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QUALIFYING EXAMINATION / ALGEBRA **August 23, 2002**

• *If you have any difficulty with the wording of the following problems please contact the supervisor immediately.*
• *While dealing with a certain item of a multi-part problem, you are allowed to rely on any previous items (proved or not). Nonetheless, all individual answers should be fully justified.*

• *Throughout, $\mathbb{R}$ denotes the real numbers, $\mathbb{C}$ denotes the complex numbers, and $\mathbb{Z}$ denotes the integers.*

Algebra

1. For any two subgroups H, K of a group G, let $HK = \{hk \mid h \in H, k \in K\}$. Show that HK is a subgroup of G if and only if $HK = KH$.

2. Let G be a group containing two normal subgroups A and B with p and q elements, respectively, where p and q are distinct prime numbers. Suppose that a is a generator for A and b is a generator for B. Determine the order of ab, and justify your answer.

3. Let R be a commutative ring with identity, and let I be an ideal of the polynomial ring $R[x]$. Suppose that for some monic polynomial $f \in I$, $\deg(f) \leq \deg(g)$ for all nonzero $g \in I$. Show that I is a principal ideal.

4. What is the cardinality of the quotient ring $\mathbb{Z}[x]/(x^2 - 3, 2x + 4)$? Justify your answer.

5. Let R be the ring of continuous functions $f : [0, 1] \to \mathbb{R}$. Suppose that $f_1, \ldots, f_n \in R$ have no common zero in $[0, 1]$. Prove that the ideal generated by $f_1, \ldots, f_n$ is R.

6. Let I be the ideal of $\mathbb{Z}_{211}[x]$ consisting of polynomials $p(x)$ such that $p(a) = 0$ for all $a \in \mathbb{Z}_{211}$. Show that I is principal, and determine a generator for the ideal.

Linear Algebra

A. Let V be the set of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Let $T : V \to V$ be the linear map defined by $T(f) = f'$. Determine all of the eigenvalues and the corresponding eigenvectors of T.

B. Let V be the vector space of all real polynomials of degree ≤ 2. The space V has a bilinear form defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx.$$

Find an orthonormal basis of V.

C. Let $w \in \mathbb{R}^n$ be a column vector of length 1.
(C1) Prove that the matrix $P = I - 2ww^t$ is orthogonal, where I is the identity matrix.
(C2) Prove that multiplication by P is a reflection through the space W orthogonal to w; that is, if we write an arbitrary vector v in the form $v = cw + w'$ with $w' \cdot w = 0$, then $Pv = -cw + w'$.
(C3) Let X, Y be two vectors in $\mathbb{R}^n$ with the same length. Determine a vector w (in terms of X and Y) such that $PX = Y$, where P is the matrix in question (C1).

D. Let A be a complex $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $|\lambda_1| > |\lambda_i|$ for all $i > 1$. Show that the limit

$$L_X = \lim_{k \to \infty} \lambda_1^{-k} A^k X$$

exists for all $X \in \mathbb{R}^n$, and that L_X is an eigenvector of A with eigenvalue λ_1 if and only if $L_X \neq 0$. Determine the dimension the vector space

$$V = \{X \in \mathbb{R}^n \mid L_X = 0\},$$

and state precisely what conditions on X will guarantee that $L_X \neq 0$.

E. For any complex matrix A, prove that $I + A^*A$ has a nonzero determinant, where $A^* = \overline{A^t}$.