Analysis Qualifying Exam, January 2021

Instructions: *Do all 8 problems. Use a separate sheet for each problem. Your work will be graded for correctness, completeness, and clarity.*

1. Let $\mathcal{A}$ be an algebra and let μ be a finite measure on $\sigma(\mathcal{A})$. Let $\mathcal{F}$ be the collection of sets F in $\sigma(\mathcal{A})$ such that for each $\varepsilon > 0$, there exists $A \in \mathcal{A}$ satisfying $\mu(F \Delta A) < \varepsilon$, where $F \Delta A = (F \backslash A) \cup (A \backslash F)$. Show that $\mathcal{F}$ is a σ-algebra.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that f is an integrable function. Show that for every $\varepsilon > 0$, there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and
$$\left| \int f d\mu - \int_E f d\mu \right| < \varepsilon.$$

3. Evaluate the limit
$$\lim_{n \to \infty} \int_{\mathbb{R}} \frac{e^{x/2}}{1 + e^{nx}} dx.$$

4. Let f be an integrable function on $[0, 1]$. For $y \in \mathbb{R}$, let
$$g(y) = \int_0^1 \frac{f(x)}{\sqrt{|y - x|}} dx.$$
   a. Show that g is integrable on $[0, 1]$.
   b. Is $g(y)$ finite for almost every $y \in \mathbb{R}$?

5. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^1(X, \mathcal{M}, \mu)$. Prove that for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mu(E) < \delta$, then
$$\sup_n \int_E |f_n| d\mu < \varepsilon.$$

6a. Show that if $f : [0, 1] \to \mathbb{R}$ is absolutely continuous, then it is of bounded variation on $[0, 1]$.

b. For $a \geq 0$, define $g_a : [0, 1] \to \mathbb{R}$ by
$$g_a(x) = \begin{cases} x^a \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
For which values of a is g_a absolutely continuous?

7. Let X and Y be normed spaces and let $T : X \to Y$ be a bounded linear operator.
   a. Give the definition of the adjoint T^* of T and show that it is bounded with $\|T^*\| = \|T\|$.
   b. Show that if T^* is injective, then the range of T is dense in Y.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space, $\{f_n\} \subseteq L^p(\mu)$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that for each $g \in L^q(\mu)$, $\int f_n g d\mu \to 0$ as $n \to \infty$. Show that $\sup_n \|f_n\|_p < \infty$.