ANALYSIS QUALIFYING EXAM AUGUST 2017

Instructions:

• Do ten questions.
• Do at most one question on each sheet of paper.
• Put your name on each sheet of paper you hand in.
• Use a paper clip to put your answers together.

(1) Define the Lebesgue outer measure \( m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty] \). Show \( m^*(I) = \ell(I) \) for any interval \( I \), where \( \ell(I) \) denotes the length of \( I \). Remember to include the cases where \( I \) is closed, open, half-open, and unbounded.

(2) Give Carathéodory’s definition for a subset of \( \mathbb{R} \) to be Lebesgue measurable. Show that if \( A \) and \( B \) are measurable, so is \( A \cup B \). Show that if \( A \) and \( B \) are measurable and \( A \cap B = \emptyset \), then \( m(A \cup B) = m(A) + m(B) \).

(3) Use the axiom of choice to show the existence of a non-measurable set.

(4) Define what it means for a function \( f : \mathbb{R} \to [-\infty, \infty] \) to be measurable. Show that if \( f \) is measurable, and \( B \) is Borel, then \( f^{-1}(B) \) is measurable.

(5) Suppose that we have defined \( \int f \) for \( f \) simple and supported on a set of finite measure, and that we know that this definition is additive. Show how to define \( \int f \) for any measurable \( f \geq 0 \). Show that \( \int (f + g) = \int f + \int g \) whenever \( f, g \geq 0 \).

(6) Give the statement of Vitali’s Convergence Theorem for uniformly integrable sequences of functions. Define what it means for a function to be absolutely continuous. Show that if \( f \) is absolutely continuous and increasing on \( [a, b] \), then \( \int_a^b f' = f(b) - f(a) \).

(7) Define the spaces \( L^p(E) \) for \( 1 \leq p < \infty \) where \( E \) is a measurable subset of \( \mathbb{R} \). State and prove Minkowski’s inequality. You may quote Hölder’s inequality without proof.

(8) Show that the simple functions are dense in \( L^p(E) \) for \( 1 \leq p < \infty \) for any measurable \( E \). Show that the step functions are dense in \( L^p(I) \) for \( 1 \leq p < \infty \) for any interval bounded \( I \).

(9) Consider the sequence of functions \( f_n(x) = \sin(nx) \) on \( [0, 1] \). Show that \( f_n \) converges weakly to the zero function on \( L^p([0, 1]) \). (Hint, first show that \( \int_{[0,1]} f_n I_{[a,b]} \to 0 \) for any \( 0 \leq a \leq b \leq 1 \).)

(10) State the Tonelli and Fubini Theorems. You may assume the construction of the product measure without proof. Give a counterexample to Tonelli’s Theorem that shows the necessity of the assumptions that the measure spaces are \( \sigma \)-finite.

(11) Define what is meant by a signed measure on a sigma-algebra. Define what it means for a measurable set to be positive with respect to a signed measure. Show that for any signed measure, then there exists a positive measurable set, whose complement is also positive. (For simplicity, you may assume throughout that the signed measure is bounded.)
(12) State and prove the Radon-Nikodym Theorem. You may assume that all measures are finite and unsigned. You may assume the Riesz Representation Theorem for Hilbert spaces, or the Jordan/Hahn Decomposition Theorem for signed measures, without proof.

(13) State the Closed Graph Theorem. Show that a closed subspace $Y$ of a Banach space $X$ has a closed complement if and only if there is a continuous projection $X \to Y$. 