Qualifying Examination in Analysis

August 2023

- If you have any difficulty with the wording of the following problems please contact the supervisor immediately.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- Solve all problems.

1. Let $h : \mathbb{R} \to \mathbb{R}$ be defined as

$$h(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}.$$

   (a) Explain why h is Borel measurable.

   (b) Let $g(x) = \sqrt{\log \frac{1}{x}}$, for $x \in (0, 1/2]$. Show that $e^{g-nh} \in L^1((0, 1/2])$, for all $n \in \mathbb{N}$.

   (c) Show that

$$\lim_{n\to\infty} \int_0^{1/2} e^{g(x)-nh(x)} dx = 0.$$

2. (a) Define $\|f\|_\infty$ for a Borel measurable function $f : [0, 1] \to \mathbb{R}$. Define $L^\infty([0, 1])$.

   (b) Let $g \in L^1([0, 1])$ with $\|g\|_1 > 0$. Show that if $f \in L^\infty([0, 1])$ is such that $\|f\|_\infty > 0$ and $|\{x \in [0, 1] : |f(x)| = \|f\|_\infty\}| = 0$ then $fg \in L^1([0, 1])$ and

$$\int_0^1 f(t)g(t)\, dt < \|g\|_1 \|f\|_\infty.$$

   (c) For each Lebesgue measurable set $E \subseteq [0, 1]$ such that $|E| > 0$ find $g \in L^1([0, 1])$, $f \in L^\infty([0, 1])$ such that $\|g\|_1 > 0, \|f\|_\infty > 0$, $|f| = \|f\|_\infty$ on E, and $\int_0^1 f(t)g(t)dt = \|g\|_1\|f\|_\infty$.

3. (a) State Fatou's lemma.

   (b) Let $f_{n,m}$ and f_m $(n, m = 1, 2, \dots)$ be complex-valued measurable functions on a measure space (X, μ) such that for all $x \in X$ and each m we have $\lim_{n\to\infty} f_{n,m}(x) = f_m(x)$. Suppose that for every $n = 1, 2, \dots$ we have

$$\int_X \sup_m |f_{n,m}(x)| \, d\mu(x) \leq 2 + (-1)^n.$$

   Prove that

$$\int_X \sup_m |f_m(x)| \, d\mu(x) \leq 1.$$

   (c) Under the assumption of part (b) show that there is a set E of measure zero such that for all $x \in X \setminus E$ and all $n = 1, 2, \dots$ we have

$$\sup_m |f_{n,m}(x)| + \sup_m |f_m(x)| < \infty.$$

4. Let $g : [0, \infty) \to \mathbb{R}$ be a Lebesgue measurable function such that

$$|g(t)| \leq \frac{t^a}{t+1}, \qquad \forall t \geq 0$$

for some $a \in (0, 1)$. Define $G(x, t) = e^{-xt} g(t)$ on $[0, \infty)^2$. Show that G is integrable on $[0, \infty)^2$.

5. On $\mathbb{R}$ consider the positive measures

$$\mu_1(A) = \int_A \frac{dx}{1+x^2}, \quad \mu_2(A) = \int_A \frac{dx}{1+x^4}, \quad \mu_3(A) = |A| + \delta_0(A),$$

where $|A|$ is the Lebesgue measure of A and δ_0 Dirac mass at $\{0\}$.

(a) For each of the following assertions establish whether or not the assertion is true, justifying your answer:

$$\mu_1 \ll \mu_2, \;\; \mu_2 \ll \mu_1, \;\; \mu_1 \ll \mu_3, \;\; \mu_3 \ll \mu_1, \;\; \mu_2 \ll \mu_3, \;\; \mu_3 \ll \mu_2.$$

(b) Compute the Radon-Nikodym derivative in the cases where absolute continuity holds.

6. Let H be a Hilbert space over $\mathbb{R}$ and let $T : H \to H$ be a linear map such that $\langle Tx, x\rangle \geq 0$ for all $x \in H$. Prove the following:

   (a) For all $u, v \in H$ we have

   $$|\langle Tu, v\rangle + \langle u, Tv\rangle|^2 \leq 4\langle Tu, u\rangle\langle Tv, v\rangle.$$

   [Hint: Try $x = u + \lambda v$.]

   (b) T is a bounded linear operator from H to H. [Hint: Use part (a) for an arbitrary $v \in H$ and the Closed Graph Theorem.]

7. (a) State the Uniform Boundedness Principle.

   (b) Let $\{x_n\}$ be a sequence of vectors in a normed space X. Let X' be the space of all bounded linear functionals on X. Assume that for any $f \in X'$ we have

   $$\sum_{n=1}^{\infty} |f(x_n)| < \infty.$$

   Prove that there exists a constant $K > 0$ with the property

   $$\sum_{n=1}^{\infty} |f(x_n)| \leq K\|f\|,$$

   for all $f \in X'$.
   [Hint: Consider the bounded linear operators $T_n : X' \to \ell^1$ defined by

   $$T_n(f) = (f(x_1), \ldots, f(x_n), 0, 0 \ldots), \quad f \in X'.]$$