Instructions: Do all 8 problems. Use a separate sheet for each problem.

1. For $x, t$ real let $\varphi(t, x) = t \sin(x) + \cos(x)$.
   (a) Prove that
   \[
   \int_{1/n}^{n} \frac{\sin x}{x} \, dx = \int_{0}^{\infty} e^{-t/n} \varphi(t, 1/n) - e^{-nt} \varphi(t, n) \cdot \frac{t}{t^2 + 1} \, dt.
   \]
   Hint: Use that $1/x = \int_{0}^{\infty} e^{-xt} \, dt$ for $x > 0$ on the left hand side.
   (b) Find the limit of the expression in part (a) as $n \to \infty$.
   Justify your steps in parts (a) and (b).

2. (a) Let $\{B_n\}_{n=1}^{\infty}$ be a decreasing sequence of measurable subsets of a measure space $(X, S, \mu)$, in the sense that $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$. Let $B = \bigcap_{n=1}^{\infty} B_n$. Is it true that
   \[
   \lim_{n \to \infty} \mu(B_n) = \mu(B)?
   \]
   Prove this fact or provide a counterexample.
   (b) Suppose that $\mu(X) < \infty$. Assume that $C_n \in S$, $n = 1, 2, \ldots$, satisfy $\mu(C_n) \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. Prove that there exists a set $S_0 \in S$ such that $\mu(S_0) > 0$ and that every $x \in S_0$ lies in $C_j$ for infinitely many $j \in \mathbb{N}$. (Hint: Put $B_n = \bigcup_{k=n}^{\infty} C_k$.)

3. Let $\lambda$ be Lebesgue measure on the $\sigma$-algebra $L$ of all Lebesgue measurable subsets of $[-1, 1]$. Define a measure $\nu$ on $([-1, 1], L)$ by setting
   \[
   \nu = \delta_0 + h \, d\lambda,
   \]
   where $\delta_0$ is Dirac mass at zero and $h(t) = e^t$, $t \in [-1, 1]$. Which of the two statements $\nu \ll \lambda$, $\lambda \ll \nu$ is true and which one is false? Justify your answers in both statements. For the true statement find the associated Radon-Nikodym derivative.

4. (a) State Tonelli’s theorem and Fubini’s theorem.
   (b) Let $f$ be Lebesgue integrable on $(0, 1)$. For $0 < x < 1$, let
   \[
   g(x) = \int_{x}^{1} t^{-1} f(t) \, dt.
   \]
   Prove that $g$ is Lebesgue integrable on $(0, 1)$ and that
   \[
   \int_{0}^{1} g(x) \, dx = \int_{0}^{1} f(x) \, dx.
   \]
5. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for $\mathcal{H}$. For a bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ on $\mathcal{H}$ satisfying:

$$
(\alpha) \quad \langle Af, f \rangle \geq 0 \quad (f \in \mathcal{H}) \\
(\beta) \quad \langle Af, g \rangle = \langle f, Ag \rangle \quad (f, g \in \mathcal{H})
$$

we define $tr(A^2)$, the trace of $A^2$, by

$$
tr(A^2) = \sum_{n=1}^{\infty} \langle A^2 e_n, e_n \rangle \quad (= \sum_{n=1}^{\infty} \langle Ae_n, Ae_n \rangle).
$$

Prove that the definition of $tr(A^2)$ above is not dependent on the orthonormal basis, i.e. prove that if $\{d_m : m \in \mathbb{N}\}$ is another orthonormal basis on $\mathcal{H}$, then

$$
\sum_{n=1}^{\infty} \langle A^2 e_n, e_n \rangle = \sum_{m=1}^{\infty} \langle A^2 d_m, d_m \rangle.
$$

**Hint:** Expand $Ae_n$ in terms of the basis $\{d_m\}$.

6. (a) State the Lebesgue differentiation theorem.

(b) Prove that there does not exist a Lebesgue measurable subset $E$ of $[0,1]$ such that the Lebesgue measure of $E \cap [x,1]$ is equal to $(1-x)^2$ for all $x \in [0,1]$.

7. Let $f, g$ be elements of an inner product space $V$ over $\mathbb{R}$. Prove that $\langle f, g \rangle = 0$ if and only if for all $\lambda \in \mathbb{R}$

$$
\|f\| \leq \|f + \lambda g\|.
$$

8. (a) State (but do not prove) Fatou’s lemma.

(b) Suppose that $f_n$ is a sequence of measurable functions on a measure space that satisfy $f_n \leq g$ for some $g \in L^1$. Using part (a) show that

$$
\limsup_{n \to \infty} \int f_n \, d\mu \leq \int \limsup_{n \to \infty} f_n \, d\mu.
$$

(c) For the sequence $f_n(x) = (-1)^n \chi_{[0,1]} + (-1)^{n+1} \chi_{(1,3]}$ defined on $[0,3]$, show that

$$
\int_0^3 \limsup_{n \to \infty} f_n \, dx - \limsup_{n \to \infty} \int_0^3 f_n \, dx = \liminf_{n \to \infty} \int_0^3 f_n \, dx - \int_0^3 \liminf_{n \to \infty} f_n \, dx
$$