Qualifying Examination

(*May 2018*)

- You are allowed to rely on a previous part of a multi-part problem even if you do not prove the previous part.

- Let n denote a positive natural number, i.e., $n \in \mathbb{N}^+$.

- Let S_n denote the symmetric group on n letters.

- Let $\mathbb{Z}$ denote the group or ring of integers with the usual operations and let $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the groups or fields of rational, real, and complex numbers, respectively, with the usual operations.

- All rings are assumed to be commutative with identity.

- Let K denote a field and $\mathcal{M}_{n\times n}(K)$ denote the set of all $n \times n$ matrices with entries in K. Let $(V, \mathcal{B}, f)$ stand for the n-dimensional K-vector space V with basis $\mathcal{B}$ and bilinear form $f : V \times V \to K$. Let $[f]_{\mathcal{B}}$ be the representing matrix of the bilinear form f on V with respect to the basis $\mathcal{B}$. Let I denote the $n \times n$ identity matrix over the field K.

Algebra Qualifying Exam

(I)Groups

(**1a**) (3*points*) State Cauchy's Theorem.
(**1b**) (4*points*) Give a self-contained proof of Cauchy's Theorem for the case that the group in question is *abelian*. (Obviously, your proof must not invoke Cauchy's Theorem.)

(**2a**) (5*points*) Let H be a subgroup of G such that $[G : H] = n$. Prove that there exists a group homomorphism $\phi : G \to S_n$ with $Ker(\phi)$ a subgroup of H.
(**2b**) (5*points*) Prove that there is no simple group of order 36.

(**3**) (3*points*) Determine if the following statement is true or false and *substantiate* your answer.
The symmetric group S_5 is a solvable group.

(II)Rings

(**4a**) (5*points*) State Eisenstein's Criterion.
(**4b**) (5*points*) Determine if the following statement is true or false and *substantiate* your answer.
Let $f(X) = 2X^5 - 6X^3 + 9X^2 - 15 \in \mathbb{Z}[X]$. Then $f(X)$ is irreducible in $\mathbb{Q}[X]$ *and* in $\mathbb{Z}[X]$.

(**5**) Let R be a commutative ring with identity 1_R. Let S be a multiplicatively closed subset of the ring R such that $0_R \notin S$.
(**a**) (5*points*) Prove that there exists an ideal I of R, maximal with respect to $I \cap S = \emptyset$.
(**b**) (5*points*) Prove that the ideal I in (5a), above, is a prime ideal of R.

(III)Fields

(**6**) (5*points*) Let F be a finite-dimensional field extension of the field K. Prove that F is an algebraic and finitely generated field extension of K.

(**7a**) (5*points*) Let K be a field such that $|K| = q < \infty$. Let $f(X) := X^q - X \in K[X]$. Prove that $f(a) = 0_K$ for each $a \in K$.
(**7b**) (5*points*) Determine if the following statement is true or false and *substantiate* your answer.
Let K be a field such that $|K| = 4$ and let F be a field such that $|F| = 8$. Then K cannot be a subfield of F.

(**8**) (5*points*) Let K be a field and $g(X) \in K[X]$ be a monic, irreducible polynomial. Let $E = K(u_1, u_2)$ be a splitting field of $g(X)$ over K such that $u_1 \neq u_2$ and $g(X) = (X - u_1)^{n_1}(X - u_2)^{n_2}$ for some $n_1, n_2 \in \mathbb{N}^+$. What is the relationship between $deg(g(X))$ and n_1? Give a detailed proof and state precisely any results that you are using.

(IV)Modules

(**9**)
(**a**) (5*points*) Give the definition of a divisible R-module.
(**b**) (5*points*) Let E be an injective R-module. Prove that E is a divisible R module.

(**10**) Determine if the following statement is true or false and *substantiate* your answer.
(**a**) (5*points*) R is a flat R-module.
(**b**) (5*points*) Let L, M, N be non-zero R-modules such that $L \otimes_R M = 0$. Then $Hom_R(M, Hom_R(L, N)) = 0 = Hom_R(L, Hom_R(M, N))$.

(V)LinearAlgebra

(**11**) (5*points*) Determine if the following statement is true or false and *substantiate* your answer.
Let $(V, \mathcal{B}, f)$ be a *real* (i.e., $K = \mathbb{R}$), bilinear form such that

$$[f]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Then f has an orthonormal basis $\mathcal{B}'$ for V.

(**12**) Substantiate fully your answers in the following. Let

$$A := \begin{pmatrix} 3 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 1 & 1 \end{pmatrix}$$

(**a**) (5*points*) Is A diagonalizable over $\mathbb{R}$?
(**b**) (5*points*) Compute the matrix $A^3 - 5A^2 + 8A - 3I$.

(**13**) (5*points*) Substantiate fully your answers in the following. Let $A \in \mathcal{M}_{5\times 5}(\mathbb{R})$ be of minimal rank with respect to having $\{0, 1, 2\}$ as its set of all characteristic values. Write down (up to similarity) all the possible Jordan Canonical Forms of A.