begin{cases} 
1 & \text{if } n = p \in \mathcal{P} \text{ and } \delta_p = h, \\
0 & \text{otherwise}.
\end{cases}
\]
Clearly,
\[
\pi(x; \mathcal{B}, \hat{\mathcal{B}}) = \sum_{h \leq (\log x)^3} \pi_h(x; \mathcal{B}, \hat{\mathcal{B}}) + O\left( \frac{x}{(\log x)^3} \right). \tag{3.1}
\]

Fixing an even integer \( h \in [1, (\log x)^3] \) for the moment, our initial goal is to express \( \pi_h(x; \mathcal{B}, \hat{\mathcal{B}}) \) in terms of the function
\[
S_h(x) := \sum_{n \leq x} f_h(n)
\]
recently introduced by Lemke Oliver and Soundararajan \([3, \text{Equation (2.5)}]\). In view of (2.2) we can write
\[
\pi_h(x; \mathcal{B}, \hat{\mathcal{B}}) = \sum_{n \leq x} \psi_a(an + b)\hat{\psi}_a(\hat{a}(n + h) + \hat{b})f_h(n). \tag{3.2}
\]

According to a classical result of Vinogradov (see \([8, \text{Chapter I, Lemma 12}]\)), for any \( \Delta \) such that
\[
0 < \Delta < \frac{1}{8} \quad \text{and} \quad \Delta \leq \frac{1}{2} \min\{a, 1 - a\}
\]
there is a real-valued function \( \Psi_a \) with the following properties:

(i) \( \Psi_a \) is periodic with period one;
(ii) \( 0 \leq \Psi_a(t) \leq 1 \) for all \( t \in \mathbb{R} \);
(iii) \( \Psi_a(t) = \psi_a(t) \) if \( \Delta \leq \{t\} \leq a - \Delta \) or if \( a + \Delta \leq \{t\} \leq 1 - \Delta \);
(iv) \( \Psi_a \) is represented by a Fourier series
\[
\Psi_a(t) = \sum_{k \in \mathbb{Z}} g_a(k)e(kt),
\]
where \( g_a(0) = a \), and the Fourier coefficients satisfy the uniform bound
\[
|g_a(k)| \ll \min\{\{k\}^{-1}, \{k\}^{-2}\Delta^{-1}\} \quad (k \neq 0). \tag{3.3}
\]

For convenience, we denote
\[
\mathcal{I}_a := [0, \Delta) \cup (a - \Delta, a + \Delta) \cup (1 - \Delta, 1),
\]
so that \( \Psi_a(t) = \psi_a(t) \) whenever \( \{t\} \notin \mathcal{I}_a \). Defining \( \Psi_{\hat{a}} \) and \( \mathcal{I}_{\hat{a}} \) similarly with \( \hat{a} \) in place of \( a \), and taking into account the properties (i)–(iii), from (3.2) we deduce that
\[
\pi_h(x; \mathcal{B}, \hat{\mathcal{B}}) = \sum_{n \leq x} \Psi_a(an + b)\Psi_{\hat{a}}(\hat{a}(n + h) + \hat{b})f_h(n) + O(V(x)), \tag{3.4}
\]
where \( V(x) \) is the number of positive integers \( n \leq x \) for which
\[
\{ an + b \} \in \mathcal{I}_a \quad \text{or} \quad \{ \hat{a}(n + h) + \hat{b} \} \in \mathcal{I}_{\hat{a}}.
\]

Since \( \mathcal{I}_a \) and \( \mathcal{I}_{\hat{a}} \) are unions of intervals with overall measure \( 4\Delta \), it follows from the definition (2.1) and Lemma 2.1 that
\[
V(x) \ll \Delta x + x^{1-1/\tau+o(1)} \quad (x \to \infty).
\]

Now let \( K \geq \Delta^{-1} \) be a large real number, and let \( \Psi_{a,K} \) be the trigonometric polynomial given by
\[
\Psi_{a,K}(t) := \sum_{|k| \leq K} g_a(k)e(kt).
\]

