Qualifying Examination

(*August 2006*)

- If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam.
- You are allowed to rely on a previous part of a multi-part problem even if you do not prove the previous part.
- $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.
- All matrices are over fields.
- The rank of a linear transformation is the dimension of its image.

Abstract Algebra

1. Let G denote a group.

(i) (2 points) State the Fundamental Theorem of Finitely Generated Abelian Groups;

(ii) (4 points) Let G be a finite abelian group. Prove that G is not cyclic if and only if G contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime number p;

(iii) (3 points) Let H be a subgroup of G, and N be a normal subgroup of G. Prove that HN is a subgroup of G;

(iv) (4 points) Let H be a subgroup of G, and N be a normal subgroup of G. If $|H| = 2006$ and $|N| = 2007$, compute $|H \cap N|$ and $|HN|$. You must justify your answer.

2. (2 points) Let $n \geq 3$ and S_n be the symmetric group of n letters. Prove that the only element σ of S_n satisfying $\sigma\gamma = \gamma\sigma$ for all $\gamma \in S_n$ is $\sigma = \mathrm{Id}$, the identity permutation.

3. Let R denote a ring and F denote a field.

(i) (3 points) A ring R is *Boolean* if $r^2 = r$ for every $r \in R$. Prove that every Boolean ring is commutative;

(ii) (4 points) Let $f(x) \in F[x]$ be a degree-3 polynomial. Prove that $f(x)$ is irreducible if and only if $f(x)$ does not have any roots in F;

(iii) (4 points) Construct a field F with 125 elements. You must justify your answer.

4. Let $\mathbb{C}[x, y, z, w]$ be the polynomial ring with variables x, y, z, w. Let

$$I = \{f \in \mathbb{C}[x, y, z, w] |\ f(-2, -1, 1, 2) = 0\}.$$

(i) (2 points) Prove that I is a maximal ideal of $\mathbb{C}[x, y, z, w]$;

(ii) (2 points) Determine a set of generators for I and justify your answer.

Linear Algebra

1. (3 points) Let $n \geq 2$, and let N be an $n \times n$-matrix over a field F such that $N^n = 0$ but $N^{n-1} \neq 0$. Show that N has no square root. That is, there is no matrix B such that $B^2 = N$.

2. (5 points) For $d \geq 1$, let P_d be the space of all real polynomials of degree less than d. Then P_4 has an inner product defined by

$$f * g = f(0)g(0) + f(-1)g(-1) + f(2)g(2) + f(-2)g(-2).$$

(i) Use Gram Schmidt to find an orthogonal basis for P_4.

(ii) Find a basis for the orthogonal complement of P_2 in P_4.

3. (i) (2.5 points) Let P_4 be from Problem 2. Does there exist a linear transformation $L : \mathbb{R}^3 \to P_4$ such that

$$\begin{aligned} L\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right) &= x^3 + 4x - 2, \\ L\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right) &= x^3 + 7x + 2, \\ L\left(\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right) &= 3x + 4. \end{aligned}$$

Justify your answer completely.

(ii) (2.5 points) Let V and W be two real vector spaces of dimension m and n respectively. Let $L : V \to W$ be a linear transformation. Assume that Rank $L = 1$. Show that L can be factored via $\mathbb{R}$. That is, there are linear transformations $S : V \to \mathbb{R}$ and $T : \mathbb{R} \to W$ such that $L = T \circ S$.

4. (5 points) Let $T = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 1 & -1 \end{bmatrix}$ be over $\mathbb{R}$.

(i) Is T diagonalizable? If yes, diagonalize it and if not show why it is not diagonalizable.

(ii) Find T^{10}.

5. State True or False for each of the following statements. If true, prove it. If false, give an example to disprove it. You cannot just state a theorem to disprove anything.

(i) (4 points) If A and B are two invertible $n \times n$-matrices over a field F, then $A + xB$ is invertible for all but finitely many $x \in F$.

(ii) (4 points) The eigenvalues of AB are the same as the eigenvalues of BA for any two $n \times n$ matrices A and B.

(iii) (4 points) Let A be a real $n \times n$-matrix such that $AA^T = A^T A$. If the eigenvalues of A are all real, then A must be symmetric.