Qualifying Examination II **January 12, 2001**

If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam.

Throughout, $\mathbb{R}$ denotes the real numbers, and $\mathbb{C}$ denotes the complex numbers.

Real Analysis I

1. (a) (3 points) Show that if $a < b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ is continuous, then f attains a maximum on $[a, b]$. (We will denote this maximum $\max_{x\in[a,b]} f(x)$.)

(b) (4 points) Suppose that $a, b \in \mathbb{R}$, that $f_k : [a, b] \to \mathbb{R}$ are continuous functions for $k \in \mathbb{N}$, and that $f_k \to f$ uniformly on $[a, b]$. Let $m_k = \max_{x\in[a,b]} f_k(x)$. Show that $m_k \to \max_{x\in[a,b]} f(x)$.

(c) (3 points) Show that in part (b) that the assumption of uniform convergence is necessary, that is, give an example of a sequence of continuous functions $f_k : [a, b] \to \mathbb{R}$ with $f_k \to f$ pointwise, but $m_k \not\to \max_{x\in[a,b]} f(x)$, where again $m_k = \max_{x\in[a,b]} f_k(x)$. (Be sure to identify f_k, m_k and f.)

2. (a) (3 points) Write Taylor's Series Formula, and provide a complete and accurate set of hypotheses guaranteeing its validity.

(b) (4 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, and that $|f(x)| \le M_0$ and $|f''(x)| \le M_2$ for all $x \in \mathbb{R}$. Show that

$$|f'(x)| \le 2M_0 + \frac{1}{2}M_2 \qquad (x \in \mathbb{R}).$$

(Hint: Expand $f(x + 1)$ in a Taylor series about the point x.)

(c) (3 points) Under the same hypotheses as part (b), show that

$$|f'(x)|^2 \le 4M_0M_2 \qquad (x \in \mathbb{R}).$$

(Hint: Apply (b) to the function $g(x) = f(\lambda x)$ for $\lambda > 0$, and optimize over λ.)

Real Analysis II

3. (a) (3 points) State precisely Fubini's theorem for a continuous function $f : R \to \mathbb{R}$ in a two dimensional rectangle $R = [a, b] \times [c, d]$.

(b) (4 points) Prove that if $f \in C^2(\mathbb{R}^2)$ then

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

at each point $(x_0, y_0) \in \mathbb{R}^2$. (Hint: Reason by contradiction and then find a way to use part (a).)

(c) (3 points) Show that for the function $f(0,0) = 0$ and

$$f(x,y) = \frac{x^3 y}{x^2+y^2} \qquad (x,y) \neq (0,0),$$

that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

4. (a) (3 points) Give a precise statement of the one variable 2nd derivative test for maxima and minima.

Let $D = \{(x,y) \in \mathbb{R}^2 : |(x,y)| \leq 1\}$, and let $u : D \to \mathbb{R}$ be a continuous function with three continuous derivatives in the interior of D. Denote

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

(b) (4 points) Show that if there is a number $c > 0$ such that $\Delta u \geq c$, then u attains its maximum on $\{(x,y) \in \mathbb{R}^2 : |(x,y)| = 1\}$, i.e. the boundary of D. (Hint: Suppose that u attains its maximum at (x_0, y_0), $|(x_0,y_0)| < 1$. Consider the functions $f(t) = u(x_0 + t, y_0)$ and $g(t) = u(x_0, y_0 + t)$.)

(c) (3 points) Show that if $\Delta u \geq 0$, then u attains its maximum on $\{(x,y) \in \mathbb{R}^2 : |(x,y)| = 1\}$. (Hint: consider the functions $u_n(x,y) = u(x,y) + (x^2+y^2)/n$.)

Complex Analysis

5. (a) (5 points) Use the calculus of residues to evaluate the contour integral

$$\oint_C \frac{e^z}{(z-1)\sin z}\, dz,$$

where $C = \{z : |z| = 4\}$ is oriented counterclockwise and traced once.

(b) (5 points) Evaluate the integral

$$\int_0^\infty \frac{\cos x}{1+x^4}\, dx$$

using contour integration in the complex plane.

6. (a) (5 points) Let $f(z) = e^{\frac{z+1}{z-1}}$.

(i) Show that $f(z)$ maps the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto itself.

(ii) Let $0 < |a| < 1$. Prove that all isolated singular points of $\frac{1}{f(z)-a}$ in $\mathbb{D}$ are simple poles. Enumerate the poles explicitly.

(b) (5 points) Given an entire $f(z)$, show that if $f(z)f(1/z)$ is bounded on $\mathbb{C}$, then $f(z) = a\, z^n$ for some $a \in \mathbb{C}$. (Hint: Show first that $f(z)f(1/z)$ is entire and hence is constant. Use this to check that the only possible zero of $f(z)$ is at $z = 0$.)