Using (3.3) it is clear that the estimate
\[
\Psi_a(t) = \Psi_{a,K}(t) + O(K^{-1}\Delta^{-1})
\]
holds uniformly for all \( t \in \mathbb{R} \). Defining \( \Psi_{\hat{a},K} \) in a similar way, combining (3.6) with (3.4), and taking into account (3.5), we derive the estimate
\[
\pi_h(x; B, \hat{B}) = \Sigma_h + O(\Delta x + x^{1-1/\tau+\epsilon} + K^{-1}\Delta^{-1} x),
\]
where
\[
\Sigma_h := \sum_{n \leq x} \Psi_{a,K}(an + b)\Psi_{\hat{a},K}(\hat{a}(n + h) + \hat{b})f_h(n)
\]
\[
= \sum_{n \leq x} \sum_{|k|,|\ell| \leq K} g_a(k)e(k(an + b))g_{\hat{a}}(\ell)e(\ell(\hat{a}(n + h) + \hat{b}))f_h(n)
\]
\[
= \sum_{|k|,|\ell| \leq K} g_a(k)e(k b)g_{\hat{a}}(\ell)e(\hat{b})e(\ell \hat{a}) \sum_{n \leq x} e((ka + \ell \hat{a})n)f_h(n).
\]

Therefore
\[
\pi_h(x; B, \hat{B}) = \sum_{|k|,|\ell| \leq K} g_a(k)e(k b)g_{\hat{a}}(\ell)e(\hat{b})e(\ell \hat{a}) \int_{3^{-}}^{x} e((ka + \ell \hat{a})u) d(S_h(u))
\]
\[
+ O(\Delta x + x^{1-1/\tau+\epsilon} + K^{-1}\Delta^{-1} x),
\]
which completes our initial goal of expressing \( \pi_h(x; B, \hat{B}) \) in terms of the function \( S_h \). To proceed further, it is useful to recall certain aspects of the analysis of \( S_h \) that is carried out in [3]. First, writing \( \tilde{1}_p(n) := 1_p(n) - 1/\log n \), up to an error term of size \( O(x^{1/2+\epsilon}) \) the quantity \( S_h(x) \) is equal to
\[
\sum_{n \leq x} \left( \tilde{1}_p(n) + \frac{1}{\log n} \right) \left( \tilde{1}_p(n + h) + \frac{1}{\log n} \right) \prod_{0 < t < h} \left( 1 - \frac{1}{\log n} - \tilde{1}_p(n + t) \right)
\]
\[
= \sum_{\mathcal{A} \subseteq \{0,h\}} \sum_{\mathcal{T} \subseteq [1,h-1]} (-1)^{|\mathcal{T}|} \sum_{n \leq x} \left( \frac{1}{\log n} \right)^{2-|\mathcal{T}|} \left( 1 - \frac{1}{\log n} \right)^{h-1-|\mathcal{T}|} \prod_{t \in \mathcal{A} \cup \mathcal{T}} \tilde{1}_p(n + t);
\]
see [3, Equations (2.5) and (2.6)]. By the modified Hardy-Littlewood conjecture (2.3) the estimate

\[ \sum_{n \leq x} (\log n)^{-c} \prod_{t \in H} \tilde{\alpha}(n + t) = \int_{3}^{x} (\log u)^{-c} d \left( \sum_{n \leq u} \prod_{t \in H} \tilde{\alpha}(n + t) \right) \]

\[ = \mathcal{S}_0(H) \int_{3}^{x} (\log u)^{-c} \nu(H) du + O(x^{1/2+\varepsilon}) \]

holds uniformly for any constant \( c > 0 \); consequently, up to an error term of size \( O(x^{1/2+\varepsilon}) \) the quantity \( S_h(x) \) is equal to

\[ \sum_{A \subseteq \{0, h\}} \sum_{\tau \subseteq [1, h-1]} (-1)^{|\tau|} \mathcal{S}_0(A \cup \tau)(\nu(u) \log u)^{-|\tau|} \nu(u)^h, \]

where

\[ \nu(u) := 1 - \frac{1}{\log u} \quad (u > 1) \]

(note that \( \nu(u) \) is the same as \( \alpha(u) \) in the notation of [3]). For every integer \( L \geq 0 \) we denote

\[ D_{h,L}(u) := \sum_{A \subseteq \{0, h\}} \sum_{\tau \subseteq [1, h-1]} (-1)^{|\tau|} \mathcal{S}_0(A \cup \tau)(\nu(u) \log u)^{-|\tau|} \nu(u)^h, \]

so that

\[ S_h(x) = \sum_{L=0}^{h+1} \int_{3}^{x} \nu(u)^{-1} (\log u)^{-2} D_{h,L}(u) du. \]

We now combine this relation with (3.7), sum over the even natural numbers \( h \leq (\log x)^3 \), and apply (3.1) to deduce that the quantity \( \pi(x; B, \hat{B}) \) is equal to

\[ \sum_{h \leq (\log x)^3} \sum_{L=0}^{h+1} \sum_{|\ell| = L} g_a(k)e(kb)g_a(\ell)e(\ell\hat{\alpha}) \cdot e(\ell\hat{\alpha}h) \int_{3}^{x} \frac{e((ka + \ell\hat{\alpha})u)}{\nu(u) (\log u)^2} D_{h,L}(u) du \]

up to an error term of size

\[ \ll \frac{x}{(\log x)^3} + (\Delta x + x^{1-\tau+\varepsilon} + K^{-1} \Delta^{-1} x) (\log x)^3. \]

Choosing \( \Delta := (\log x)^{-6} \) and \( K := (\log x)^{12} \) the combined error is \( O(x/(\log x)^3) \), which is acceptable.

