Qualifying Examination
May 2019

- You are allowed to rely on a previous part of a multi-part problem even if you do not prove the previous part.
- $\mathbb{N} = \{0, 1, 2, \ldots\}$
- $\mathbb{Z}$ denotes the group or ring of integers with the usual operations.
- $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ denote the groups or fields of rational, real, and complex numbers, respectively, with the usual operations.
- For a field F and positive integer n, $\mathrm{Mat}_{n\times n}(F)$ denotes the space of $n \times n$ matrices with entries in F. The identity $n \times n$ matrix is denoted by I_n.
- If n is a positive integer, the ring of integers modulo n is denoted by $\mathbb{Z}_n$; if m is an arbitrary integer, its class modulo n is denoted by $\overline{m} \in \mathbb{Z}_n$.
- If $n \geq 3$ is an integer, we denote by D_{2n} the dihedral group with $2n$ elements.
- For a square-free integer D other than 1, define $\mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} \mid a, b \in \mathbb{Z}\}$.
- Given a finite dimensional vector space V over a field F and an F-linear operator $T : V \to V$, we endow V with the structure of an $F[X]$-module such that $X \cdot v = T(v)$ for all $v \in V$.
- If R is a ring and M a (left) R-module, the annihilator of M in R is $Ann_R(M) = \{r \in R \mid r \cdot m = 0, \forall m \in M\}$.

ALGEBRA QUALIFYING EXAM

(I) Groups

(1) Determine if the following statements are true or false and *substantiate* your answer.

(a) [2 points] There are no non-constant group homomorphisms from A_8 to D_8.

(b) [3 points] Let G be a group and let H, K be two normal subgroups of G such that $G/H \simeq K$. Then G/K is isomorphic to H.

(2a) [2 points] Let G be a non-trivial group whose only subgroups are $\{1\}$ and G. Show that $G \simeq \mathbb{Z}_p$ for a prime number p.

(2b) [2 points] Let G be a group and N a normal subgroup of G with $N \neq \{1\}$ and $N \neq G$. Show that N is a maximal subgroup of G if and only if $|G : N|$ is a prime number.

(3a) [4 points] Classify all groups of order p^2 where p is a prime number.

(3b) [7 points] Show that a group of order 72 is not simple.

(II) Rings

(4) Determine if the following statements are true or false and *substantiate* your answer.

(a) [2 points] The polynomial $f(X, Y) = X^2 + Y^2 - 1$ is irreducible in $\mathbb{Q}[X, Y]$.

(b) [2 points] If R is a PID and $p \in R$ is irreducible then (p) is a maximal ideal of R.

(5) Let $R = \mathbb{Z}[\sqrt{-5}]$ and define the ideal $I = (3, 2 + \sqrt{-5})$.

(a) [4 points] Show that I is not a principal ideal.

(b) [4 points] Show that $1 + \sqrt{-5}$ is an irreducible but not a prime element of R.

(6) Let R be a commutative ring with $1 \neq 0$ and let

$$\mathfrak{N}(R) = \{a \in R \mid a^n = 0 \text{ for some positive integer } n\}.$$

(a) [5 points] Show that $\mathfrak{N}(R)$ is the intersection of all prime ideals of R.

(b) [3 points] *Use part (a)* to show that if $p(x) \in R[x]$ is a unit then all of its coefficients, except the constant term, must be nilpotent elements in R.

(III) Fields

7) Determine if the following statements are true or false and *substantiate* your answer.

(a) [1 points] Any algebraically closed field is infinite.

(b) [2 points] Suppose that $F = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ where $\alpha_i^2 \in \mathbb{Q}$ for $i \in \{1, 2, \ldots, n\}$. Prove that $\sqrt[3]{2} \notin F$.

(8) [8 points] Let $K = \mathbb{Q}(\sqrt[8]{2}, i)$ and $F = \mathbb{Q}(i)$. Show that K/F is a Galois extension and $Gal(K/F) \simeq \mathbb{Z}_8$.

(9) Let F be a field of characteristic $\neq 2$.
(a) [4 points] If E/F is a quadratic extension, i.e. $[E : F] = 2$, prove that $E = F(\sqrt{D})$ where D is a square-free element of F.
(b) [5 points] If $K = F(\sqrt{D_1}, \sqrt{D_2})$ where $D_1, D_2 \in F$ have the property that none of D_1, D_2, or $D_1 \cdot D_2$ is a square in F, prove that K/F is a Galois extension with $Gal(K/F)$ isomorphic to the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(IV) Modules and Linear Algebra

(10) Determine if the following statements are true or false and *substantiate* your answer.
(a) [1 points] Let R be a ring and M a left R-module. The torsion subset of M is a submodule of M.
(b) [4 points] Let V be a real vector space of odd dimension and $T : V \to V$ a linear operator on V. If the minimal polynomial of T is irreducible then T is diagonalizable (i.e. there exists a basis $\mathcal{B}$ of V such that the matrix representation of T with respect to $\mathcal{B}$ is a diagonal matrix).

(11) [5 points] Determine all possible Jordan canonical forms for a linear transformation with characteristic polynomial $(x-2)^3(x-3)^2$.

(12a) [5 points] Let N_1 and N_2 be two 5×5 nilpotent matrices over a field F. Show that if N_1 and N_2 have the same rank and the same minimal polynomial then N_1 and N_2 are similar.
(12b) [5 points] Let A and B be two $n \times n$ matrices over a field F such that A and B have the same characteristic polynomial

$$c(X) = (X - \lambda_1)^{d_1} \cdot \ldots \cdot (X - \lambda_l)^{d_l},$$

with $\lambda_1, \ldots, \lambda_l \in K$ pairwise distinct. Suppose furthermore that A and B have the same minimal polynomial, and that the matrices $A - \lambda_i \cdot I_n$ and $B - \lambda_i \cdot I_n$ have the same rank for all $i \in \{1, \ldots, l\}$, . If $d_i \leq 5$ for all $i \in \{1, \ldots, l\}$ show that A and B are similar.