Qualifying Examination in Analysis

January 2015

- If you have any difficulty with the wording of the following problems, please contact the supervisor immediately.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- Notation: $\mathbb{R}$ denotes the real numbers, $\mathbb{N}$ the positive integers, and $\mathbb{C}$ the complex numbers. Also, if $z \in \mathbb{C}$ then $Re(z)$ and $Im(z)$ denote the real and imaginary parts of z, respectively.

Real Analysis I: One-Dimensional Calculus

1. **(a)** (4 points) Let $f : (0, \infty) \to (0, \infty)$ be a continuously differentiable function satisfying $f' \leqslant 0$ on $(0, \infty)$. Fix $a \in (0, \infty)$ and define

$$f_a(t) := \frac{f(t+a) - f(t)}{a} \qquad \text{for } t \in (0, \infty). \tag{1}$$

Prove that $\lim_{t \to \infty} f_a(t) = 0$.

(b) (4 points) Let $f : (0, \infty) \to (0, \infty)$ be a function of class C^2 with $f' \leqslant 0$ and f'' bounded on $(0, \infty)$. Prove that $\lim_{t\to\infty} f'(t) = 0$.

Hint: You may use Taylor expansion and the result from **(a)**.

(c) (2 points) Show that the conclusion in **(b)** is false if one allows f to take arbitrary real values. That is, prove that there exists $f : (0, \infty) \to \mathbb{R}$ of class C^2 with $f' \leqslant 0$ and f'' bounded on $(0, \infty)$ for which $\lim_{t\to\infty} f'(t) \neq 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that there exists $C \in (0, \infty)$ such that

$$|f(x)| \leqslant \frac{C}{1 + x^2} \qquad \forall\, x \in \mathbb{R}. \tag{2}$$

(a) (2 points) Prove that for each $x \in \mathbb{R}$ the series

$$\sum_{n=-\infty}^{n=\infty} f(x+n)$$

is absolutely convergent.

(b) (3 points) Define the function

$$F(x) := \sum_{n=-\infty}^{n=\infty} f(x+n) \qquad \forall\, x \in \mathbb{R}.$$

Prove that F is continuous on $\mathbb{R}$.

(c) (1 point) Let F be the function defined in part **(b)**. Prove that F is periodic of period 1.

(d) (1 point) Prove that if $g : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function with period 1 and f is as above, then $\int_{-\infty}^{\infty} f(x)g(x)\,dx$ is absolutely convergent.

(e) (3 points) Prove that if $g : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function with period 1 and f, F are as above, then

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = \int_0^1 F(x)g(x)\,dx.$$

Real Analysis II: Multi-dimensional Calculus

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set whose boundary $\partial\Omega$ is of class C^1. In this problem, we write $\operatorname{div} \mathbf{F}$ to denote the divergence of a differentiable function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$. Likewise, $\operatorname{grad} G$ denotes the gradient of a differentiable function $G : \mathbb{R}^n \to \mathbb{R}$.

(a) (2 points) State the divergence theorem as it applies to Ω.

(b) (3 points) Prove that if $\mathbf{v} \in C^1(\overline{\Omega}; \mathbb{R}^n)$, and $\phi \in C^1(\overline{\Omega}; \mathbb{R})$, then

$$\int_\Omega \mathbf{v} \cdot \operatorname{grad} \phi \, d\mathbf{x} = \int_{\partial\Omega} \phi \mathbf{v} \cdot \mathbf{n} \, dS - \int_\Omega \phi \operatorname{div} \mathbf{v} \, d\mathbf{x},$$

where $\mathbf{n}$ is the outward unit normal on $\partial\Omega$.

(c) (3 points) Show that $\mathbf{v} \in C^1(\overline{\Omega}; \mathbb{R}^n)$ is divergence free in Ω, i.e.

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } \Omega,$$

if and only if

$$\int_\Omega \mathbf{v} \cdot \operatorname{grad} \phi \, d\mathbf{x} = 0, \qquad \text{for all } \phi \in C^1(\overline{\Omega}; \mathbb{R}) \text{ with } \phi|_{\partial\Omega} = 0,$$

where $\phi|_{\partial\Omega}$ denotes the pointwise restriction of ϕ to $\partial\Omega$.

(d) (2 points) Prove that if $\phi \in C^2(\overline{\Omega};\mathbb{R})$ satisfies

$$\operatorname{div}(\operatorname{grad}\phi) = 0 \text{ in } \Omega, \qquad \phi|_{\partial\Omega} = 0,$$

then $\phi \equiv 0$.

