In this talk, we will discuss a recent breakthrough addressing the Lipschitz continuous dependence of solutions on initial data for a quasi-linear wave equation

$$u_{tt} - c(u)[c(u)u_x]_x = 0.$$ 

Our earlier results showed that this equation determines a unique flow of conservative solution within the natural energy space $H^1(\mathbb{R})$. However, this flow is not Lipschitz continuous with respect to the $H^1$ distance, due to the formation of singularity first found by Glassy–Hunter–Zheng. To prove the desired Lipschitz continuous property, we construct a new Finsler type metric, where the norm of tangent vectors is defined in terms of an optimal transportation problem. For paths of piece-wise smooth solutions, we carefully estimate how the distance grows in time. To complete the construction, we prove that the family of piece-wise smooth solutions is dense, following by an application of the Thom’s transversality theorem. Applications to the Camassa-Holm equation will also be introduced in the end of the talk. This is a collaboration work with Alberto Bressan.