Qualifying Examination in Analysis

January, 2011

- If you have any difficulty with the wording of the following problems please contact the supervisor immediately.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- Notation: $\mathbb{R}$ denotes the real numbers, $\mathbb{N}$ the positive integers, and $\mathbb{C}$ the complex numbers.

Real Analysis I: One-dimensional Calculus

1. **(a)** (3 points) Suppose that $\{a_n\}$ is a nonnegative decreasing sequence, defined for $n \geq 2$. Prove that for every integer $N \geq 1$ the following inequality holds:

$$\sum_{k=2}^{2^N} a_k \geq \sum_{k=1}^{N} 2^{k-1} a_{2^k}.$$

(b) (3 points) Prove that if $\{a_n\}$ is a nonnegative decreasing sequence and $\sum_{k=2}^{\infty} a_k$ converges then $\sum_{k=1}^{\infty} 2^k a_{2^k}$ also converges.

[Hint: Series with nonnegative terms have increasing partial sums.]

(c) (4 points) Show that the following series diverges:

$$\sum_{k=2}^{\infty} k^{-1-\frac{1}{1+\ln k}}.$$

2. For parts (a) and (b), let $f_1,\ f_2$ be functions defined on $(0,1)$. Set, for $x \in (0,1)$, $f(x) = \max\{f_1(x),\ f_2(x)\}$.

(a) (6 points) Prove that if $f_1,\ f_2$ are continuous at $x = a \in (0,1)$, then $f(x)$ is continuous at $x = a$.

(b) (4 points) Give an example with functions $f_1,\ f_2$ which are differentiable at $x = a \in (0,1)$, but f is not differentiable at a. Clearly indicate your functions $f_1,\ f_2,\ f$ and your value of a.

Real Analysis II: Multi-dimensional Calculus

3. A set $E \subseteq \mathbb{R}^2$ is closed if for any sequence $\{\mathbf{a}_n\}$ in E with $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{c} \in \mathbb{R}^2$ we have $\mathbf{c} \in E$. For each $A \subseteq \mathbb{R}^2$ and $B \subseteq \mathbb{R}^2$ let

$$A + B = \{\mathbf{a} + \mathbf{b},\ \mathbf{a} \in A,\ \mathbf{b} \in B\}.$$

(a) (4 points) Let A and B be closed with $A \subseteq [-M, M] \times [-M, M]$, some $M > 0$. Show that $A + B$ is closed.

(b) (6 points) Let

$$A = \left\{(x, y) :\ x > 0,\ y \geq \frac{1}{x}\right\}, \qquad B = \left\{(x, y) :\ x < 0,\ y = \frac{1}{2x}\right\}.$$

Show that A and B are closed, but $A + B$ is not closed.

[Hint: for $A + B$ find suitable sequences $\mathbf{a}_n \in A, \mathbf{b}_n \in B$ such that $\mathbf{a}_n + \mathbf{b}_n \to (0,0)$].

4. **(a)** (7 points) Find the absolute maximum and absolute minimum of $f(x,y) = xy - (1 - x^2 - y^2)^{3/2}$ on $\{(x,y) : x^2 + y^2 \leq 1\}$.

(b) (3 points) Prove that $(0,0)$ is a saddle point for the function $g(x,y) = x^2y^5 + x^4y^4$. [Recall that a saddle point is a critical point which is neither a maximum nor a minimum.]

Complex Analysis

5. **(a)** (5 points) For this problem, let $\sqrt{z} = r^{1/2}e^{i\theta/2}$, where $z = re^{i\theta}$, $r \geq 0$, and $0 \leq \theta < 2\pi$. Let $B(0,1)$ denote the open ball of radius 1 and center at the origin. Suppose $f(z)$ is holomorphic on $B(0,1)$, and $g(z) = f(z)+f(-z)$. Show that $g(\sqrt{z})$ is holomorphic on $B(0,1)$.

   (b) (5 points) Let f be an entire function on $\mathbb{C}$. Show that if for all natural numbers n,
$$\left| f\left(\frac{1}{n}\right)\right| \leq \frac{1}{n^n},$$
then $f(z) \equiv 0$. [Hint: Show that all derivatives of f at 0 vanish.]

6. **(a)** (1 point) Give the definition of the complex derivative of a function $f : \mathbb{C} \to \mathbb{C}$ at a point $a \in \mathbb{C}$.

   (b) (5 points) Show that the function defined by
$$f(z) = \begin{cases} \dfrac{z^5}{|z|^4}, & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$
satisfies the Cauchy-Riemann equations at 0, but is not complex-differentiable at 0.

   (c) (4 points) Assume that a function g is entire, with a zero of order k at 0. Classify (with explanation) the singularity at $z = 0$ of each of the following functions:
$$\text{i)}\ \frac{g'(z)}{g(z)}, \qquad \text{ii)}\ \frac{\sin g(z)}{g(z)}.$$
If the function has a pole at 0, state the order of the pole and find the residue.