Algebra Qualifying Examination

January 2021

You are allowed to rely on a previous part of a multi-part problem even if you do not prove the previous part. There are 100 points total.

Groups

1. (12 points) Let S_n be the symmetric group of permutations on n letters. Let σ be an odd permutation in S_n, and G be a subgroup of S_n such that $\sigma \in G$. Prove that the order of G is even, and that exactly half of the elements in G are odd.

2. Suppose that H and K are subgroups of a finite group G.

a) (7 points) Show that $[H : H \cap K] \leq [G : K]$.

b) (6 points) Show that $[H : H \cap K] = [G : K]$ if and only if $G = KH$.

Rings

3. (13 points) Let $\alpha \in \mathbb{C}$ be a root of a monic polynomial $f(x) \in \mathbb{Z}[x]$. Prove that the minimal polynomial $p(x)$ of α over $\mathbb{Q}$ lies in $\mathbb{Z}[x]$.

4. (12 points) Let A be a commutative ring (with identity). Suppose that for every $x \in A$, there exists $n > 1$ (which depends on x) such that $x^n = x$. Show that every prime ideal of A is maximal.

Modules

5. (12 points) Let X be a subspace of $\mathrm{M}_n(\mathbb{C})$, the $\mathbb{C}$-vector space of all $n \times n$ complex matrices. Assume that every nonzero matrix in X is invertible. Prove that $\dim_{\mathbb{C}}(X) \leq 1$.

6. Let V be a vector space over a field K, and let $(\, , \,) : V \times V \to K$ be a bilinear form on V. A subspace A of a vector space B is proper if A is not the zero vector space and A is not equal to B.

a) (6 points) Suppose that V is finite dimensional and W is a proper subspace of V. Show that there exists a nonzero vector $v \in V$ such that $(w, v) = 0$ for all $w \in W$.

b) (7 points) Suppose that V is infinite dimensional, and $\mathcal{B}$ is a basis of V. Let $(\, , \,)$ be the unique bilinear form on V such that for all $a, b \in \mathcal{B}$, we have that $(a, b) = 0$ if $a \neq b$ and $(a, b) = 1$ if $a = b$. (You do not need to prove that this is a bilinear form.) If W is the subspace of V spanned by all vectors of the form $a - b$ with $a, b \in \mathcal{B}$, show that W is a proper subspace of V and that there is no nonzero vector $v \in V$ with $(w, v) = 0$ for all $w \in W$.

Fields

7. (13 points) Let $f(x)$ be irreducible in the polynomial ring $F[x]$, where F is a field of characteristic $p > 0$. Show that $f(x)$ can be written as $g(x^{p^e})$ where $g(x)$ is irreducible and separable. Use this to show that every root of $f(x)$ has the same multiplicity p^e in a splitting field.

8. (12 points) Construct a splitting field of $x^5 - 2$ over $\mathbb{Q}$. Find its dimension over $\mathbb{Q}$.