Instructions: Do all 8 problems. Use a separate sheet for each problem.

1. Let
   \[ f(x, y) = \begin{cases} 
   x^{-4} & \text{if } y \leq x^2, \ x \neq 0 \\
   -y^{-2} & \text{if } y > x^2 \\
   0 & \text{if } x = 0.
   \end{cases} \]

   Apply Tonelli’s Theorem to prove that \( f \) is not integrable on \( A = [0, \infty)^2 \), but it is integrable on \( B = [0, \infty) \times [1, \infty) \). Apply Fubini’s Theorem to compute the integral
   \[ \int \int_B f(x, y) dx dy. \]

2. State the dominated convergence theorem. Let
   \[ f_n(x) = \frac{\sin(nx^{1/3})}{x(n + x^{1/3})}, \quad x > 0 \]
   Show that \( f_n \in L^1([0, \infty)) \) and that
   \[ \lim_{n \to +\infty} \int_0^\infty f_n(x) dx = 0. \]

3. Let \( A \) be a measurable subset of \([0, 1]\) of positive measure. Show that there exist \( x_1, x_2 \in A \) with \( x_1 \neq x_2 \) such that \( x_1 - x_2 \) is a rational number.

4. Let \( f \) be a measurable function on a measure space \((X, \mu)\).
   (a) Prove that if \( f \) is integrable then the series
   \[ \sum_{n=1}^\infty \mu(\{x \in X : |f(x)| > n^2 \}) \]
   converges. Provide an example to show that the converse implication is false.
   (b) Show that if \( \mu(X) < \infty \) and
   \[ \sum_{n=1}^\infty n^2 \mu(\{x \in X : |f(x)| > n^2 \}) \]
   converges, then \( f \) must be integrable over \( X \).
5. Let $T$ be a linear operator from a Hilbert space $V$ (over $\mathbb{C}$) to $V$ that satisfies
\[ \|f\| = \frac{1}{5} \|T(f)\| \]
for all $f \in V$. Prove that for all $f, g \in V$ we have
\[ \langle T(f), T(g) \rangle = 25 \langle f, g \rangle. \]

6. Consider the space $C[-1,1]$ of continuous functions on $[-1,1]$, equipped with the $L^1$ norm with respect to the Lebesgue measure, and the linear functional $T_0 : C[-1,1] \to \mathbb{R}$ defined by
\[ T_0(f) = \int_{-1}^{1} |t| f(t) dt. \]
(a) Show that $T_0$ is bounded, and compute $\|T_0\|$.
(b) Let $f_0 = \chi_{[0,1]}$ and consider the operator $T : C[-1,1] + \mathbb{R}f_0 \to \mathbb{R}$ defined as
\[ T(f + \lambda f_0) = \int_{-1}^{1} |t| f(t) dt + \lambda, \quad f \in C[-1,1], \lambda \in \mathbb{R}. \]
(i) Show that $T$ is a linear functional that extends $T_0$.
(ii) Show that $T$ is unbounded on $C[-1,1] + \mathbb{R}f_0$, equipped with the $L^1$ norm.

7. Let $g(t)$ be a nonnegative measurable function on the real line. Show that
\[ \left( \int_{1}^{\infty} g(t) dt \right)^2 \leq 3 \int_{1}^{\infty} t^{4/3} g(t)^2 dt. \]
Assuming that the right-hand side is finite, find all functions $g$ for which equality holds in the preceding inequality.

8. Let $f$ be a complex-valued measurable function on a measure space $(X, \mu)$ that satisfies:
\[ \int_{|f|<n} \frac{|f|^2}{\frac{1}{n} + |f|} d\mu \leq 1 \]
for all $n = 1, 2, \ldots$. Find the limit of
\[ \int_{|f|\geq n} \frac{|f|^2}{\frac{1}{n} + |f|} d\mu \]
as $n \to \infty$. 