Next, arguing as in [3] and noting that

\[ \sum_{|k|, |\ell| \leq K} \left| g_a(k)g_a(\ell) \right| \ll (\log x)^2, \]

one sees that the contribution to \( \pi(x; B, \hat{B}) \) coming from terms with \( L \geq 3 \) does not exceed \( O(x/(\log x)^{5/2}) \). Since \( D_{h,1} \) is identically zero (as \( \mathcal{S}_0 \) vanishes on singleton sets), this leaves only the terms with \( L = 0 \) or \( L = 2 \). The function \( D_{h,2} \)
splits naturally into four pieces according to whether \( A = \emptyset, \{0\}, \{h\} \) or \( \{0, h\} \). Consequently, up to \( O(x/(\log x)^{5/2}) \) we can express the quantity \( \pi(x; B, \hat{B}) \) as

\[
\sum_{j=1}^{5} \sum_{|k|, |l| \leq K} g_a(k)e(kb)g_a(l)e(\ell \hat{b}) \int_{3}^{x} \frac{e((ka + \ell \hat{a})u)}{\nu(u)(\log u)^2} F_{j, \ell}(u) \, du,
\]

(3.8)

where (taking into account Lemma 2.3) we have written

\[
\sum_{h \leq (\log x)^{4/3} \atop 2 \nmid h} e(\ell \hat{a}h)D_{h, L}(u) = \sum_{j=1}^{5} F_{j, \ell}(u) + O(x^{-1})
\]

with

\[
F_{1, \ell}(u) := \sum_{h \geq 1 \atop 2 \nmid h} \nu(u)^h e(\ell \hat{a}h),
\]

\[
F_{2, \ell}(u) := \sum_{h \geq 1 \atop 2 \nmid h} \mathcal{G}_0(\{0, h\}) \nu(u)^h e(\ell \hat{a}h),
\]

\[
F_{3, \ell}(u) := \frac{(-1)}{\nu(u) \log u} \sum_{h \geq 1 \atop 2 \nmid h} \sum_{1 \leq t \leq h-1} \nu(0, t) \nu(u)^h e(\ell \hat{a}h),
\]

\[
F_{4, \ell}(u) := \frac{(-1)}{\nu(u) \log u} \sum_{h \geq 1 \atop 2 \nmid h} \sum_{1 \leq t \leq h-1} \nu(t, h) \nu(u)^h e(\ell \hat{a}h),
\]

\[
F_{5, \ell}(u) := \frac{1}{(\nu(u) \log u)^2} \sum_{h \geq 1 \atop 2 \nmid h} \sum_{1 \leq t_1 < t_2 \leq h-1} \nu(t_1, t_2) \nu(u)^h e(\ell \hat{a}h).
\]

First, we show that certain terms in (3.8) make a negligible contribution that does not exceed \( O(x/(\log x)^{3/2-\varepsilon}) \).

For any \( \ell \neq 0 \), using Lemma 2.4 with \( \lambda = \ell \hat{a} \) we have

\[
F_{1, \ell}(u) = R_{0,0; \ell \hat{a}}(u) \ll \ell^{-4}
\]

provided that \( |\ell \hat{a}| \geq (\log u)^{-1} \), and for this it suffices that \( u \geq \exp(\hat{a}) \). Thus,

\[
\int_{3}^{x} \frac{e((ka + \ell \hat{a})u)}{\nu(u)(\log u)^2} F_{1, \ell}(u) \, du \ll 1 + \ell^{-4} \frac{x}{(\log x)^{2}}.
\]

In view of (3.3), the contribution to (3.8) from terms with \( j = 1 \) and \( \ell \neq 0 \) is

\[
\ll \sum_{|k|, |l| \leq K \atop \ell \neq 0} |g_a(k)| \cdot |\ell|^{-1} \left( 1 + \ell^{-4} \frac{x}{(\log x)^{2}} \right) \ll \frac{x \log x}{(\log x)^2} \ll \frac{x}{(\log x)^{3/2-\varepsilon}}.
\]

