Analysis Qualifying Exam - May 2018

- *Work through all parts included in the 8 items below.*
- *Start each item on a clean sheet of paper.*
- *Write only on one side of each sheet.*
- *Put your name on each sheet of paper you hand in.*
- *Your work will be graded for correctness, completeness, and clarity.*

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Define completeness of the measure μ, with respect of the σ-algebra $\mathcal{M}$. Prove that there is a σ-algebra $\overline{\mathcal{M}}$ over X and a measure $\overline{\mu}$ on $\overline{\mathcal{M}}$, such that $\mathcal{M} \subseteq \overline{\mathcal{M}}$, $\overline{\mu}$ is complete, and $\mu(E) = \overline{\mu}(E)$ for all $E \in \mathcal{M}$. Prove that $\overline{\mu}$ is the unique extension of μ which is complete on $\overline{\mathcal{M}}$.
2. Define outer measure. Define μ^*-measurability. State the Hahn-Kolmogorov-Caratheodory Theorem. Outline briefly the construction of the Lebesgue measure.
3. State and prove the Monotone Convergence Theorem.
4. Make use of the Dominated Convergence Theorem, and not any other method, to evaluate the following (dm =Lebesgue measure):
$$a) \lim_{n\to+\infty} \int_{[0,\infty)} \frac{x+ne^x}{1+ne^{\pi x}}\, dm, \qquad b) \lim_{n\to+\infty} \int_{[0,\infty)} \frac{\sin(nx)}{nx^{3/2}+x^2}\, dm.$$
5. Define the average operator A_r on locally integrable functions. Prove that if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ then $\lim_{r\to 0} A_r f(x) = f(x)$ for a.e. x in $\mathbb{R}^n$ (you can assume the maximal theorem, and the density of continuous functions with compact support in $L^1(\mathbb{R}^n)$.)
6. State the Hahn-Banach Theorem for vector spaces over $\mathbb{R}$. Give a proof of the theorem in the case of extensions from a vector space to a larger vector space of codimension 1.
7. Consider the space $\mathcal{X}_0 = C[0,2] \subseteq \mathcal{X} := L^1[0,2]$, both endowed with L^1 norm w.r. to the Lebesgue measure, and the linear functional $T_0 : \mathcal{X}_0 \to \mathbb{C}$ defined as
$$T_0 f = \int_0^2 t^2 f(t)dt,$$
   a) Show that T_0 is bounded, and compute $\|T_0\|$ from the definition of operator norm (and not any other method).
   b) Show that if $f_0 = \chi_{[0,1]}$, an element of $\mathcal{X} \setminus \mathcal{X}_0$, then there is a linear extension T of T_0 defined on the space $\mathcal{X}_0 + \mathbb{R}f_0$ which is not bounded (find such T explicitly).
8. On a $\sigma-$finite measure space $(X, \mathcal{M}, \mu)$ consider the map $J : g \to \phi_g$
$$\phi_g(f) = \int_X fg\, d\mu$$
Assume as given that for $1 \le p \le \infty$ the map in J is an isometry from L^q to $(L^p)^*$, where $p^{-1} + q^{-1} = 1$, and that if for some g measurable ϕ_g is well defined and in $(L^p)^*$ then $g \in L^q$. Prove that if $1 \le p < \infty$ then J is surjective. Show that J is not in general surjective if $p = \infty$ (provide a counterexample).