Riemann-Liouville Operators of Varying Order

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Abstract: We present continuity and compactness properties of the integration operator

$$(R^{\alpha(\cdot)}f)(t) := \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} f(s)\,\mathrm{d}s\,, \quad 0 \le t \le 1\,.$$

Here $\alpha(\cdot)$ is a given measurable function on $[0,1]$ possessing a.e. positive values. Operators $R^{\alpha(\cdot)}$ are generalizations of classical Riemann-Liouville operators R^α of order $\alpha > 0$ which correspond to $\alpha(t) \equiv \alpha$. Thus $R^{\alpha(\cdot)}$ may be viewed as fractional integration operator of varying order.

Our interest to investigate operators $R^{\alpha(\cdot)}$ stems from the theory of multi-fractional random processes. These are fractional Brownian motions $\{B_H(t) : t \ge 0\}$ with time depending Hurst index $H = H(t)$.

In the talk we will treat the following problems:

- Under which conditions on $\alpha(\cdot)$ is $R^{\alpha(\cdot)}$ bounded from $L_p[0,1]$ into $L_q[0,1]$?
- In which cases is $R^{\alpha(\cdot)}$ not only bounded but even a compact operator?
- How does the degree of compactness (measured by the behavior of its entropy numbers) depend on properties of the function $\alpha(\cdot)$?

References:

M. Lifshits, Linde, W.: *Fractional integration operators of variable order: continuity and compactness properties*, Math. Nachr. **287**, 980 – 1000 (2014).

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