Similarly, for \( \ell \neq 0 \) and \( u \geq \exp(\hat{a}) \) we have \( F_{2, \ell}(u) = S_{\ell \hat{a}}(u) \ll \ell^{-4} \) by Lemma 2.4, so the contribution to (3.8) from terms with \( j = 2 \) and \( \ell \neq 0 \) is also \( O(x/(\log x)^{3/2-\varepsilon}) \).
For any \( \ell \in \mathbb{Z} \), by Lemma 2.2 and Lemma 2.4 we have

\[
\max \left\{ \left| F_{3,\ell}(u) \right|, \left| F_{4,\ell}(u) \right| \right\} \ll \frac{1}{\log u} \sum_{h \geq 1 \atop 2 \nmid h} h^{1/2+\epsilon/2} \nu(u)^h \ll (\log u)^{1/2+\epsilon/2},
\]

hence for \( j = 3, 4 \) we see that

\[
\int_3^x \frac{e((ka + \ell\hat{a})u)}{\nu(u)(\log u)^2} F_{j,\ell}(u) \, du \ll \frac{x}{(\log x)^{3/2-\epsilon/2}}.
\]

By (3.3), it follows that the contribution to (3.8) from terms with \( j = 3, 4 \) is

\[
\ll \frac{x}{(\log x)^{3/2-\epsilon/2}} \sum_{|k|,|\ell| \leq K} |g_a(k)g_a(\ell)| \ll \frac{x(\log x)^2}{(\log x)^{3/2-\epsilon/2}} \ll \frac{x}{(\log x)^{3/2-\epsilon}}.
\]

Finally, for any \( \ell \in \mathbb{Z} \) and \( u \geq \exp(\hat{a}) \), by Lemma 2.2 and Lemma 2.4 we have

\[
F_{5,\ell}(u) = \frac{1}{(\nu(u) \log u)^2} \sum_{h \geq 1 \atop 2 \nmid h} \left(-\frac{1}{2} h \log h + \frac{1}{2} Ah + O(h^{1/2+\epsilon/2})\right) \nu(u)^h e(\ell\hat{a}h)
\]

\[
= -\frac{1}{2} R_{1,1;\ell\hat{a}}(u) + \frac{1}{2} A R_{1,0;\ell\hat{a}}(u) + O(R_{1/2+\epsilon/2,0,0}(u))
\]

\[
\ll \frac{\lambda^{-4} + (\log u)^{3/2+\epsilon/2}}{(\log u)^2},
\]

and arguing as before we see that the contribution to (3.8) coming from terms with \( j = 5 \) does not exceed \( O(x/(\log x)^{3/2-\epsilon}) \).

Applying the preceding bounds to (3.8) we see that, up to \( O(x/(\log x)^{3/2-\epsilon}) \), the quantity \( \pi(x; B, \hat{B}) \) is equal to

\[
\hat{a} \sum_{j=1,2} \sum_{|k| \leq K} g_a(k) e(kb) \int_3^x \frac{e(kau)}{\nu(u)(\log u)^2} F_{j,0}(u) \, du,
\]

where we have used the fact that \( g_a(0) = \hat{a} \). By Lemma 2.4 we have

\[
F_{1,0}(u) = \sum_{h \geq 1 \atop 2 \nmid h} \nu(u)^h = R_{0,0,0}(u) = \frac{1}{2} \log u + O(1)
\]

and

\[
F_{2,0}(u) = \sum_{h \geq 1 \atop 2 \nmid h} S_0(\{0,h\}) \nu(u)^h = S_0(u) = \frac{1}{2} \log u - \frac{1}{2} \log \log u + O(1);
\]

therefore,

\[
\int_3^x \frac{e(kau)}{\nu(u)(\log u)^2} F_{j,0}(u) \, du = \frac{1}{2} \int_3^x \frac{e(kau)}{\nu(u) \log u} \, du + O\left(\frac{x \log \log x}{(\log x)^2}\right)
\]

\((j = 1, 2)\).

Consequently, up to \( O(x/(\log x)^{3/2-\epsilon}) \) we can express the quantity \( \pi(x; B, \hat{B}) \) as

\[
\hat{a} \sum_{|k| \leq K} g_a(k) e(kb) \int_3^x \frac{e(kau)}{\nu(u) \log u} \, du.
\]

(3.9)
To complete the proof of Theorem 1.1, we apply Lemma 2.5, which shows that the term $k = 0$ in (3.9) contributes

$$a \hat{\alpha} \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right) = (\alpha \hat{\alpha})^{-1} \pi(x) + O\left(\frac{x}{(\log x)^2}\right)$$

to the quantity $\pi(x; B, \hat{B})$ (and thus accounts for the main term), whereas the terms in (3.9) with $k \neq 0$ contribute altogether only a bounded amount.

References


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