2. Consider the function $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\mathbf{F}(x,y,z) := \big(x + y + f(z),\ g(x,y) + y + z\big),$$

where $f \in C^1(\mathbb{R})$, and $g \in C^1(\mathbb{R}^2;\mathbb{R})$ satisfies

$$\sup_{(x,y)\in\mathbb{R}^2} \big|(\operatorname{grad} g)(x,y)\big| < \frac{1}{4}.$$

Define $S := \mathbf{F}^{-1}\big(\{(0,0)\}\big)$.

(a) (2 points) State the implicit function theorem as it applies to a function $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^2$.

(b) (3 points) Prove that if $(x_0, y_0, z_0) \in S$, then there is an open neighborhood $\mathcal{O} \subset \mathbb{R}^3$ of (x_0, y_0, z_0), and $\delta > 0$, such that

$$S \cap \mathcal{O} = \big\{(X(z), Y(z), z) : z \in (z_0 - \delta, z_0 + \delta)\big\},$$

for some functions $X, Y \in C^1\big((z_0 - \delta, z_0 + \delta);\mathbb{R}\big)$.

(c) (3 points) Show that, for all $\alpha \in \mathbb{R}$, every point in the set

$$S_\alpha := \big\{(x, y, \alpha) : \mathbf{F}(x, y, \alpha) = (0,0)\big\}$$

is an isolated point for S_α.

(d) (2 points) Prove that, for all $\alpha \in \mathbb{R}$, and all compact sets $K \subset \mathbb{R}^3$, $K \cap S_\alpha$ is a finite set.

Complex Analysis

1. (a) (1 point) State Rouche's Theorem.

(b) (4 points) Let $P(z) = z^8 - 5z^3 + z - 2$. Find the number of zeros of P, counting multiplicities, inside the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ of $\mathbb{C}$.

2. (a) (1 point) State Schwarz's Lemma.

(b) (4 points) Let $S = \{z \in \mathbb{C} : |Re(z)| < 1 \text{ and } |Im(z)| < 1\}$ be the unit square in $\mathbb{C}$, let $B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in $\mathbb{C}$, and let $f : B(0,1) \to S$ be a holomorphic function. Suppose that f is both surjective and injective and that $f(0) = 0$. Show that

$$f(iz) = if(z)$$

for all $z \in B(0,1)$. You may assume the following theorem:

Let U and V be connected open sets in $\mathbb{C}$ and let $f : U \to V$ be holomorphic. If f is bijective, then f' does not vanish in U and its inverse $f^{-1} : V \to U$ is also holomorphic and $(f^{-1})'(f(z)) = \dfrac{1}{f'(z)}$ for each $z \in U$.

Hint: You may want to consider the composite map $f^{-1} \circ \mu \circ f$, where $\mu : S \to S$ is the map $\mu(z) = iz$ for all $z \in S$.

3. Let $B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$. Bierberbach's Theorem states that:

Suppose $g : B(0,1) \to \mathbb{C}$ is an injective holomorphic function satisfying $g(0) = 0$ and $g'(0) = 1$, so that its Maclaurin series is of the form

$$z + \beta_2 z^2 + \beta_3 z^3 + \cdots.$$

Then $|\beta_2| \leqslant 2$.

In what follows you may assume Bieberbach's Theorem.

(a) (4 points) Let $f : B(0,1) \to \mathbb{C}$ be an injective holomorphic function with $f(0) = 0$ and $f'(0) = 1$. Fix $w \notin f(B(0,1))$ and define the function $g : B(0,1) \to \mathbb{C}$ by

$$g(z) := \frac{wf(z)}{w - f(z)}$$

for all $z \in B(0,1)$. Prove that g satisfies the hypothesis of Bieberbach's Theorem.

(b) (4 points) Show that $|w| \geqslant \frac{1}{4}$.

(c) (2 points) In Part **(b)** you have proved the following assertion:

Suppose $f : B(0,1) \to \mathbb{C}$ is an injective holomorphic function satisfying $f(0) = 0$ and $f'(0) = 1$. If $w \notin f(B(0,1))$, then we must have

$$|w| \geqslant \frac{1}{4}.$$

Prove that the number $\frac{1}{4}$ in this inequality cannot be improved.

Hint: You may want to consider the function $f(z) := \dfrac{z}{(1-z)^2}$ for $z \in B(0,1)$.